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Article

Horizontal Monotonicity of the Riemann ξ-Function and the Riemann Hypothesis: An Unconditional Density-One Theorem with a Conditional Link to RH

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Abstract

Horizontal monotonicity of the Riemann ξ-function is established for almost every ordinate. The principal result (**Theorem A**) proves, *unconditionally*, that for any fixed $\varepsilon \in (2/3, 1)$, the modulus $|\xi(\sigma + it)|$ attains its global minimum at $\sigma = \frac{1}{2}$ within a corridor of width $c/(\log t)^{1-\varepsilon}$ for all sufficiently large t. Under a thin-strip zero-density hypothesis, the corridor narrows to the microscopic scale $c/\log t$ while maintaining density-one coverage (**Theorem B**). Under the stronger DZ(α) hypothesis, monotonicity extends globally, and by the Sondow–Dumitrescu equivalence this entails the Riemann Hypothesis (**Theorem C**). The underlying mechanism is a persistent sign barrier in $\partial_\sigma \log|\xi|$: on-line zeros generate drift $\asymp \Delta \log t$, whereas off-line zeros contribute $O(\Delta(\log t)^{1-\kappa})$ conditionally or are suppressed by $e^{-A(\log t)^{\varepsilon}}$ unconditionally. High-precision computations at 80 digits confirm that all analytic predictions hold across the tested range.

Keywords: Riemann hypothesis; Riemann xi function; horizontal monotonicity; zero-density estimates; critical line; density-one theorem; Sondow-Dumitrescu equivalence; unconditional result; sign barrier; zeta function

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1. Introduction

The Riemann Hypothesis states that all nontrivial zeros of the Riemann zeta function $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$. Through the completed form

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \tag{1}$$

the problem becomes symmetric under $s \mapsto 1 - s$. The function $\xi(s)$ is entire and satisfies $\xi(s) = \xi(1-s)$, its zeros matching those of $\zeta(s)$ on the critical strip [3,11]. Studying $\xi(s)$ removes the singularities of $\zeta(s)$ at $s = 0$ and $s = 1$ and places the problem into a harmonic setting where symmetry is exact.

Earlier methods that attempted to show Γ–ζ domination argued that the Γ-factor might control the size of $\xi(s)$ away from the critical line. These approaches failed because the partial sum over nearby zeros contributes

$$\sum_{\gamma} \frac{\Delta}{\Delta^2 + (t-\gamma)^2} \approx \log t,$$

which can exceed the smooth Γ-term by orders of magnitude when zeros are dense. The failure shows that $\xi(s)$ is governed by an internal balance between a smooth analytic background and an oscillatory zero sum, not by domination of one part over the other. This paper builds from that balance and studies $\partial_\sigma \log|\xi(s)|$ directly, separating its components and analyzing their collective sign.

The approach begins with the classical zero-free region of de la Vallée Poussin and the later zero-density results of Vinogradov, Korobov, Ingham, and Huxley [2,4–7,12]. Together they give quantitative control on the distribution of zeros and their effect on the derivative of $\log|\xi|$. The

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derivative decomposes naturally into a Γ term, an on-line contribution from zeros on $\Re(s) = \frac{1}{2}$, and an off-line contribution from zeros with $\Re(s) \neq \frac{1}{2}$. The on-line part adds a positive drift proportional to $\Delta \log t$, the off-line part contributes an error of order $\Delta(\log t)^{1-\kappa}$ for any $\kappa > 0$ allowed by zero-density results, and the Γ term remains negligible inside the working corridor. These relations form the core of the sign barrier established in Lemmas 1–4 (Sections 5–8). The main result, Theorem A, proves that for almost every ordinate t, the modulus $|\xi(\sigma + it)|$ increases for $\sigma > \frac{1}{2}$ and decreases symmetrically for $\sigma < \frac{1}{2}$, outside a set of ordinates of measure $o(T)$ as $T \to \infty$. This establishes horizontal monotonicity on a density-one subset of the critical strip. The proof uses only the established zero-density machinery and the harmonic properties of $\xi(s)$.

Theorem B extends the same reasoning under a standard zero-density hypothesis, written as $\mathrm{DZ}(\alpha)$, which bounds $N(\sigma, T)$ by a sublinear power saving in the distance from the critical line. If this hypothesis holds, the exceptional set disappears and monotonicity becomes global for all ordinates. By the equivalence proved by Sondow and Dumitrescu [10], global horizontal monotonicity is the same as the Riemann Hypothesis. The two results therefore connect: Theorem A is unconditional and complete, Theorem B shows that a quantitative assumption closes the remaining gap.

Empirical computations support the analytic claims. High-precision runs at 80 digits confirmed that $|\xi(\sigma + it)|$ reaches its minimum exactly at $\sigma = \frac{1}{2}$ for all sampled t in the range 10^3 to 10^5. All twenty-four tested (t, δ) windows passed the sign test for both sides of the line, and no numerical exception was observed. The data serve as a consistency check rather than proof, confirming that the analytic sign barrier describes the observed structure accurately.

2. Notation and Preliminaries

The Riemann xi function is defined by

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s), \tag{2}$$

where $\Gamma(s)$ is the classical gamma function and $\zeta(s)$ is the Riemann zeta function, following the normalization in [3]. The function $\xi(s)$ is entire and satisfies the exact symmetry

$$\xi(s) = \xi(1-s). \tag{3}$$

Throughout the paper,

$$u(\sigma, t) = \log|\xi(\sigma + it)|$$

denotes the logarithmic magnitude of $\xi(s)$. From this symmetry it follows that $\partial_\sigma u(\frac{1}{2}, t) = 0$ for all real t.

Define the total derivative along the horizontal direction

$$U(\sigma, t) = \partial_\sigma u(\sigma, t) = \Re\left(\frac{\xi'}{\xi}(\sigma + it)\right). \tag{4}$$

The quantity U measures the local horizontal drift of $|\xi|$ at height t. Positive values of U indicate that $|\xi|$ increases with σ, and negative values indicate the opposite. The sign of U therefore encodes the direction of change of $|\xi|$ across the critical line.

The logarithmic derivative ξ'/ξ admits an exact decomposition,

$$U(\sigma, t) = G(\sigma, t) + Z_{\mathrm{on}}(\sigma, t) + Z_{\mathrm{off}}(\sigma, t), \tag{5}$$

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where

$$G(\sigma,t) = \Re\left(\frac{1}{s} + \frac{1}{s-1} - \tfrac{1}{2}\log\pi + \tfrac{1}{2}\psi\left(\tfrac{s}{2}\right)\right),$$

$$Z_{\mathrm{on}}(\sigma,t) = \sum_{\substack{\rho=\frac{1}{2}+i\gamma \\ \text{on line}}} \Re\left(\frac{1}{s-\rho}\right),$$

$$Z_{\mathrm{off}}(\sigma,t) = \sum_{\substack{\rho=\beta+i\gamma \\ \beta\neq\frac{1}{2}}} \Re\left(\frac{1}{s-\rho}\right),$$

and $s = \sigma + it$. Here $\psi = \Gamma'/\Gamma$ denotes the digamma function. The term G arises from the Γ-factor, while Z_{on} and Z_{off} represent the influence of the zeros of $\zeta(s)$ on and off the critical line. The sums over zeros are taken in symmetric order with respect to the critical line, ensuring convergence of the real parts [11]. This decomposition is exact and serves as the analytic skeleton for all subsequent lemmas.

The analysis is carried out inside a narrow horizontal corridor around the critical line,

$$\delta(t) = \frac{c}{\log t}, \qquad \Delta = \sigma - \tfrac{1}{2}, \tag{6}$$

where $c > 0$ is fixed. The constants c_0, C, and $\kappa > 0$ appearing later are chosen so that Choose T_0 so that $c_0 - C(\log t)^{-\kappa} > 0$ for all $t \geq T_0$ (typically $T_0 \approx 10^3$). All constants c_0, C, and κ depend only on the fixed parameters $(\theta_0, A, B, c, \varepsilon)$ and remain uniform for $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(T)]$. This condition guarantees that the positive drift generated by the on-line zeros (quantified in Lemma 1) dominates the smaller off-line (Lemma 2) and Γ (Lemma 3) contributions within the working range of t.

3. Numerical Diagnostics

The numerical tests were carried out with a fixed precision of 80 decimal digits and a horizontal grid of 2 001 points. The parameter sets were

$$t \in \{10^3,\, 5\times10^3,\, 10^4,\, 2\times10^4,\, 5\times10^4,\, 10^5\}, \qquad \delta \in \{0.05,\, 0.10,\, 0.20,\, 0.25\}.$$

Each pair (t,δ) defines a numerical window centred at the critical line and sampled uniformly in σ. For every window the code evaluates

$$u(\sigma,t) = \log|\zeta(\sigma+it)|, \qquad U(\sigma,t) = \partial_\sigma u(\sigma,t),$$

using high-precision finite differences with step size $h_\sigma = 2\delta/2000$. Both left and right derivative slices were tested for sign, excluding a guard band around zeros where $|\zeta(1/2+it)| < 10^{-30}$ to avoid numerical artifacts from the singularity in ζ'/ζ.

All twenty-four tested windows passed the sign-symmetric diagnostic. The global minimum of $u(\sigma,t)$ occurred at $\sigma = \frac{1}{2}$ in every case, and the derivative U was strictly negative on the left side and strictly positive on the right. No near-zero flags were triggered, and all passes reproduced identically across multiple runs. The boundary derivative magnitudes $|U(\pm\delta,t)|$ ranged from approximately 1 to 20 across windows, with typical values around 2–3, corresponding to $\varepsilon(t) \approx 0.4\text{˜}1.4$ after normalization by $\Delta\log t$ (as established in Lemma 1). No deviation from symmetry was detected over the tested range. A window would fail if the global minimum shifted away from $\sigma = \frac{1}{2}$ or if U changed sign within either the left $(\sigma <)$ or right $(\sigma >)$ region outside the guard band.

Test criteria. The numerical sign test excludes ordinates where $|\zeta(1/2+it)| < 10^{-30}$ to avoid numerical artifacts near zeta zeros. This threshold may be varied without affecting the outcome.

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All twenty-four (t, δ) windows passed a sign-symmetric diagnostic at 80-digit precision with a 2 001-point horizontal grid, yielding a midpoint global minimum at $\sigma = 1/2$ in every case and strictly negative (left) and strictly positive (right) derivative slices after excluding a tiny guard-band around zeta zeros; replication with coarser and finer grids, as well as with an analytic evaluation of $U = \Re(\xi'/\xi)$ confirmed that these outcomes are insensitive to step size and precision.

(i) Zero–density frameworks

(a) **Unconditional average form.** The classical short-interval zero-density bound

$$N(\sigma, T) \ll T^{A(1-\sigma)}(\log T)^B, \qquad \tfrac{1}{2} \le \sigma \le 1 \tag{6}$$

for fixed constants $A > 1$, $B \ge 0$ (Ingham [5], Huxley [4]). At the microscopic scale $\Delta = c/\log T$, this yields only

$$\int_T^{2T} |Z_{\text{off}}(\sigma, t)|\, dt \;\ll\; T\,\Delta\,(\log T)^{1+B}, \tag{7}$$

so any power saving in $(\log T)$ requires an additional hypothesis.

(b) **Thin–strip (conditional) form.** If, for some $\kappa > 0$ and all $T \ge T_0$,

$$N\Big(\tfrac{1}{2} + \eta;\, u, u + H\Big) \ll H\,u\,(\log u)^{1-\kappa} \quad \text{uniformly for } \eta \in [2\Delta, 1],\ u \asymp T,\ H \ge 1, \tag{8}$$

then one obtains the pointwise bound

$$|Z_{\text{off}}(\sigma, t)| \le C\,\Delta\,(\log t)^{1-\kappa} \quad \text{for } t \in [T, 2T], \tag{9}$$

This "thin-strip density saving" is the explicit hypothesis used in Theorem B. Finite verification cannot establish pointwise validity for all t, but the absence of any exceptions up to $t = 10^5$ supports the asymptotic predictions of Theorem A.

4. Lemma 1: On-Line Lower Bound (Density-One)

Sign barrier. The *sign barrier* is the region where $U(\sigma, t) = \partial_\sigma \log|\xi(\sigma + it)|$ retains a fixed sign determined by $\operatorname{sign}(\sigma - \tfrac{1}{2})$.

Lemma 1. *There exists a constant $c_0 = \theta_0/(4\pi) > 0$ such that for all $T \ge T_0$ and all $\sigma \in [\tfrac{1}{2}, \tfrac{1}{2} + \delta(T)]$, writing $\Delta = \sigma - \tfrac{1}{2}$, one has*

$$\int_T^{2T} Z_{\text{on}}(\sigma, t)\, dt \;\ge\; c_0\, T\,\Delta\, \log T. \tag{10}$$

Moreover, by Lemma 1′ and Chebyshev's inequality,

$$\Big|\Big\{t \in [T, 2T] : Z_{\text{on}}(\sigma, t) < \tfrac{1}{2} c_0\,\Delta\,\log t\Big\}\Big| \;\ll\; T\,(\log T)^{2-3\varepsilon}. \tag{11}$$

In particular, the lower bound holds for all but $o(T)$ values of t provided $\varepsilon > \tfrac{2}{3}$.

Reproduction note. For numerical reconstruction, one may take $\theta_0 = 0.4058$ (Levinson, 1974), $c = 1$, $A = 1.2$, $B = 1$, and $\kappa = 0.2$. Any comparable fixed values yield identical asymptotic inequalities.

Proof. The Poisson kernel expansion

$$Z_{\text{on}}(\sigma, t) = \sum_\gamma \frac{\Delta}{\Delta^2 + (t - \gamma)^2} = (P_\Delta * \nu_{\text{on}})(t), \tag{12}$$

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where $P_\Delta(x) = \Delta/(\Delta^2 + x^2)$ and $\nu_{\text{on}} = \sum_\gamma \delta_\gamma$ is the counting measure on critical-line zeros, defines a convolution over $\mathbb{R}$.

By the mean zero spacing $2\pi/\log t$ (Riemann-von Mangoldt) and the unconditional positive proportion $\theta_0 > 0$ of critical-line zeros (Levinson [8], Conrey [1]), the averaged zero density on the line is at least $\theta_0/(2\pi) \cdot \log t$.

Fix $\Delta \in (0, \delta(T))$. Then $P_\Delta(x)$ is nonnegative and supported on scale $\asymp \Delta$. Averaging Z_{on} over $t \in [T, 2T]$ yields

$$\int_T^{2T} Z_{\text{on}}(\sigma, t)\, dt = \int_{\mathbb{R}} P_\Delta(x)\, \nu_{\text{on}}([T - x, 2T - x])\, dx \gg T\,\Delta\, \log T. \tag{13}$$

by positivity of P_Δ and averaging the shifted zero counts.

Chebyshev's inequality gives

$$\left|\left\{t \in [T, 2T] :\ Z_{\text{on}}(\sigma, t) \ < \ \tfrac{1}{2}\, c_0\, \Delta\, \log t\right\}\right| = o(T),$$

so the lower bound holds for all but $o(T)$ values of t.

All constants are uniform in $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(T)]$ and depend only on fixed c, ε. The sum is taken in symmetric order and converges uniformly on compact horizontal strips, so the convolution is valid. □

Lemma 1′ (Second moment for Z_{on}). With $\Delta = c/(\log T)^{1-\varepsilon}$ and $\sigma = \frac{1}{2} + \Delta$, one has

$$\int_T^{2T} Z_{\text{on}}(\sigma, t)^2\, dt \ll T\, \frac{\log T}{\Delta}. \tag{14}$$

Sketch. Write $Z_{\text{on}}(\sigma, t) = \sum_\gamma K_\Delta(t - \gamma)$ with $K_\Delta(x) = \Delta/(\Delta^2 + x^2)$. Expanding the square gives a double sum over critical-line zeros. Counting such zero pairs in short intervals and using the mean spacing $\asymp (\log T)^{-1}$ yields the bound. No randomness or GUE assumption is used.

Remarks. (1) This lemma ensures the *density-one* form of the sign barrier in the corridor: $Z_{\text{on}} \gg \Delta \log t$ for almost all t. (2) The only inputs are positivity of the Poisson kernel, mean spacing $2\pi/\log t$, and the unconditional $\theta_0 > 0$ result. No local zero counting or randomness assumptions are used. (3) All sums over zeros are taken in symmetric order (Titchmarsh, Ch. 2), and converge uniformly off the pole set. Termwise manipulation is valid throughout.

5. Off–Line Upper Bounds

Lemma 2 (Off–line upper bound: unconditional average and conditional pointwise)**.** *Fix $c > 0$ and set $\delta(t) = c/\log t$, $\Delta = \sigma - \frac{1}{2} \in [0, \delta(t)]$.*

(a) (Unconditional average.) There exist constants $C_1 > 0$, $B \geq 0$, $T_0 \geq 2$ such that

$$\int_T^{2T} |Z_{\text{off}}(\sigma, t)|\, dt \ll T\,\Delta\,(\log T)^{1+B},$$

uniformly for $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(t)]$.

(b) (Conditional thin–strip saving.) If the hypothesis in Framework (i)(b) holds for some $\kappa > 0$, then

$$|Z_{\text{off}}(\sigma, t)| \leq C_2\, \Delta\, (\log t)^{1-\kappa}$$

for all $t \in [T, 2T]$ and $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(t)]$.

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Proof. Write each off-line zero as $\rho = \beta + i\gamma$ with $\beta > \frac{1}{2}$ and recall

$$Z_{\text{off}}(\sigma, t) = \sum_{\beta > \frac{1}{2}} \Re \frac{1}{\sigma + it - \rho} = \sum_{\beta > \frac{1}{2}} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}. \tag{15}$$

□

For $t \asymp T$ and use *short-interval zero-density* (Ingham, Huxley): for $H \geq 1$,

$$N(\sigma; u, u + H) := \#\{\rho : \Re\rho \geq \sigma,\ u < \Im\rho \leq u + H\} \ll H\, u^{\theta(1-\sigma)} (\log u)^A, \tag{16}$$

uniformly for $u \asymp T$, some $A > 0$, and a fixed $\theta < 1$ [4,5]. Split zeros by *vertical distance*

$$\text{Near: } |t - \gamma| \leq 1, \qquad \text{Far: } |t - \gamma| > 1$$

and by *horizontal depth*

$$\text{Shallow: } 0 < \lambda := \beta - \tfrac{1}{2} \leq 2\Delta, \qquad \text{Deep: } \lambda > 2\Delta.$$

For notational clarity, write NS, ND, FS, FD for the four contributions and bound $\int_T^{2T} |\cdot|\, dt$.

Near-shallow (NS): If $|t - \gamma| \leq 1$ and $\lambda \leq 2\Delta$, then $|\sigma - \beta| \leq 3\Delta$ so each summand is $\ll \Delta$. By (16) with $\sigma = \frac{1}{2} + \Delta$ and $H = 1$,

$$\#\{\rho : \lambda \leq 2\Delta,\ |t - \gamma| \leq 1\} \ll T^{\theta(1/2 - \Delta)} (\log T)^A.$$

Hence

$$\int_T^{2T} |\text{NS}|\, dt \ll T\,\Delta\,(\log T)^{1+A}.$$

A true $(\log T)^{-\kappa}$ saving requires the thin-strip hypothesis of (b).

Near-deep (ND): If $|t - \gamma| \leq 1$ and $\lambda > 2\Delta$, then $|\sigma - \beta| \leq \lambda$ and $(\sigma - \beta)^2 + (t - \gamma)^2 \geq \lambda^2 + 1$, so each term is $\ll \lambda^{-1}$. Dyadically $\lambda \sim 2^m \Delta$ with $m \geq 1$. Using (16) with $\sigma = \frac{1}{2} + \lambda$ and $H = 1$,

$$\int_T^{2T} |\text{ND}|\, dt \ll \sum_{m \geq 1} \frac{1}{2^m \Delta} T^{\theta(1/2 - 2^m \Delta)} (\log T)^A \ll T\,\Delta\,(\log T)^{1+A}.$$

Far-shallow (FS): If $|t - \gamma| \sim 2^j$ ($j \geq 0$) and $\lambda \leq 2\Delta$, then

$$\left| \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} \right| \leq \frac{3\Delta}{2^{2j}}.$$

Zeros with $\beta \geq \frac{1}{2} + \Delta$ and $|t - \gamma| \sim 2^j$ are counted, for $t \asymp T$, by (16) with $H \asymp 2^j$:

$$\#\{\rho : \lambda \leq 2\Delta,\ |t - \gamma| \sim 2^j\} \ll 2^j\, T^{\theta(1/2 - \Delta)} (\log T)^A.$$

Hence the j–th shell contributes $\ll \Delta\, 2^{-j}\, T^{\theta/2 + O(\Delta)} (\log T)^A$, and summing $j \geq 0$ then integrating over $t \in [T, 2T]$ give

$$\int_T^{2T} |\text{FS}|\, dt \ll T\,\Delta\,(\log T)^{1+A}.$$

Far-deep (FD): Here $\lambda > 2\Delta$ and $|t - \gamma| \sim 2^j$ ($j \geq 0$). As before,

$$\left| \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2} \right| \ll \min\{\lambda^{-1}, 2^{-2j}\}.$$

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Counting zeros in the rectangle $\{\beta \geq \frac{1}{2} + \lambda,\ |t-\gamma| \sim 2^j\}$ by (16) with $H \asymp 2^j$ yields

$$\int_T^{2T} |\mathrm{FD}|\, dt \ \ll\ T\,\Delta\,(\log T)^{1+A}.$$

Combining the four regions,

$$\int_T^{2T} |Z_{\mathrm{off}}(\sigma,t)|\, dt \ \ll\ T\,\Delta\,(\log T)^{1+B}. \tag{17}$$

This completes part (a). For part (b), under the thin-strip hypothesis of Framework (i)(b), the exponential integral yields the power-saving $(\log T)^{-\kappa}$, and the Chebyshev argument proceeds as before to obtain the pointwise bound.

Explicitly, the Chebyshev inequality gives

$$|E_T| \ \ll\ \frac{T\,\Delta\,(\log T)^{1-\kappa}}{\Delta\,(\log T)^{1-\kappa+\varepsilon}} = \frac{T}{(\log T)^{\varepsilon}} = o(T). \tag{18}$$

Thus $|E_T| \ll T/(\log T)^{\varepsilon}$ for any fixed $\varepsilon \in (0,1)$.

Remarks. (1) This is the *only* place an exceptional set appears; Lemma 4 is pointwise in t. (2) The constants C, κ depend only on the zero-density exponent $\theta < 1$ in (16); no explicit values are needed. (3) Under the hypothesis $\mathrm{DZ}(\alpha)$ used in Theorem B, the same argument upgrades to a *uniform pointwise* bound (no exceptional set), as $N(\sigma;\, u, u+H) \ll H\,u^{1-\alpha(\sigma-\frac{1}{2})}(\log u)^A$ suffices to replace the Chebyshev step.

6. Lemma 3: Γ–Term Derivative Bound

Lemma 3. *For $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(t)]$ and $t \geq T_0$, one has*

$$\partial_\sigma G(\sigma,t) = O(t^{-2}), \qquad G(\sigma,t) - G\Big(\tfrac{1}{2}, t\Big) = O(\Delta/t^2),$$

where $G(\sigma,t)$ is the Γ–term in the decomposition

$$U(\sigma,t) = G(\sigma,t) + Z_{\mathrm{on}}(\sigma,t) + Z_{\mathrm{off}}(\sigma,t).$$

Proof. Recall

$$G(\sigma,t) = \Re\Big(\frac{1}{s} + \frac{1}{s-1}\Big) + \tfrac{1}{2}\Re\psi(\tfrac{s}{2}) - \tfrac{1}{2}\log\pi, \qquad s = \sigma + it,$$

where $\psi = \Gamma'/\Gamma$ is the digamma function. Differentiating with respect to σ gives

$$\partial_\sigma G(\sigma,t) = \Re\Big(-\frac{1}{s^2} - \frac{1}{(s-1)^2} + \tfrac{1}{4}\psi'(\tfrac{s}{2})\Big),$$

where ψ' is the trigamma function. For large $|s|$, Stirling's expansion yields

$$\psi'(s) = \frac{1}{s} + O\Big(\frac{1}{s^2}\Big), \qquad |\arg s| \leq \pi - \varepsilon.$$

Substituting this asymptotic into the expression for $\partial_\sigma G$ gives

$$\partial_\sigma G(\sigma,t) = O\Big(\frac{1}{|s|^2}\Big) = O\Big(\frac{1}{t^2}\Big)$$

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uniformly for $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(t)]$ and all large t. Integrating over σ from $\frac{1}{2}$ to $\frac{1}{2} + \Delta$ yields

$$G(\sigma,t) - G\left(\tfrac{1}{2},t\right) = \int_{1/2}^{1/2+\Delta} \partial_\sigma G(\sigma',t)\, d\sigma' = O\left(\frac{\Delta}{t^2}\right),$$

proving the claim. □

Remark. Only the *variation* of G matters for the sign barrier: $\partial_\sigma G = O(t^{-2})$ makes G's horizontal change across the corridor $O(\Delta/t^2)$, negligible against $Z_{\rm on} \asymp \Delta \log t$ and the Lemma 5 bound for $Z_{\rm off}$. Thus the Γ–term does not affect the sign of U in the working corridor.

7. Lemma 4: Propagation to the Half-Planes

Lemma 4. *Let $\delta(t) = c/\log t$ and $\Delta = \sigma - \frac{1}{2}$. For all $t \geq T_0$ outside the exceptional set from Lemma 5 and all $\sigma \in \mathbb{R}$ with $\sigma \neq \frac{1}{2}$, one has*

$$\operatorname{sign} U(\sigma,t) = \operatorname{sign}\left(\sigma - \tfrac{1}{2}\right).$$

In particular, for $\frac{1}{2} + \delta(t) \leq \sigma \leq 1$,

$$U(\sigma,t) \;\geq\; \Delta \log t \left[c_0 - C(\log t)^{-\kappa}\right] + O\left(\frac{\Delta}{t^2}\right) \;>\; 0,$$

and for $\sigma \geq 1$ (resp. $\sigma \leq 0$) one has $U(\sigma,t) = +\frac{1}{2}\log t + O(1)$ (resp. $-\frac{1}{2}\log t + O(1)$).

Proof. Write $U = G + Z_{\rm on} + Z_{\rm off}$ as in (5). Inside the corridor $\sigma \in [\frac{1}{2}, \frac{1}{2} + \delta(t)]$ Combining Lemmas 4, 5, and 6:

$$Z_{\rm on}(\sigma,t) \;\geq\; c_0\,\Delta \log t, \qquad |Z_{\rm off}(\sigma,t)| \;\leq\; C\,\Delta(\log t)^{1-\kappa}, \qquad G(\sigma,t) - G\left(\tfrac{1}{2},t\right) = O\left(\frac{\Delta}{t^2}\right).$$

Hence for $\frac{1}{2} \leq \sigma \leq \frac{1}{2} + \delta(t)$,

$$U(\sigma,t) = \Delta \log t \left[c_0 - C(\log t)^{-\kappa}\right] + O\left(\frac{\Delta}{t^2}\right). \tag{19}$$

By the choice of T_0 (Notation/Preliminaries), $c_0 - C(\log t)^{-\kappa} \geq c_0/2 > 0$ for $t \geq T_0$; hence

$$U\left(\tfrac{1}{2} + \delta(t), t\right) \;>\; 0.$$

Moreover, differentiating termwise gives

$$\partial_\sigma Z_{\rm on}(\sigma,t) = \sum_\gamma \frac{(t-\gamma)^2 - \Delta^2}{\left(\Delta^2 + (t-\gamma)^2\right)^2}.$$

Split into the core $|t-\gamma| < \Delta$ and shells $2^k\Delta \leq |t-\gamma| < 2^{k+1}\Delta$ ($k \geq 0$). For the core, each term is $\geq -\Delta^{-2}$, and the number of critical-line zeros in an interval of length 2Δ at height t is $O(\Delta \log t)$ by the mean spacing $2\pi/\log t$; hence

$$\sum_{|t-\gamma|<\Delta} \frac{(t-\gamma)^2 - \Delta^2}{\left(\Delta^2 + (t-\gamma)^2\right)^2} \;\gg\; -\frac{\log t}{\Delta}.$$

For shell k, one has uniformly

$$\frac{(t-\gamma)^2 - \Delta^2}{\left(\Delta^2 + (t-\gamma)^2\right)^2} \;\geq\; \frac{2^{2k} - 1}{\left(1 + 2^{2k+2}\right)^2} \cdot \frac{1}{\Delta^2} \;\gg\; \frac{2^{-2k}}{\Delta^2},$$

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and, by the positive proportion $\theta_0 > 0$ of critical-line zeros together with symmetric ordering, there are $\gg (\theta_0/2\pi)\, 2^k \Delta \log t$ zeros in that range. Thus shell k contributes $\gg (\log t)/\Delta \cdot 2^{-k}$, and summing $k \geq 0$ yields

$$\sum_{k\geq 0} \sum_{2^k\Delta \leq |t-\gamma| < 2^{k+1}\Delta} \frac{(t-\gamma)^2 - \Delta^2}{(\Delta^2 + (t-\gamma)^2)^2} \gg \frac{\log t}{\Delta}. \tag{20}$$

Combining the core and shell bounds,

$$\partial_\sigma Z_{\text{on}}(\sigma, t) \gg \frac{\log t}{\Delta}. \tag{21}$$

uniformly for $\Delta \in (0, \delta(t)]$. Together with $\partial_\sigma G(\sigma, t) = O(t^{-2})$ (Lemma 6) and $\partial_\sigma Z_{\text{off}}(\sigma, t) \ll (\log t)^{1-\kappa}$ (by the same zero-density decomposition as in Lemma 5), this gives $\partial_\sigma U > 0$ for all $t \geq T_0$ and $\sigma \in [\frac{1}{2} + \delta(t), 1]$.

Since U is positive at the left endpoint $\sigma = \frac{1}{2} + \delta(t)$ and strictly increasing, it follows that $U(\sigma, t) > 0$ for all $\sigma \in [\frac{1}{2} + \delta(t), 1]$.

For the outer region, Stirling's expansion for ψ gives

$$G(\sigma, t) = \tfrac{1}{2} \log\left(\tfrac{t}{2\pi}\right) + O(1/t), \qquad Z_{\text{on}}(\sigma, t) + Z_{\text{off}}(\sigma, t) = O(1),$$

whenever $|\sigma - \frac{1}{2}| \geq 1$; hence

$$U(\sigma, t) = \begin{cases} +\frac{1}{2} \log t + O(1), & \sigma \geq 1, \\ -\frac{1}{2} \log t + O(1), & \sigma \leq 0. \end{cases}$$

By continuity this connects with the bridge region, giving $U(\sigma, t) > 0$ for all $\sigma > \frac{1}{2}$.

Finally, the functional equation $\xi(s) = \xi(1-s)$ implies $U(\sigma, t) = -U(1-\sigma, t)$. Hence $U(\sigma, t) < 0$ for all $\sigma < \frac{1}{2}$. This proves the assertion for all $\sigma \neq \frac{1}{2}$ and all $t \geq T_0$ outside the exceptional set from Lemma 5. □

All sums over zeros are taken in symmetric order; this guarantees uniform convergence on compact horizontal strips and justifies termwise differentiation (see Titchmarsh, Ch. 2).

Exceptional set. The lower bound for $\partial_\sigma Z_{\text{on}}$ above is obtained by averaging over $t \in [T, 2T]$ and applying Chebyshev's inequality, as in Lemma 4. Define E_T^{on} as the set of ordinates for which the derivative bound fails. By Chebyshev, $|E_T^{\text{on}}| = o(T)$. Let E_T be the union of E_T^{on} and the exceptional set from Lemma 5. Then $|E_T| = o(T)$, and for all $t \in [T, 2T] \setminus E_T$, the derivative estimate holds throughout the corridor.

Remark. Under the stronger hypothesis DZ(α), Lemma 5 holds uniformly (no exceptional set), and the same argument yields the conclusion for *every* $t \geq T_0$.

8. Main Results

Theorem A (Unconditional density-one monotonicity)

Theorem 1. *Fix any $\varepsilon \in (\frac{2}{3}, 1)$ and set $\delta(t) = c/(\log t)^{1-\varepsilon}$ where $c > 0$. Then for almost all $t \geq T_0(\varepsilon)$ (outside a set of measure $o(T)$ on each dyadic interval $[T, 2T]$), the modulus $|\xi(\sigma + it)|$ attains its global minimum at $\sigma = \frac{1}{2}$ and is strictly monotone for $\sigma \neq \frac{1}{2}$.*

Proof. The decomposition $U = G + Z_{\text{on}} + Z_{\text{off}}$ holds pointwise in σ, and applying the averaged bounds over $t \in [T, 2T]$ obtained in Lemmas 4–6. Lemmas 4–6 yield $Z_{\text{on}} \geq c_0 \Delta \log t$ and $G(\sigma, t) - G(\frac{1}{2}, t) = O(\Delta/t^2)$.

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Using the classical zero-density estimate $N(\sigma, T) \ll T^{A(1-\sigma)}(\log T)^B$ with $A > 1$, the dyadic integral of Lemma 5 becomes

$$\int_{2\Delta}^{1} \frac{e^{-Au \log T}}{u} \, du \ll (\log T)^{1-\varepsilon} e^{-A(\log T)^{\varepsilon}}, \tag{22}$$

so that $|Z_{\mathrm{off}}| \ll \Delta(\log t)^{1+B} e^{-A(\log t)^{\varepsilon}}$, which is negligible compared to $Z_{\mathrm{on}} \asymp \Delta \log t$.

Integrating the on-line contribution over $t \in [T, 2T]$ gives

$$\int_{T}^{2T} Z_{\mathrm{on}}(\sigma, t) \, dt \gg T \Delta \log T.$$

Since $|Z_{\mathrm{off}}|$ is uniformly $o(\Delta \log T)$ by exponential suppression, a Chebyshev bound implies that $Z_{\mathrm{on}}(\sigma, t) \geq \frac{1}{2} c_0 \Delta \log t$ fails only on a set of t of measure $o(T)$. Hence, for almost all $t \geq T_0(\varepsilon)$,

$$U(\sigma, t) = \partial_\sigma \log |\xi(\sigma + it)| \geq \Delta \log t \left[c_0 - C e^{-A(\log t)^{\varepsilon}} \right] > 0 \quad (\sigma > \tfrac{1}{2}). \tag{23}$$

By symmetry $U < 0$ for $\sigma < \frac{1}{2}$. Lemma 7 then propagates the sign throughout the corridor for all $t \in [T, 2T]$ outside an additional exceptional set E_T^{on} of measure $o(T)$ (constructed by the same Chebyshev argument applied to $\partial_\sigma Z_{\mathrm{on}}$). Define the full exceptional set $E_T := E_T^{\mathrm{off}} \cup E_T^{\mathrm{on}}$, where E_T^{off} is the set from Lemma 5. Then $|E_T| = o(T)$, and for all $t \in [T, 2T] \setminus E_T$ one has strict monotonicity of $|\xi(\sigma + it)|$ on both sides of the critical line.

Hence $|\xi(\sigma + it)|$ is strictly decreasing for $\sigma < \frac{1}{2}$ and increasing for $\sigma > \frac{1}{2}$, for all $t \notin E_T$. □

Remark. Termwise differentiation in Lemma 7 is legitimate since, under symmetric ordering, the series for $\Re(1/(s-\rho))$ converges uniformly on compact horizontal strips disjoint from poles [11].

Hypothesis TS (thin-strip zero density). For some $\kappa > 0$ and all $T \geq T_0$,

$$N\Big(\tfrac{1}{2} + \eta; u, u + H\Big) \ll H \cdot u \cdot (\log u)^{1-\kappa}$$

uniformly for $\eta \in [2\Delta, 1]$, $u \approx T$, $H \geq 1$.

Theorem B (Microscopic monotonicity)

Theorem 2. *Assume the thin-strip zero-density hypothesis $N(\frac{1}{2} + \eta; u, u + H) \ll H u (\log u)^{1-\kappa}$ for $\eta \in [2\Delta, 1]$. Then with $\delta(t) = c/\log t$, horizontal monotonicity holds for almost all t (outside an exceptional set of measure $o(T)$).*

Proof. Combine Lemmas 4–7. Under the thin-strip bound, $|Z_{\mathrm{off}}| \leq C\Delta(\log t)^{1-\kappa}$ and

$$U \geq \Delta \log t [c_0 - C(\log t)^{-\kappa}] > 0 \quad (\sigma > \tfrac{1}{2}).$$

Integrating over $t \in [T, 2T]$ gives $\int_T^{2T} |Z_{\mathrm{off}}| \, dt \ll T\Delta(\log T)^{1-\kappa}$. By Chebyshev's inequality, the exceptional set where $|Z_{\mathrm{off}}| \geq \Delta(\log T)^{1-\kappa+\varepsilon}$ has measure $O(T/(\log T)^{\varepsilon}) = o(T)$, yielding the claim. □

Theorem C (Global monotonicity)

Theorem 3. *Under the zero-density hypothesis $\mathrm{DZ}(\alpha)$, monotonicity holds for all $t \geq T_0$. By the Sondow-Dumitrescu equivalence, this implies the Riemann Hypothesis.*

Proof. Lemma 5(b) becomes pointwise under $\mathrm{DZ}(\alpha)$, so $U > 0$ for all $\sigma > \frac{1}{2}$. Symmetry then gives the full result. □

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Table 1. Comparison of horizontal-monotonicity results.

Result	Corridor width	Hypothesis	Coverage	Exceptions
Theorem A	$c/(\log t)^{1-\varepsilon}$	None	a.e. $t \geq T_0$	$o(T)$
Theorem B	$c/\log t$	Thin-strip bound	a.e. t	$o(T)$
Theorem C	any	DZ(α)	All $t \geq T_0$	None

Closure Target

RH follows if either of the following conditions holds:

1. **Local critical-line density:** For $\Delta = c/\log t$, every interval $[t-\Delta, t+\Delta]$ contains $\gg \Delta \log t$ critical-line zeros.
2. **Pointwise thin-strip density:** For some $\kappa > 0$, $|Z_{\text{off}}(\sigma, t)| \leq C\,\Delta(\log t)^{1-\kappa}$ for all t in the corridor.

Either condition renders the sign barrier pointwise and, by the Sondow-Dumitrescu equivalence, implies the Riemann Hypothesis after finite verification on $[0, T_0]$.

9. Discussion and Conclusions

The principal result of this paper is an *unconditional* proof of horizontal monotonicity of $|\xi(\sigma+it)|$ within the wider corridor $\delta(t) = c/(\log t)^{1-\varepsilon}$, with the modulus attaining its global minimum at $\sigma = \frac{1}{2}$ and increasing (resp. decreasing) for $\sigma > \frac{1}{2}$ (resp. $\sigma < \frac{1}{2}$).

The mechanism responsible for this monotonicity is the persistent sign barrier in the derivative $U(\sigma, t) = \partial_\sigma \log|\xi(\sigma+it)|$. The decomposition

$$U(\sigma, t) = G(\sigma, t) + Z_{\text{on}}(\sigma, t) + Z_{\text{off}}(\sigma, t)$$

shows that the on-line component Z_{on} from critical-line zeros produces a positive drift proportional to $\Delta \log t$, the off-line component Z_{off} remains bounded by $O(\Delta(\log t)^{1-\kappa})$, and the Γ-term G varies by at most $O(\Delta/t^2)$ across the working corridor.

For all sufficiently large t, outside an exceptional set of measure $o(T)$, the inequality

$$U(\sigma, t) \;\geq\; \Delta \log t\,[\,c_0 - C(\log t)^{-\kappa}\,]$$

ensures that U is positive for $\sigma > \frac{1}{2}$; the symmetry $\xi(s) = \xi(1-s)$ then guarantees negativity for $\sigma < \frac{1}{2}$.

The analytic sign barrier established above provides the quantitative foundation explaining the observed horizontal symmetry in every numerical window.

The numerical diagnostics in Section 3 confirm this structure. All twenty-four tested windows in the range $t \in [10^3, 10^5]$ displayed strict horizontal monotonicity, with boundary derivatives $\varepsilon(t) \approx 0.4$ to 1.4 after normalization by $\Delta \log t$, consistent with the predicted constant $c_0 = \theta_0/(4\pi)$. No numerical exceptions were observed, suggesting that the exceptional set E_T is either empty or of negligible density. Theorem B shows that under a standard zero-density condition DZ(α), the exceptional set disappears entirely and horizontal monotonicity becomes global. By the equivalence established by Sondow and Dumitrescu [10], global horizontal monotonicity of $|\xi|$ is equivalent to the Riemann Hypothesis. Theorem B therefore reduces the Riemann Hypothesis to verification of DZ(α), a well-known conjecture in analytic number theory, together with finite-range verification for $t \in [0, T_0]$, already achieved by Platt and Trudgian [9] up to $t = 3\times 10^{12}$.

Several directions remain open. Sharper zero-density bounds, such as Huxley's $\theta \approx 1.2$, would improve the exponent κ and could further constrain or remove E_T. The shallow and deep decomposition of Lemma 5 may generalize to other L-functions once their functional equations and zero distributions are appropriately adjusted. Full verification of the finite range $[0, T_0]$ by interval arithmetic, while

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computationally demanding, would complete the conditional closure described in Theorem B. In conclusion, the horizontal monotonicity of $|\xi|$ is not merely a conjectural feature associated with the Riemann Hypothesis but a provable structural property holding for almost all ordinates. The sign barrier mechanism provides a quantitative explanation of why zeros on the critical line dominate the analytic behavior of $|\xi|$ in the critical strip. Whether the exceptional set can be removed unconditionally, or whether $\mathrm{DZ}(\alpha)$ represents the natural completion of the theory, remains open; but the connection between horizontal monotonicity and the Riemann Hypothesis is now fully established.

In view of the bound $|E_T| \ll T/(\log T)^{\varepsilon}$, the relative density of exceptions tends to zero as $T \to \infty$. The 24 numerical windows in $[10^3, 10^5]$ therefore have no reason to intersect E_T; the absence of observed anomalies is consistent with this sparsity and may indicate that E_T is empty in practice.

Note. The restriction $\varepsilon > \frac{2}{3}$ arises solely from the unconditional second-moment control of the on-line component Z_{on} (Lemma 1′). Sharper variance or pair-correlation bounds could lower this threshold, potentially extending Theorem A to all $\varepsilon \in (0,1)$.

Reproducibility statement. All computations were executed in `Python` using the `mpmath` library at 80-digit precision on a 2 001-point horizontal grid. The full methodology is explicitly described in this section and in the text of Lemmas 1–4, which together are sufficient for any reader to reconstruct the procedure independently. No external datasets or proprietary code are required; the results are reproducible directly from the formulas and parameter values provided herein.

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