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Article

# Quantum Wave Probability Derive Thermodynamic Distribution

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## Abstract

Using the concept of quantum wave probability, combined with the identity principle, we can derive the Boltzmann distribution, Fermi distribution, and Bose distribution. Different distributions correspond to different conditions. The Boltzmann condition corresponds to the Boltzmann distribution. The Fermi condition corresponds to the Fermi distribution. The Bose condition corresponds to the Bose distribution. This demonstrates that the foundation of these three statistical distributions is quantum wave probability, all originating from quantum mechanics. The Boltzmann distribution is also an independent quantum distribution and is not simply a sparse limit of the Fermi or Bose distributions. The essence of the Boltzmann distribution is a uniform distribution. The Fermi and Bose distributions are deviations from the uniform distribution. Particles that follow the Boltzmann distribution can be called Boltzmannons. Boltzmann entropy based on the quantum wave probability can resolve the Gibbs paradox. We need to rethink the fundamentals of statistical mechanics.

**Keywords:** quantum wave probability; identical particle; Boltzmann distribution; Fermi distribution; Bose distribution; uniform distribution; Boltzmann entropy; Gibbs paradox; Boltzmannon

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## Introduction

In current quantum mechanics, there exist two statistical methods: Bose-Einstein statistics (referred to as Bose statistics) and Fermi-Dirac statistics (referred to as Fermi statistics). These two statistical methods also correspond to two particle distribution models, the Bose distribution and the Fermi distribution. The Bose distribution corresponds to bosons, while the Fermi distribution corresponds to fermions. Under the sparse limit conditions, both distributions can be derived into the Boltzmann distribution [1–3].

In current quantum mechanics, Bose statistics and Fermi statistics are introduced as fundamental principles in the theory and cannot explain the physical origin of these two statistical methods. Is there a more fundamental concept that can derive both the Bose and Fermi distributions? Is the Boltzmann distribution really just the sparse limit result of quantum distributions? Without using the sparse limit condition, is it possible to derive the Boltzmann distribution directly from quantum mechanics? How should the Boltzmann distribution be more comprehensively understood within quantum mechanics? In current quantum mechanics theory, the understanding of these questions still seems rather superficial, with logical inconsistencies and lack of clarity. We need new thinking and new insights into quantum statistical mechanics.

In previous papers, the author proposed a new concept of quantum wave probability [4]. By introducing this new concept, the author derived a new concept of quantum wave entropy. The author found that, based on the concept of quantum wave probabilities, we can derive the Boltzmann distribution more simply and consistently, thus resolving many questions in Boltzmann entropy.

This paper provides a completely new perspective, new thinking, and new answers, inspiring people to rethink the fundamentals of statistical mechanics.

## 1. Quantum Wave Probability Derive Boltzmann Distribution

In the previous paper, the author proposed an entirely new concept of quantum wave probability [4]. In quantum mechanics, the wavelength of a monochromatic particle wave represents a new distribution probability. Within a fixed space range, the smaller the wavelength, the higher the probability that the particle will be excited within that range; conversely, the larger the wavelength, the lower the probability of particle excitation within that range. This new distribution property of particle probability has a probability density that is inversely proportional to the wavelength. The density of quantum wave probability is expressed by the following formula (1.1).

$$dp = \varrho \frac{dr}{\lambda} \quad (1.1)$$

Among them,  $\varrho$  is a probability proportionality constant,  $\lambda$  is the wavelength of the particle wave, and  $dp$  represents the probability density over a length  $dr$ . The numerical value of the proportionality constant  $\varrho$  is  $\pi$ . In order to distinguish it from the commonly used symbols in thermodynamics, the probability proportionality constant in previous papers is denoted by the symbol  $\varrho$ . Please take care of this distinction.

For example, within a range of length  $L$ , the total probability of particle excitation is the integral of the probability density.

$$p = \int_0^L dp = \int_0^L \varrho \frac{dr}{\lambda} = \varrho \frac{L}{\lambda} \quad (1.2)$$

In fact, here the particle wavelength  $\lambda$  can be regarded as a unit size. One particle wavelength is equivalent to one unit length. A unit can actually be equivalently viewed as a phase space unit. This corresponds to the phase space cell in classical statistical mechanics. The particle wavelength is the unit length of a one-dimensional phase space cell.

Similar to the phase space cells in classical statistical mechanics, in three-dimensional space, the particle wavelengths along different dimensions are independent of each other, so the probability densities of particles in different dimensions are also independent. The probability density within a three-dimensional space unit is given by formula (1.3).

$$dp = \varrho \frac{dr_1}{\lambda_1} \varrho \frac{dr_2}{\lambda_2} \varrho \frac{dr_3}{\lambda_3} = \varrho^3 \frac{dV}{\lambda_1 \lambda_2 \lambda_3} \quad (1.3)$$

We simplify the model by assuming that the particle has the same wavelength in the three dimensions, which means the momentum components in the three dimensions are the same, leading to formula (1.4).

$$dp = \varrho^3 \frac{dV}{\lambda^3} \quad (1.4)$$

Formula (1.4) is completely analogous to the phase space in classical statistical mechanics. For a 3-dimensional space region of volume  $V$ , the total probability of a particle excited with wavelength  $\lambda$  is given by formula (1.5).

$$p = \int_0^V dp = \int_0^V \varrho^3 \frac{dV}{\lambda^3} = \varrho^3 \frac{V}{\lambda^3} \quad (1.5)$$

Discussions in three-dimensional space are very complicated and difficult to understand. We simplify the problem by first discussing probability in one-dimensional space. The total excitation probability of a particle in a one-dimensional space of length  $L$  is given by formula (1.2). In formula (1.2), the particle's wavelength  $\lambda$  serves as a natural space unit boundary. Each distance of length  $\lambda$  corresponds to one space cell. The total number of cells within the length  $L$  is given by formula (1.6), denoted by  $m$  as the total number of cells. For a free particle in a monochromatic wave, the total excitation probability within the length  $L$  is given by formula (1.6). This result implies an assumption that only one particle can be excited in one cell, and that the excitation probability of a particle is the same in each cell.

$$m = \varrho \frac{L}{\lambda} \quad (1.6)$$

Now suppose there are  $n$  free particles of monochromatic waves with the same wavelength  $\lambda$ . Each particle can be excited in each cell, meaning each particle has  $m$  excitation probabilities. Therefore, the total excitation probability of these  $n$  particles within the length  $L$  is given by formula (1.7).

$$\Omega = \underbrace{m \times m \times m \times \dots \times m}_{n \text{ times}} = m^n \quad (1.7)$$

In quantum mechanics, particles are identical, so these  $n$  particles are identical particles. These  $n$  particles have the same wavelength  $\lambda$ , which actually means they have the same energy and belong to the same energy level.

We assume a condition where each cell is allowed to excite only one particle, so there exist probabilities of duplication in formula (1.7). The probability that more than one particle is excited in one cell belongs to the duplication probability. The total number of excited particles is  $n$ , so the total number of probabilities where more than one particle is excited in one cell is  $n!$ . These duplicate probabilities need to be removed from formula (1.7).

For example, suppose there are 10 cells, and at a certain moment, 4 particles are randomly excited, with each cell able to excite only one particle. The situation where more than one particle is excited in a single cell is an impossible probability. If we calculate the total probability according to formula (1.7), the probability that more than one particle is excited in a single cell is the duplicate probability. If 4 particles are excited in a single cell, the number of duplicates is 4, so the total probability must be divided by 4. If 3 particles are excited in a single cell, the number of duplicates is 3, so the total probability must be divided by 3. If 2 particles are excited in a single cell, the number of duplicates is 2, so the total probability must be divided by 2. Therefore, all the probabilities that need to be removed is  $4! = 4 \times 3 \times 2$ .

Therefore, the actual total excitation probability is given by formula (1.8).

$$\Omega = \frac{m^n}{n!} \quad (1.8)$$

Formula (1.8) is exactly the same as the expression for the number of microstates at a given energy level in the Boltzmann distribution of classical statistical mechanics [1–3]. The excitation probability of a particle in  $m$  cells, and the possible number of microstates of a particle in  $m$  phase cells, these are just two different conceptual representations. But the two representations are mathematically completely equivalent.

However, in the Boltzmann distribution of classical statistical mechanics, the phase space cell is introduced as a hypothetical condition and cannot explain the physical origin of this phase space cell. Moreover, in the Boltzmann distribution, it is assumed that the phase space cell sizes of particles at all energy levels are the same. From the derivation process above, we can see that the physical origin of the phase space cell actually arises from the wavelength of particles' waves. Since the excitation probability of particles is expressed by formula (1.1), there is a relationship between the excitation probability of particles and their wavelength. It is precisely formula (1.1) that gives rise to the existence of this phase space cell. Furthermore, from the derivation above, we also find another result: particle at different energy levels have different momentum and different wavelength, so particle at different energy levels have different phase space cell sizes. Therefore, we find that starting from the quantum wave probability, we can use a simpler, clearer, and more self-consistent method to derive the Boltzmann distribution for identical particles. Thus, the foundation for the validity of the Boltzmann distribution in classical statistical mechanics originates from quantum mechanics. It is precisely because particles in quantum mechanics have wave properties and the attribute of quantum wave probability that the Boltzmann distribution and Boltzmann statistics exist.

In formula (1.8), because  $n!$  is omitted, this means that the number of particles  $n$  must be less than the number of cells  $m$ , so  $n < m$ .

So we find that starting from the quantum wave probability, we can derive the Boltzmann distribution. At the same time, we can also understand the physical essence of the Boltzmann distribution: for  $n$  particles, with  $n$  less than the number of cells  $m$ , each cell has a probability of particle excitation, and the number of excited particles in each cell does not exceed one. This can be referred to as the Boltzmann condition. This condition is essentially a uniform condition. If a cell allows the excitation of more than one particle, or if a cell allows the probability of exciting zero particles, then this situation is considered non-uniform.

We can also find that the conditions for the validity of the Boltzmann distribution do not necessarily require the total number of particles to be very sparse. As long as the Boltzmann condition is satisfied, the Boltzmann distribution holds. Therefore, the classical limit condition for transitioning from the Bose distribution (or Fermi distribution) to the Boltzmann distribution, also called the particle sparsity condition, is not a necessary condition. However, a sparse particle situation does satisfy the Boltzmann condition. A sparse particle situation is just one of the scenarios under the Boltzmann condition. In actual physical processes, other situations that satisfy the Boltzmann condition cannot be ruled out.

In classical statistical mechanics, the factor of  $1/n!$  in the Boltzmann distribution originates from the fact that exchanging distinguishable particles within the same energy level does not generate new microstates. However, in quantum mechanics, particles are identical, so this factor cannot arise from particle exchange. In quantum mechanics, this factor comes from the restriction imposed by the Boltzmann condition, which allows only one particle to occupy each cell. Therefore, the origin of this factor is completely different in quantum statistics compared to classical statistics. In fact, the origin of this factor is related to the truth of the Boltzmann distribution. We find that this restriction is actually a constraint of a uniform distribution. The essence of the Boltzmann distribution is a form of uniform distribution.

The probability of particle distribution on a single energy level is given by formula (1.8), and the total probability of all particles over multiple energy levels is given by formula (1.9).

$$\Omega = \prod \frac{m_i^{n_i}}{n_i!} \quad (1.9)$$

Formula (1.9) represents the number of microstates of identical particles distributed according to the Boltzmann distribution across all energy levels.

Because particles are identical in quantum mechanics, exchanging particles between different energy levels does not produce new microstates, so there is no  $N!$  term.

Similarly, all particles and all energy levels satisfy the following conditions. At any given moment, the total number of excited particles  $N$  remains unchanged and is a constant. The total energy  $E$  of all excited particles remains unchanged and is a constant.

$$N = \sum n_i \quad (1.10)$$

$$E = \sum n_i E_i \quad (1.11)$$

Using standard methods of statistical mechanics, we define entropy  $S$ , which is the Boltzmann entropy, satisfying the following formula (1.12).

$$S = \kappa_B \ln \Omega = \kappa_B \ln \left( \prod \frac{m_i^{n_i}}{n_i!} \right) \quad (1.12)$$

Similarly, the equilibrium state is the case of extreme entropy, satisfying condition (1.13).

$$\delta S = \kappa_B \delta(\ln \Omega) = 0 \quad (1.13)$$

$$\delta S = \kappa_B \delta[\sum(n_i \ln m_i - \ln n_i!)]$$

$$\delta S = \kappa_B \delta[\sum(n_i \ln m_i - n_i \ln n_i + n_i)]$$

$$\delta S = \kappa_B \delta \sum(n_i \ln m_i) - \kappa_B \delta \sum(n_i \ln n_i - n_i)$$

For a multi-particle system, the total probability  $m_i$  of each energy level is a fixed value, the only variable is the number of particles  $n_i$  excited in each energy level. So we get

$$\delta S = \kappa_B \sum (\delta n_i \ln m_i - \delta n_i \ln n_i) = \kappa_B \sum \ln \frac{m_i}{n_i} \delta n_i$$

Combining (1.10) and (1.11), according to the Lagrange multiplier method, we can obtain

$$\begin{aligned} \kappa_B \delta S - \kappa_B \alpha \sum \delta n_i - \kappa_B \beta \sum E_i \delta n_i &= 0 \\ \kappa_B \sum \ln \frac{m_i}{n_i} \delta n_i - \kappa_B \alpha \sum \delta n_i - \kappa_B \beta \sum E_i \delta n_i &= 0 \\ \sum (\ln \frac{m_i}{n_i} - \alpha - \beta E_i) \delta n_i &= 0 \end{aligned}$$

So get (1.14).

$$\ln \frac{m_i}{n_i} - \alpha - \beta E_i = 0 \quad (1.14)$$

We then derive the Boltzmann distribution (1.15).

$$n_i = m_i e^{-\alpha - \beta E_i} \quad (1.15)$$

The  $\alpha$  in formula (1.15) is a constant. The constant  $\beta$  is related to temperature and satisfies formula (1.16).

$$\beta = \frac{1}{\kappa_B T} \quad (1.16)$$

Define the partition function  $Z$ , satisfying formula (1.17).

$$Z = \sum m_i e^{-\beta E_i} \quad (1.17)$$

Formula (1.10) is transformed into formula (1.18).

$$N = \sum n_i = \sum m_i e^{-\alpha - \beta E_i} = e^{-\alpha} \sum m_i e^{-\beta E_i} \quad (1.18)$$

So get (1.19).

$$e^{-\alpha} = \frac{N}{Z} \quad (1.19)$$

$$\alpha = \ln Z - \ln N \quad (1.20)$$

By combining formulas (1.15) and (1.17), formula (1.21) can be obtained.

$$\begin{aligned} E &= \sum n_i E_i = e^{-\alpha} \sum E_i m_i e^{-\beta E_i} = e^{-\alpha} \sum m_i \left( -\frac{\partial e^{-\beta E_i}}{\partial \beta} \right) = e^{-\alpha} \left( -\frac{\partial \sum m_i e^{-\beta E_i}}{\partial \beta} \right) \\ E &= -\frac{N}{Z} \frac{\partial Z}{\partial \beta} = -N \frac{\partial \ln Z}{\partial \beta} \end{aligned} \quad (1.21)$$

Formula (1.12) is converted into the following formula.

$$S = \kappa_B \ln \left( \prod \frac{m_i^{n_i}}{n_i!} \right) = \kappa_B \sum \ln \left( \frac{m_i^{n_i}}{n_i!} \right) = \kappa_B \sum (n_i \ln m_i - \ln n_i!) \quad (1.22)$$

It can be obtained from equation (1.15).

$$\begin{aligned} m_i &= n_i e^{\alpha + \beta E_i} \\ \ln m_i &= \ln(n_i e^{\alpha + \beta E_i}) = \alpha + \beta E_i + \ln n_i \\ \ln n_i! &= n_i \ln n_i - n_i \end{aligned}$$

Take into the above formula (1.22), and can get.

$$\begin{aligned} S &= \kappa_B \sum [n_i(\alpha + \beta E_i + \ln n_i) - (n_i \ln n_i - n_i)] = \kappa_B \beta \sum n_i E_i + \kappa_B \alpha \sum n_i + \kappa_B \sum n_i \\ S &= \kappa_B \beta E + \kappa_B \alpha N + \kappa_B N = \kappa_B \beta E + \kappa_B N(\ln Z - \ln N) + \kappa_B N \end{aligned} \quad (1.23)$$

$$S = \kappa_B N \left( \ln Z - \beta \frac{\partial \ln Z}{\partial \beta} \right) - \kappa_B (N \ln N - N) \quad (1.24)$$

$$S = \kappa_B N \left( \ln Z - \beta \frac{\partial \ln Z}{\partial \beta} \right) - \kappa_B \ln N! \quad (1.25)$$

Formula (1.25) is the Boltzmann entropy. Compared with the Boltzmann entropy in classical statistical mechanics [1-3], formula (1.25) has an extra term of  $-\ln N!$ . This additional term arises from the identity particles in quantum mechanics. By deriving it from the quantum wave probability, we can logically obtain the  $-\ln N!$  term without assuming its existence. Therefore, by taking quantum wave probability as the foundation, the entire theory becomes simpler and more logically consistent.

The above derivation is in the case of a simple one-dimensional space. For an ideal gas in three-dimensional space, the probability density within a three-dimensional space cell is given by formula (1.3).

In quantum mechanics, the particle wavelength satisfies the following formula (1.26).

$$\lambda = \frac{h}{p} \quad (1.26)$$

$p$  is the momentum of the particle. So (1.3) changes to (1.27).

$$dp = \varrho^3 \frac{dV P_1 P_2 P_3}{h^3} \quad (1.27)$$

For the case of the Boltzmann distribution, the particle distribution satisfies formula (1.15). Convert it into differential form.

$$dn_i = dm_i e^{-\alpha - \beta E_i} = \varrho^3 e^{-\alpha - \beta E} \frac{dV dP_1 dP_2 dP_3}{h^3} \quad (1.28)$$

The partition function formula (1.17) is then transformed into the integral formula (1.29).

$$Z = \int \int \int \int \varrho^3 e^{-\beta E} \frac{dV dP_1 dP_2 dP_3}{h^3} \quad (1.29)$$

For an ideal gas in three-dimensional space, the energy  $E$  of each particle satisfies formula (1.30).

$$E = \frac{p^2}{2m} = \frac{P_1^2 + P_2^2 + P_3^2}{2m} \quad (1.30)$$

$$Z = \frac{\varrho^3}{h^3} \int \int \int \int e^{-\frac{\beta P_1^2}{2m} - \frac{\beta P_2^2}{2m} - \frac{\beta P_3^2}{2m}} dV dP_1 dP_2 dP_3 \quad (1.31)$$

$$Z = \frac{\varrho^3 V}{h^3} \int e^{-\frac{\beta P_1^2}{2m}} dP_1 \int e^{-\frac{\beta P_2^2}{2m}} dP_2 \int e^{-\frac{\beta P_3^2}{2m}} dP_3$$

$$Z = \frac{\varrho^3 V}{h^3} \left( \frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \quad (1.32)$$

We thus derive the partition function of an ideal gas. The result differs from that in classical statistical mechanics only by a constant  $\varrho^3$ . The constant  $\varrho^3$  is a probability ratio constant. However, in this derivation, there is no need to assume the existence of phase space or phase cells. Formulas (1.27) and (1.29) are entirely results obtained from taking differentials in 3-dimensional space and 3-dimensional momentum, independent of phase space. Yet, the differential form brings effects equivalent to phase space cells. Nevertheless, we do not need to assume the existence of phase space, which reduces theoretical assumptions and makes the derivation simpler and more straightforward.

So we derive the result of the Boltzmann distribution from the quantum wave probability. This proves that the Boltzmann distribution actually originates from quantum mechanics. The foundation of Boltzmann statistics is actually quantum mechanics. Boltzmann statistics is also essentially a form of quantum statistics.

Combining formulas (1.23) and (1.28), a result can be calculated and proven. When two ideal gases of the same type, at the same temperature, with the same number of particles, and the same volume are mixed together, the number of particles doubles, the volume doubles, the energy doubles, and the Boltzmann entropy also doubles, meaning there is no Gibbs paradox [5,6].

According to formula (1.32), we obtain.

$$\ln Z = \ln \left( \frac{\varrho^3 V}{h^3} \left( \frac{2\pi m}{\beta} \right)^{\frac{3}{2}} \right) = A + \ln V$$

$$A = \ln \frac{\varrho^3}{h^3} \left( \frac{2\pi m}{\beta} \right)^{\frac{3}{2}}$$

$$\frac{\partial \ln Z}{\partial \beta} = \frac{\partial A}{\partial \beta}$$

According to formula (1.24), before mixing, the Boltzmann entropy of the ideal gas is.

$$S_1 = \kappa_B N A + \kappa_B N \ln V - \kappa_B N \beta \frac{\partial A}{\partial \beta} - \kappa_B N \ln N + \kappa_B N$$

After mixing, the number of particles  $N \rightarrow 2N$ , and  $V \rightarrow 2V$ . The temperature remains unchanged, so  $A$  remains unchanged, and  $\frac{\partial A}{\partial \beta}$  remains unchanged. Therefore, we obtain the entropy of the ideal gas after mixing.

$$S = \kappa_B 2N \left( A + \ln 2V - \beta \frac{\partial A}{\partial \beta} \right) - \kappa_B (2N \ln(2N) - 2N)$$

$$S = \kappa_B 2N A + \kappa_B 2N \ln 2 + \kappa_B 2N \ln V - \kappa_B 2N \beta \frac{\partial A}{\partial \beta} - \kappa_B 2N \ln N - \kappa_B 2N \ln 2 + \kappa_B 2N$$

$$S = \kappa_B 2N A + \kappa_B 2N \ln V - \kappa_B 2N \beta \frac{\partial A}{\partial \beta} - \kappa_B 2N \ln N + \kappa_B 2N$$

$$S = 2 \left( \kappa_B N A + \kappa_B N \ln V - \kappa_B N \beta \frac{\partial A}{\partial \beta} - \kappa_B N \ln N + \kappa_B N \right) = 2S_1$$

Therefore, Gibbs paradox does not exist.

To summarize the derivation process above, what are the differences compared with the derivation in classical statistical mechanics? First, particles are completely identical. Second, there is no need to assume phase space cells. Because the quantum fluctuation probabilities satisfy formula

(1.2), the particle wavelength introduces a space unit equivalent to a phase cell. Not all particles have the same size unit; particles at different energy levels have different wavelengths and thus have different unit sizes. Third, the probability that each unit excites a particle satisfies the Boltzmann condition. Fourth, there are no fixed, permanent particles. Particles are randomly excited in the unit, each unit has an excitation probability, and then these particles quickly annihilate. The excitation and annihilation continue repeatedly. However, at any given moment, the total number of excited particles satisfies the constraints of formulas (1.10) and (1.11). Deriving the Boltzmann distribution and Boltzmann entropy based on quantum wave probability starts entirely from the fundamental probabilities of quantum mechanics, and the entire derivation naturally conforms to the concepts and methods of quantum mechanics. This fully integrates the Boltzmann distribution and Boltzmann statistics into the theoretical framework of quantum mechanics.

## 2. Quantum Wave Probability Derive Fermi Distribution and Bose Distribution

In the derivation of the Boltzmann distribution above, a key assumption was made. Each particle has the same excitation probability at each cell. There are total of  $m$  cells, and each particle has  $m$  possible excitation probabilities. Based on this condition, we can obtain the total excitation probabilities given in formulas (1.7) and (1.8).

Now let's assume another condition. The total number of cells is  $m$ , which is still given by formula (1.6). We assume that at a certain moment, among these  $m$  cells, some cells may excite one particle, while some cells may excite zero particles. The total number of excited particles at a certain moment in these  $m$  cells is  $n$ . These  $n$  particles are identical particles. The number of cells  $m$  is greater than the number of excited particles  $n$ ,  $n < m$ . Under such conditions, if we regard the  $m$  cells as  $m$  degenerate quantum states, this condition is actually the Pauli exclusion principle: a quantum state can either contain one particle or contain zero particles. Therefore, this assumption is essentially the condition for Fermi distribution. Hence, under this assumption, the total particle excitation probability of  $m$  cells and  $n$  particles is given by formula (2.1), which is essentially the number of microstates for Fermi distribution [1–3]. We can refer to this assumption as the Fermi condition.

$$\Omega = C_m^n = \frac{m!}{n!(m-n)!} \quad (2.1)$$

Formula (2.1) is the excitation probability for a single energy level. The total excitation probability for all particles and all energy levels is given by formula (2.2).

$$\Omega = \prod \frac{m_i!}{n_i!(m_i-n_i)!} \quad (2.2)$$

By using the same derivation method as for the Boltzmann distribution above, the Fermi distribution formula (2.3) can be derived.

$$n_i = m_i \frac{1}{e^{\alpha+\beta E_i} + 1} \quad (2.3)$$

Similarly, the  $\alpha$  in formula (2.3) is a constant. The constant  $\beta$  is related to temperature and satisfies formula (1.16).

In the process of deriving the Boltzmann distribution above, we assumed that at any given moment, a cell can only excite one particle. What would happen if we allow a cell to excite multiple particles, or even none at all? We still consider  $m$  cells as  $m$  degenerate quantum states, which is essentially the case of the Bose distribution. The number of particles that can be excited on one cell is unlimited, with a minimum of zero and a maximum equal to the total number of particles  $n$ . Therefore, under this assumption, the total excitation probability of  $m$  cells and  $n$  particles is given by formula (2.4), which corresponds to the number of microstates in the Bose distribution [1–3]. We can refer to this assumption as the Bose condition.

$$\Omega = C_{m+n-1}^{n-1} = \frac{(m+n-1)!}{n!(m-1)!} \quad (2.4)$$

Formula (2.4) is the excitation probability for a single energy level. The total excitation probability for all particles and all energy levels is given by formula (2.5).

$$\Omega = \prod \frac{(m_i+n_i-1)!}{n_i!(m_i-1)!} \quad (2.5)$$

Using the same derivation method, the Bose distribution (2.6) can be derived.

$$n_i = m_i \frac{1}{e^{\alpha + \beta E_i - 1}} \quad (2.6)$$

Similarly, the  $\alpha$  in formula (2.6) is a constant. The constant  $\beta$  is related to temperature and satisfies formula (1.16).

Through the above process, it is thus proven that based on quantum wave probability, we can derive the Boltzmann distribution, Fermi distribution, and Bose distribution. These three statistical mechanics distribution models can all be derived from quantum wave probability. Their foundation is quantum mechanics. Similar to the Fermi and Bose distributions, the Boltzmann distribution is also essentially a quantum distribution and originates from quantum mechanics. Based on quantum wave probability, different distributions can be derived using different conditions. Using Boltzmann conditions, the Boltzmann distribution is derived. Using Fermi conditions, the Fermi distribution is derived. Using Bose conditions, the Bose distribution is derived.

By comparing formulas (1.9) and (2.1), we can see that in formula (2.1), under the condition where  $n \ll m$ , formula (2.1) can be approximated by formula (1.9), and the Fermi distribution transitions to the Boltzmann distribution. Similarly, under the condition  $n \ll m$ , formula (2.4) can also be approximated by formula (1.9), and the Bose distribution likewise transitions to the Boltzmann distribution. This is the conventional classical limit condition, also known as the sparse condition. Therefore, conventional statistical mechanics considers Boltzmann statistics to be the classical limit of quantum statistics. However, from the derivation above, we can reach a conclusion that this statement is incorrect. The Boltzmann distribution is actually a type of quantum distribution, completely independent of both Fermi and Bose distributions. As long as the Boltzmann condition is satisfied, the Boltzmann distribution still holds even when  $n$  is close to  $m$ . It is not only under the condition  $n \ll m$  that the Boltzmann distribution exists. The correct statement is that under the condition  $n \ll m$ , these three distributions converge to the same distribution.

We observe the Boltzmann condition again. For  $n$  particles, where  $n$  is less than the number of cells  $m$ , each cell has a probability of particle excitation, and the number of excited particles in each cell is one. Each particle has a probability of being excited in each cell. This condition is essentially a uniform condition. Therefore, the Boltzmann distribution is actually a kind of uniform distribution pattern.

In contrast, Bose distribution allows all particles to be excited onto a single cell. Obviously, such a distribution pattern is not uniform. Although Fermi statistics restrict each cell to at most one particle, but it allows some cells to have zero particles, its uniformity is still not as high as that of the Boltzmann distribution. When the number of particles  $n$  is much smaller than the number of cells  $m$ , both Bose and Fermi distributions are actually close to a uniform distribution, and thus can be approximated by the Boltzmann distribution.

In the existing theory of quantum mechanics, both Bose and Fermi distributions are related to the symmetry of the wave function. Particles with symmetric wave functions are bosons and follow the Bose distribution, while particles with antisymmetric wave functions are fermions and follow the Fermi distribution. This can be understood as the symmetry of the wave function causing an effect among particles. This effect disrupts the uniformity of particle distribution, causing the distribution of particles to deviate from uniform distribution, and thus deviate from the Boltzmann distribution. Bose and Fermi distributions are two different types of deviation effects. In quantum mechanics, under certain extreme conditions, there may be other types of deviation effects that could lead to new distribution patterns. The deviation effect from the uniform distribution is a topic worth further study.

In current quantum statistical mechanics, a popular viewpoint exists. For Fermi and Bose distributions, under sparse conditions, the waves of different particles no longer overlap, and they can be approximately considered distinguishable, so under sparse conditions, they can be approximated by the Boltzmann distribution. The biggest doubts and logical inconsistencies in existing statistical mechanics also stem from this viewpoint. Now we can assert that this is incorrect. Particles in the Boltzmann distribution are also identical; even under sparse conditions, they remain identical particles and are still indistinguishable. The actual truth is that under sparse conditions, the

interactions between particles become very small, and the distribution of particles becomes nearly uniform, so under sparse conditions, the Fermi and Bose distributions approximately converge to the Boltzmann distribution. The key to the Boltzmann distribution lies in the absence of interactions between particles. Particles are approximately ideal single particles, with no aggregation or exclusion, and their distribution becomes very uniform. Particles following the Boltzmann distribution are ideal single particles, but they are still identical particles. Ideal single particles are particles that have absolutely no influence on each other. Similar to fermions and bosons, we refer to ideal single particle as Boltzmannon. The Boltzmannons are quantized particles that follows the Boltzmann distribution.

Let's further look at the quantum wave probability. For the quantum wave probability of a single particle, it is expressed by formulas (1.1) and (1.6). From these two formulas, no close relationship with statistical distributions is apparent. However, when many particles combine together, the overall probability undergoes a magical change, and we have derived three statistical distribution patterns. Therefore, in quantum mechanics, because particles are identical, the overall properties of combined particles are not simply equal to the sum of the properties of these particles. Identicalness leads to changes in the overall properties. This is similar to an emergent phenomenon.

## Conclusion

From the derivation process above, we obtain a result. By using the concept of quantum wave probability, combined with the identity principle of particles, we can derive the Boltzmann distribution, Fermi distribution, and Bose distribution. Different distributions correspond to different conditions. Using the Boltzmann condition, we derive the Boltzmann distribution. Using the Fermi condition, we derive the Fermi distribution. Using the Bose condition, we derive the Bose distribution. This proves that the basis of these three statistical distribution patterns is quantum wave probability and all originate from quantum mechanics. The Boltzmann distribution is also an independent quantum distribution and is not merely a sparse limit of the Fermi and Bose distributions. The essence of the Boltzmann distribution is uniform distribution. The Fermi and Bose distributions represent deviations from the uniform distribution. In the sparse limit, these three distributions tend to converge to the same distribution pattern. Based on the Boltzmann distribution, we calculate Boltzmann entropy, compute the entropy of an ideal gas, and resolve the Gibbs paradox. This demonstrates that quantum wave probability has very important physical significance and is well worth further in-depth study.

Through the discussion in this paper, we are inspired to rethink the fundamental of statistical mechanics. The foundations of statistical mechanics are not entirely problem-free. We can still approach the study of its fundamentals from a completely new perspective for in-depth research.

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