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A Localized Axisymmetric Solution of the Hilbert-Einstein Equations for Spacetime Without Mass Sources

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Abstract

Usually, the Hilbert-Einstein equations are considered for systems with mass-energy sources, and if one were to reduce the mass-parameter towards zero, the spacetime curvature would also diminish towards zero. However, in systems without mass-energy sources, the spacetime may locally exist as non-flat. We present a novel result—a localized axisymmetric solution of nonlinear GR equations for such massless systems—and provide step-by-step derivation for the solution.

Keywords: curved space-time; general relativity; metrics; tensor ricci; localized field structures

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1. Introduction

In the general theory of relativity (GR) the basic equations—the so-called Einstein's equations—connect the geometry of the spacetime with the distribution of the matter within it (see, among other superb works, books [1,2]). As a reminder: in his version of the GR, Einstein identified gravity with the metric tensor of a pseudo-Riemann spacetime. However, one of the results of this approach is that the "gravitational field" is not a physical field in the usual sense; thus, the laws of conservation are difficult to fulfill without some creative manipulations. Nevertheless, this theory remains the most popular among the two-three dozens of other theories of gravity (see, for example, Refs. [3–9], and bibliography within). And, apparently, the GR will keep its dominance until some new experimental facts decisively refute its main predictions.

To repeat the basics for no-specialized readers: the main GR equation is the tensor equation [10] which relates the local spacetime curvature (expressed by some combination of the Ricci curvature tensor R_{ik} determined by the metric g_{ik} and the contracted expression of the tensor; and typically placed on the left side of the equation) with the stress momentum-energy tensor T_{ik} of the matter (multiplied by some dimensional parameter; and typically written on the right side of the equation). In the traditional units, each term of the equation has the dimension $[length]^{-2}$, because this is the dimension of the curvature tensor. Thus, this tensor equation may be interpreted as a set of equations dictating how stress-energy-momentum determines the curvature of the spacetime, or as if the folds/singularities in the spacetime/field generate the matter, because the spacetime itself is an "entity" which is not "empty". If the energy-momentum tensor is zero in the region under consideration (outside of the domain where the matter is localized), then these field equations are often referred to as the "vacuum field equations". But we consider the entire spacetime without mass sources. The flat Minkowski spacetime is the simplest example of such massless-system solution; in this paper, we obtain a non-flat, localized, solution for the massless-system.

The most elegant derivation of the Einstein's equations is obtained from the principle of least action (S) for the gravitational field (g), the matter (without the field, m) and their interaction (mg), respectively, producing a variation with respect to δg_{ik}: $\delta(S_g + \lambda S_{gm} + S_m) = 0$. Here, obviously, it is presumed that the variational derivative of the "material part" of the action is equal to zero:

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$\delta S_m / \delta g_{ik} = 0$. The dimensional constant λ is found, in accordance with the correspondence principle, after obtaining the evolution equations and passing to the case of weak fields, when the Newtonian approximation for the gravity is valid. Parameter λ in usual units is proportional to the newtonian gravity constant κ: $\lambda = 8\pi\kappa/c^4$.

The principle of least action for a "gravitational field" was first formulated by D. Hilbert [11] (see [12], p. 354, since in [1] this mentioning is dropped; see also the detailed physical-historical review [13]). In the empty space: $T_{ik} = 0$ and, consequently, $T = 0$. Thus, the main equation is reduced to $R_{ik} = 0$ (which is formula (95.9) in [1]).

Einstein's equations without the term with a cosmological constant were derived almost simultaneously in November 1915 by D. Hilbert (see his November 20, 1915, correspondence with the derivation based on the principle of least action [11]) and by A. Einstein (see his November 25, 1915, derivation based on the principle of general covariance of the gravity equations in combination with local conservation of energy-momentum [10]). Hilbert's work [14] was published by editors later than Einstein's (1916). There are different opinions about the precedence, but according to some researchers, such as [13], Hilbert himself never claimed own precedence and considered the GR as Einstein's creation. (However, after the end of World War I, in 1924, Hilbert republished his famous 1916 paper, where he wrote: *Einstein [...] kehrt schließlich in seinen letzten Publikationen geradewegs zu den Gleichungen meiner Theorie zurück*. In English translation: *"Einstein [...] in his most recent publications, returns directly to the equations of my theory"* [15]. We will not get involved in a discussion of the contentious topic of who said "A" first—everyone sticks to own opinion regardless—instead, we refer the curious to the references [13,15–17].

The most known exact solutions of Einstein's equations are:

The Friedmann cosmological solution (for the Universe as a whole), and the exact gravitational wave solutions.

For spherically symmetric cases: the Schwarzschild solution [18] (for the spacetime surrounding a spherically symmetric, uncharged, and non-rotating massive object), the Kerr solution [19] (for a rotating massive object; see also bibliography in [20–22]), the Reissner–Nordström solution (for a charged, spherically symmetric massive object), the Kerr–Newman solution (for a charged, rotating massive object). Recent publications [23–26,34,35] offer informative discussions.

The static, cylindrically symmetric vacuum solutions (which date back to the early 20th century) are examined in the classic works by Weyl [27] and Levi-Civita [28].

During the 1980s, cylindrical solutions received broad attention when the solutions were studied as representing the spacetime outside a one-dimensional (thread-like, infinitely extended) concentration of either a matter or gauge-field energy; this 1-D structure is the so-called "cosmic string". Cosmic strings were introduced in the theoretical cosmology (see, for example, [29]) with the idea that they could be revealed by means of gravitational lensing. The physical mechanism which could have produced cosmic strings is perhaps related to the phase transition in the early Universe. Some useful information concerning basic bibliography on the subject can be found in relatively recent Refs. [30–33]. In this respect, the book by [36] about exact solutions of Einstein's field equations, may prove to be useful.

In this work, the key goal is to examine a GR model describing a structure which may be interpreted as a stationary non-spherical localized perturbation of spacetime. This model is valuable for analyzing the evolution of spacetime perturbations. (Obviously, at large distances from the perturbation localization domain the spacetime metrics tends towards the Newtonian.)

The remainder of the discussion is structured as follows: In Section 2 we lay out the setup for the task of describing the exact axisymmetric solution of the Hilbert-Einstein equations for the empty spacetime. In Section 3 we describe the solution. In Section 4, we summarize the results and discuss their features. Appendix A lists the calculated Christoffel coefficients for the model.

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2. The Setup for Model Description

We consider a spacetime characterized by the metric tensor g_{ik} with the positive signature $g_{00} > 0$ [1] in contrast to the often-used negative signature $g_{00} < 0$ (see, for example, the panoptic [2]). The question of using one or the other metric signature is only a matter of habit and does not affect the final results. Each world-point is described by the contra-variant components $q^i = (ct, q^\alpha)$, where parameter c is the speed of light, t is the time measured by a remote observer. The spacetime metric is fixed by interval $ds = (g_{ik}dq^i dq^k)^{1/2}$. The Latin indices take the values $i = 0, 1, 2, 3$ and the Greek indices take the values $\alpha = 1, 2, 3$. With these definitions, quantities s and q^i have the length dimension. Next, we assume that in the system under consideration there is some characteristic spatial scale, the meaning of which will be revealed below. We normalize all quantities to this scale; for gravitating (massive, m_g) bodies, such characteristic scale is $r_g = 2\kappa m_g/c^2$ (this quantity has the dimension of length). Here, κ is the newtonian gravity constant, c is, as noted above, the light speed. The characteristic time is then determined by the scale r_g/c. With all this in mind, we then work with dimensionless quantities.

The dimensionless Einstein equations for the gravitational field in the presence of a matter are

$$R_{ik} - \frac{1}{2}g_{ik}R = 3T_{ik}\ , \quad \text{because of} \quad r_g^2 \times \frac{8\pi\kappa}{c^4} \times m_g c^2 \frac{3}{4\pi r_g^3} = 3. \tag{1}$$

Here, T_{ik} is the dimensionless tensor of energy-momentum for the matter, R_{ik} is, as known, the spacetime Ricci curvature tensor:

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}\ . \tag{2}$$

Here, Γ^l_{ik} are the Christoffel symbols. The inverse metric tensor g^{ls} satisfies $g^{is}g_{sk} = \delta^i_k$. The scalar curvature of the spacetime R in Eq. (1) is obtained by contracting the Ricci tensor: $R = g^{ik}R_{ik}$. The comma before the index, as usual, means the partial derivative with respect to the corresponding coordinate. Obviously, for the mixed tensor R^i_k we have

$$R^i_k = g^{is}R_{sk} = g^{is}(\Gamma^l_{sk,l} - \Gamma^l_{sl,k} + \Gamma^l_{sk}\Gamma^m_{lm} - \Gamma^m_{sl}\Gamma^l_{km})\ . \tag{3}$$

We remind that expression Γ^l_{il} has the form $\Gamma^l_{il} = \partial_i(ln\sqrt{-g})$ which is important to keep in mind when calculating the components of the Ricci tensor (which is certainly symmetric). Here, g is the determinant of the matrix composed from the elements of g_{ik}. Contracting Eq. (1) on the indices i and k, with $g^{ki}g_{ik} = \delta^k_k = 4$ and $T = g^{ik}T_{ik}$, we find the dimensionless alternative form of Eq. (1):

$$R^k_i = 3(T^k_i - \frac{1}{2}\delta^k_i T). \tag{4}$$

3. The Solution

To avoid any misunderstanding, we emphasize that below we consider a completely empty spacetime, and not some limited region of it in which there is no matter. In the completely empty space, one has $T^k_i = 0$ everywhere and, consequently, $T = g^{ik}T_{ik} = 0$; and the equations of the "gravitational field" Eq. (4) become equations: $R^k_i = 0$. However, this does not at all mean that the spacetime is flat; for the spacetime to be flat, a stronger condition is needed: all components of the Riemann curvature tensor must be zero.

Let the metric tensor in the expression ds^2 be diagonal. Let there be a distinguished direction due to the shape of the source. We assign coordinate z to this direction and, thus, turn to the cylindrical coordinates. In these coordinates, the square of the interval is:

$$ds^2 = e^a dt^2 - e^b d\rho^2 - e^c \rho^2 d\phi^2 - e^d dz^2\ , \tag{5}$$

where parameters a, b, c, d are functions of now dimensionless coordinates $q^i = (ct, \rho, \phi, z)$. Sometimes this parameterization for the square of an interval is called the "Weyl's ansatz".

In this brief paper, we consider a special case of stationary and axially symmetric solution of equations $R_i^k = 0$. In this case, parameters a, b, c, d are functions only of $\rho \equiv q^1$ and $q^3 \equiv z$ and independent of time q^0. To get the equations of gravitation, we must calculate all components of the mixed tensor $R_i^k = g^{ks} R_{si}$; according to the scheme $g_{ik} \to \Gamma^i_{jk} \to R_{ik} \to R_i^k$ with Eq. (2): These calculations are cumbersome but executable.

The analysis—which we omit here— allows us to establish that a non-trivial solution is also possible in the extremely symmetric case, when $c(\rho, z) = -a(\rho, z)$ and $d(\rho, z) = b(\rho, z) + \nu$, where ν is some constant. Imposing the condition $\nu = 0$ on ν means that the same scaling factor r_g is chosen in the direction along the z-axis as in the direction perpendicular to the z-axis. For $\nu \neq 0$, simple but tedious calculations of components (3) — together with Eq. (5) — lead to the following expressions for the mixed Ricci tensor's components:

$$
\begin{aligned}
R_0^0 &= \frac{e^{-b}}{2\rho}(e^{-\nu}\rho a_{,zz} + a_{,\rho} + \rho a_{,\rho\rho}), \\
R_1^1 &= \frac{e^{-b-\nu}}{2\rho}(-2e^{\nu} a_{,\rho} + e^{\nu}\rho a_{,\rho}^2 + \rho b_{,zz} + e^{\nu}(-b_{,\rho} + \rho b_{,\rho\rho})), \\
R_2^2 = -\frac{e^{-b}}{2\rho}(e^{-\nu}\rho a_{,zz} + a_{,\rho} + \rho a_{,\rho\rho}), &\quad R_3^3 = \frac{e^{-b-\nu}}{2\rho}(\rho a_{,z}^2 + \rho b_{,zz} + e^{\nu}(b_{,\rho} + \rho b_{,\rho\rho})), \\
R_1^3 = \frac{e^{-b}}{2\rho}(-a_{,z} + \rho a_{,z} a_{,\rho} - b_{,z}), &\quad R_3^1 = \frac{e^{-b-\nu}}{2\rho}(-a_{,z} + \rho a_{,z} a_{,\rho} - b_{,z}) \, .
\end{aligned} \tag{6}
$$

The remaining components of the Ricci tensor identically vanish. The symmetry of the Ricci tensor with respect to its indices for $\nu \neq 0$ is ensured by the condition that the expression in the brackets for R_1^3 and R_3^1 is equal to zero; it gives the connection between b and a.

Equations (6) can be integrated analytically in the considered case of an axially symmetric field in vacuum. This solution may also correspond to the case where the field is produced by a localized axially-symmetric mass distribution.

By setting the energy-momentum tensor T_i^k equal to zero, and given that the exponential factor never vanishes, we obtain:

$$
e^{-\nu}\rho a_{zz} + a_\rho + \rho a_{\rho\rho} = 0, \quad b_z = -a_z + \rho a_z a_\rho, \quad b_\rho = -a_\rho + \frac{\rho}{2}(a_\rho^2 - e^{-\nu} a_z^2). \tag{7}
$$

There is no need to write down the forth equation $R_3^3 + R_1^1 = 0$, since it follows from the other three equations (7).

The arbitrariness in the choice of the constants of integration is eliminated by the requirement that, for $(\rho, z) \to \infty$, the spacetime becomes flat, its metric becomes Galilean, that is: $g_{00} \to 1$ and $g_{\alpha\beta} \to -\delta_{\alpha\beta}$. From here, it follows that function $a, b \to 0$ when $(\rho, z) \to \infty$. On the other hand, the amplitude factor in the function a cannot be arbitrary. If we require that all free constants be such that at large distances from the singularity the solution transforms into a solution corresponding to the Newtonian gravitational field, then Eqs. (7) are integrated (see [38], Chapter 10) and yield:

$$
a(\rho, z) = -(\rho^2 + e^{\nu} z^2)^{-1/2}, \quad b(\rho, z) = -\frac{1}{4}\rho^2(\rho^2 + e^{\nu} z^2)^{-2} + (\rho^2 + e^{\nu} z^2)^{-1/2} \neq -a \quad . \tag{8}
$$

Indeed, it is sufficient to substitute the first expression from Eqs, (8) into the first equation Eqs. (8) to verify that this is so. The two equations for $b_{,z}$ and $b_{,\rho}$ are solved jointly after substituting into them the expression for the function a.

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4. Discussion

For an empty spacetime, when the Ricci tensor R_{ik} becomes zero, we analytically found an exact axially-symmetric stationary solution for the components of the metric tensor g_{ik}. In the dimensionless form, the square of interval $ds^2 = g_{ik}dq^i dq^k$ is:

$$ds^2 = exp[-(\rho^2 + e^{\nu}z^2)^{-1/2}]dt^2 - exp[-\frac{1}{4}\rho^2(\rho^2 + e^{\nu}z^2)^{-2} + (\rho^2 + e^{\nu}z^2)^{-1/2}]d\rho^2 - \rho^2 exp[+(\rho^2 + e^{\nu}z^2)^{-1/2}]d\phi^2 - exp[-\frac{1}{4}\rho^2(\rho^2 + e^{\nu}z^2)^{-2} + (\rho^2 + e^{\nu}z^2)^{-1/2}]dz^2 \,. \quad (9)$$

At the infinity, the metrics automatically becomes Galilean. The determinant of the matrix composed of the elements of the metrical tensor g_{ik} is $g = -\rho^2 \exp[2b + \nu]$; it is non-zero in the spacetime, except for $\rho = 0$.

In a characteristic case when $\nu = 0$, i.e., when the characteristic transverse r_g and longitudinal $r_g \exp(-\nu/2)$ spatial scales of the considered anisotropic structure coincide, and consequently $r = \sqrt{\rho^2 + z^2} \equiv R/r_g$, then the zero-th metric tensor component becomes $g_{00} = \exp[a(\rho, z)] \simeq 1 - r^{-1}$ for large $r \gg 1$. So, at large distances, where the field is weak, Newton's law should have its hold. This allows — for a weak field $g_{00} = 1 + 2\Phi/c^2$ (see [1], p. 284) — to equate one side $2\Phi/c^2$ and the other side $-r_g/R = -2\kappa m_g/c^2 R$, and to obtain the expected expression for the Newtonian potential $\Phi = -\kappa m_g/R$. And then, a distant observer, after recording (somehow) the presence of a potential Φ or its spatial derivatives in the domain of observation, would interpret the presence of this potential as some distribution of masses m_g in the vicinity of the center of the coordinates, and not as the existence of a singularity in the spacetime metrics.

For the anisotropic case, $\nu \neq 0$, a deviation from the Newtonian approximation in Eq. (9) may potentially be detected.

Thus, the solution that we presented should be viewed as a fluctuation, a "bulge", a localized field structure, which is a new element in the context of the general problematics.

The above-obtained solution is regular over the entire spacetime, with the only exception of the center of the spatial coordinates. The components of the metric tensor g_{ik} do not turn to zero or infinity anywhere; consequently, the solution lives well without the science fiction of the black holes with their event horizons so wonderfully described in the book [37]. In the metrics Eq. (9), unlike other models, there are no "event horizons".

It is surprising that such a simple, physically acceptable solution, given by Eqs. (9), has not yet been posted in literature.

In this context, we would like to remind once again that the solution obtained is not a solution for the equation $R_{ik} = \kappa T_{ik} \sim m$, but a solution for the equation $R_{ik} = 0$ where the Ricci tensor is zero. That is, there is no mass source, m, and we find a non-zero, solitary, exact solution for this equation: a solitary spacetime structure. Obviously, it would be an egregious mistake to simply take a solution from a problem *with* a mass source and mindlessly, without any analysis, assume that it is the same for the problem *without* a mass source. That's why we conducted such a detailed analysis in the paper. The solution that we provide does not result from the solution for the $m \neq 0$ system by the continuous transition $m \to 0$. For the $m \neq 0$ system, $m \to 0$ simply yields the boring flat spacetime.

There is a fundamentally conceptual (physical) difference between the problem of spacetime deformation with a mass source and the problem of spacetime deformation in an empty World without a mass source. Indeed, it is a well-known fact that even if the Ricci tensor is zero, $R_{ik} = 0$, it doesn't mean that the curvature (Riemann tensor) of spacetime is zero. Deformation of spacetime may still take place. An obvious example is when, for a weakly perturbed metric (i.e. letting $g_{ik} = \gamma_{ik} + h_{ik}$, where γ_{ik} is the Minkowski tensor), equation $R_{ik} = 0$ reduces to the equation $R_{ik} = (1/2)\Box h_{ik} = 0$. Here, $\Box$ is the operator of d'Alembert. That is, the equation for R_{ik} becomes reduced to the usual wave equation: weak perturbations of the spacetime metric propagate as waves at the speed of light. This simple example corresponds to the linear solution for a perturbation of spacetime; in our analysis, we present a non-linear localized solution.

Appendix A. The Christoffel Symbols

Non-zero Christoffel symbols for Eq. (5) with $c = -a$ and $d = b + \nu$, are:

$$\Gamma^0_{10} = \frac{1}{2}\rho^2, \quad \Gamma^0_{30} = \frac{1}{2}a_z, \quad \Gamma^1_{00} = \frac{1}{2}a_\rho \exp(a-b), \quad \Gamma^1_{22} = \frac{1}{2}\rho(-2+\rho a_\rho)\exp(-a-b),$$
$$\Gamma^1_{33} = -\frac{1}{2}b_\rho \exp\nu, \quad \Gamma^2_{21} = -\frac{1}{\rho} - \frac{1}{2}a_\rho, \quad \Gamma^2_{32} = -\frac{1}{2}a_z, \quad \Gamma^3_{00} = -\frac{1}{2}a_z \exp(a-b-\nu),$$
$$\Gamma^3_{22} = \frac{1}{2}\rho^2 a_z \exp(-a-b-\nu), \quad \Gamma^1_{33} = -\frac{1}{2}b_\rho \exp\nu, \quad \Gamma^3_{31} = \frac{1}{2}b_\rho, \quad \Gamma^3_{33} = \frac{1}{2}b_z. \tag{A1}$$

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