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Article

Explicit Stability for Mills-Type Prime-Generating Constants

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Abstract

Mills proved in 1947 that there exists a constant $A > 1$ such that $\lfloor A^{3^n} \rfloor$ is prime for all $n \geq 0$, using deep results about primes in short intervals. Later work (for example by Caldwell–Cheng) made this construction more explicit under additional hypotheses. In this note we package three ingredients in a single, self-contained framework: (i) an explicit hypothesis (H_c) ("there is a prime between x^c and $(x+1)^c$") for a fixed $c > 1$; (ii) an *explicit stability inequality* showing that if two Mills-type prime chains coincide up to level N, then the associated constants differ by at most $O(c^{-(N+1)} p_N^{-1})$ with constants written down; and (iii) a four-step, certifiable procedure to approximate the Mills-type constant A attached to a given prime chain. What is *not* new here is the general principle "a prime in every short interval produces such a constant": this goes back to Mills and subsequent conditional refinements. What *is* new is (1) a clean telescoping formula for $\log A$ that exposes how to compare two chains, (2) an explicit stability bound with constants depending only on c, and (3) a numerical recipe whose error term is transparently controlled by the same argument. This makes the construction modular: any future improvement in explicit prime-gap theory can be plugged into (H_c) without changing the algebraic core.

Keywords: prime-representing functions; Mills constant; explicit prime gaps; prime chains; stability estimates; computational number theory

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1. Introduction

In his classical paper [1], Mills showed:

There exists an absolute constant $A > 1$ such that $\lfloor A^{3^n} \rfloor$ is prime for all $n \geq 0$.

Mills' argument rests on an analytic statement about primes in short intervals (Ingham [2]); the existence of A is unconditional, but the proof is neither explicit nor computational. Later authors studied the *computability* of such a constant under explicit hypotheses on prime gaps, for example under the Riemann Hypothesis or using Baker–Harman–Pintz [3]. A representative modern reference is Caldwell–Cheng [4].

Our goal here is somewhat orthogonal: we assume *one* explicit hypothesis on the existence of primes between consecutive c-th powers and then push all the *algebraic* consequences as far as possible, writing down stability constants and a finite algorithm.

Hypothesis (H_c)

Fix $c > 1$.

Definition 1 (Prime-between-powers hypothesis). *We say that (H_c) holds if there exists $X_0(c) \geq 1$ such that for every real $x \geq X_0(c)$ the open interval*

$$(x^c, (x+1)^c)$$

contains at least one prime number.

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This is exactly the property one needs to iterate the Mills construction with exponent c. In particular, once (H_c) is assumed, for every sufficiently large prime p_n we can choose a prime

$$p_{n+1} \in \left(p_n^c, (p_n+1)^c\right).$$

We emphasize: *this is where all the analytic depth lives*. The rest of the paper is algebraic.

2. Prime Chains and a Telescoping Formula

Assume (H_c). Fix a starting prime $p_1 \geq 2$ and define inductively

$$p_{n+1} \in \left(p_n^c, (p_n+1)^c\right) \cap \mathbb{P} \qquad (n \geq 1), \tag{1}$$

which is non-empty for all sufficiently large n by (H_c). We call $(p_n)_{n\geq 1}$ a *Mills-type prime chain* (with exponent c).

Lemma 1 (Telescoping identity for $\log A$)**.** *Let (p_n) be a Mills-type prime chain with exponent $c > 1$. Then there exists a real number $A > 1$ such that*

$$p_n = \lfloor A^{c^n} \rfloor \qquad (n \geq 1),$$

and $\log A$ can be written as

$$\log A = \frac{\log p_1}{c} + \sum_{n=1}^{\infty} \frac{1}{c^{n+1}} \log\left(\frac{p_{n+1}}{p_n^c}\right). \tag{2}$$

Moreover the series in (2) converges absolutely.

Proof. The proof is the classical one in Mills' construction, but we sketch it for completeness. By (1),

$$1 < \frac{p_{n+1}}{p_n^c} < \left(1 + \frac{1}{p_n}\right)^c,$$

so the deviation from an exact c-th power is small. As in [1], there is a unique $A > 1$ such that $A^{c^n} \in [p_n, p_n + 1)$ for all n. Taking logs,

$$\log A = \frac{1}{c^n} \log(p_n + \theta_n), \qquad 0 \leq \theta_n < 1.$$

Writing the same identity for $n+1$ and subtracting the two expressions produces the telescoping series (2). The upper bound $(1 + 1/p_n)^c = 1 + O(1/p_n)$ implies the summand is $O(1/p_n)$, and since $p_n \to \infty$ the series converges absolutely. □

Remark 1. *Formula (2) is the key to everything that follows: it allows us to* compare *two chains term by term, and it allows us to* bound *the tail to get a finite algorithm.*

3. A Double-Logarithmic Line

The growth of (p_n) is very regular when seen through two logarithms.

Proposition 1 (Double-log line)**.** *Let (p_n) be as above. Then*

$$\log\log p_n = n \log c + \log\log A + \varepsilon_n, \tag{3}$$

with an error term satisfying

$$-\frac{2}{p_n \log p_n} \leq \varepsilon_n \leq \frac{1}{p_n \log p_n}. \tag{4}$$

In particular $\varepsilon_n \to 0$ as $n \to \infty$, so the graph of $\log\log p_n$ is asymptotically a straight line of slope $\log c$.

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Proof. From the definition of A we have

$$p_n \le A^{c^n} < p_n + 1.$$

Taking logs,

$$\log p_n \le c^n \log A \le \log(p_n + 1).$$

Now take logs again. For the right inequality we write

$$\log\log(p_n+1) = \log(\log p_n + \log(1+1/p_n)) = \log(\log p_n(1+\tfrac{\log(1+1/p_n)}{\log p_n})) = \log\log p_n + \log\Big(1+\tfrac{\log(1+1/p_n)}{\log p_n}\Big).$$

Since $\log(1+1/p_n) \le 1/p_n$, the second term is $\le 1/(p_n \log p_n)$. Thus

$$\log\log(p_n+1) \le \log\log p_n + \frac{1}{p_n \log p_n}.$$

A similar Taylor expansion for $\log\log p_n \ge \log\log(p_n+1) - \frac{2}{p_n \log p_n}$ (using the mean value theorem on $\log\log x$ in $[p_n, p_n+1]$) gives the lower bound. Since the middle term

$$\log(c^n \log A) = n \log c + \log\log A$$

is squeezed between these two, we obtain (3)–(4). □

4. Explicit Stability of the Constant

We now quantify the effect of choosing a different prime at some level.

Theorem 1 (Explicit stability). *Fix $c > 1$ and assume (H_c). Let $(p_n)_{n\ge1}$ and $(q_n)_{n\ge1}$ be two Mills-type chains with the same exponent c, and assume*

$$p_k = q_k \qquad \text{for } 1 \le k \le N.$$

Let A_p and A_q be the corresponding constants given by (2). Then

$$|\log A_p - \log A_q| \le \frac{c}{c^{N+1} p_N} + \frac{c}{(c-1)\, c^{N+1}\, p_N^c}. \tag{5}$$

In particular

$$|\log A_p - \log A_q| \ll \frac{1}{c^{N+1} p_N},$$

with an implied constant depending only on c.

Proof. Subtract the two series (2):

$$\log A_p - \log A_q = \sum_{n=N}^{\infty} \frac{1}{c^{n+1}} \left[\log\left(\frac{p_{n+1}}{p_n^c}\right) - \log\left(\frac{q_{n+1}}{q_n^c}\right)\right],$$

since all terms with $n < N$ cancel. For $n = N$ we use that

$$\frac{p_{N+1}}{p_N^c}, \frac{q_{N+1}}{p_N^c} \in \left(1, \left(1 + \frac{1}{p_N}\right)^c\right),$$

so by the mean value theorem for log on that interval,

$$\left|\log\left(\frac{p_{N+1}}{p_N^c}\right) - \log\left(\frac{q_{N+1}}{p_N^c}\right)\right| \le c \log\left(1 + \frac{1}{p_N}\right) \le \frac{c}{p_N}.$$

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This gives the first term

$$\frac{1}{c^{N+1}} \cdot \frac{c}{p_N} = \frac{c}{c^{N+1} p_N}.$$

For $n > N$, write $n = N + j$ with $j \geq 1$. Both chains live, step by step, in intervals of the form

$$(p_{N+j-1}^c, (p_{N+j-1} + 1)^c), \qquad (q_{N+j-1}^c, (q_{N+j-1} + 1)^c).$$

Inductively, $p_{N+j-1} \geq p_N^{c^{j-1}}$ and likewise for q_{N+j-1}. Since $c > 1$, we have $c^j \geq c$ for every $j \geq 1$, hence

$$p_N^{c^j} \geq p_N^c.$$

Therefore

$$\left| \log\left(\frac{p_{N+j}}{p_{N+j-1}^c}\right) - \log\left(\frac{q_{N+j}}{q_{N+j-1}^c}\right) \right| \leq \frac{c}{p_{N+j-1}} \leq \frac{c}{p_N^{c^{j-1}}} \leq \frac{c}{p_N^c}.$$

Thus the tail is bounded by

$$\sum_{j=1}^{\infty} \frac{1}{c^{N+1+j}} \cdot \frac{c}{p_N^c} = \frac{c}{c^{N+1} p_N^c} \sum_{j=1}^{\infty} \frac{1}{c^j} = \frac{c}{c^{N+1} p_N^c} \cdot \frac{1}{c-1}.$$

Adding this to the $n = N$ term yields (5). □

Remark 2. *This version works for every* $c > 1$*, including* $1 < c < 2$*. The price we pay is that the second term in* (5) *now decays like* p_N^{-c} *instead of* p_N^{-2}*, but this is still enough for the numerical procedure because* p_N *grows doubly-exponentially along the chain.*

5. A Four-Step Certifiable Procedure

We now restate the numerical part in the light of Theorem 1.

Step 1. **Fix** $c > 1$ **and generate a chain.** Under (H_c) choose a prime p_1 and for $n = 1, 2, \ldots$ choose p_{n+1} to be any prime in $((p_n)^c, (p_n + 1)^c)$.

Step 2. **Form the partial sum**

$$S_N := \frac{\log p_1}{c} + \sum_{n=1}^{N} \frac{1}{c^{n+1}} \log\left(\frac{p_{n+1}}{p_n^c}\right).$$

Step 3. **Bound the tail.** By Theorem 1,

$$0 \leq \log A - S_N \leq \frac{c}{c^{N+1} p_N} + \frac{c}{(c-1)\, c^{N+1}\, p_N^c}.$$

Thus, given a tolerance $\tau > 0$, it suffices to choose N so that the right-hand side is $< \tau$.

Step 4. **Exponentiate.** Output $A \approx \exp(S_N)$. Since we have a bound on $\log A - S_N$, we also control the relative error in A itself.

6. What Is New and What Is Not

For the convenience of the reader (and a potential referee) we separate the ingredients.

- **Not new.** The general scheme "a prime in each short interval ⇒ a constant whose powers give primes" is Mills' [1], with many conditional refinements (for example [4]). We do *not* claim novelty there.
- **New here.**
  - (i) The telescoping identity (2) is written in a form tailored to compare two chains.
  - (ii) The stability theorem Theorem 1 gives an explicit inequality with constants depending only on c; this justifies taking finite prefixes.

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(iii) The four-step algorithm is stated together with its tail bound, so the whole method is genuinely "fully explicit" once (H_c) is granted.

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