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Article

Nilpotent (Anti-)Co-BRST Symmetries and Their Consequences in a 4D Field-Theoretic System

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Abstract

We demonstrate the existence of the infinitesimal, continuous and off-shell nilpotent (anti-)co-BRST symmetry transformations for the coupled (but equivalent) Lagrangian densities in the case of a four (3 + 1)-dimensional (4D) *combined* field-theoretic system of the free Abelian 1-form and 3-form gauge theories within the framework of Becchi-Rouet-Stora-Tyutin (BRST) formalism. Using the standard theoretical tricks of the Noether theorem, we derive the Noether (anti-)co-BRST currents and corresponding conserved (anti-)co-BRST charges. We establish that the *latter* are *not* nilpotent of order two due to the presence of the *non-trivial* Curci-Ferrari (CF) type restrictions on our theory (which have been deduced, in our present endeavor, from *two* different theoretical angles). We derive the off-shell nilpotent versions of the conserved (anti-)co-BRST charges and discuss the physicality criteria w.r.t. them. We demonstrate that the physical states (existing in the *total* quantum Hilbert space of states) are *those* that are annihilated by the operator forms of the *dual* versions of the first-class constraints on our theory. We comment *very* briefly on the annihilation of the physical states by the operator forms of the first-class constraints due to the physicality criteria w.r.t. the nilpotent versions of the (anti-)BRST charges because our 4D theory *also* respects the nilpotent (anti-)BRST symmetries.

Keywords: combined field-theoretic system of the 4D free Ableian 1-form and 3-form gauge theories; nilpotent (anti-)co-BRST symmetries; CF-type restrictions; noether non-nilpotent (anti-)co-BRST charges; nilpotent (anti-)co-BRST charges; physicality criteria

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1. Introduction

The ideas of symmetries of all varieties (i.e. discrete, continuous, spacetime, internal, supersymmetric, etc.) have been at the heart of modern developments in theoretical physics (see, e.g. [1–4] and references therein). In fact, it has been well-established that the requirement of the *local* continuous symmetry invariance dictates the emergence of interactions in most of the theories that are useful in the precise description of the natural phenomena. One such symmetry is the well-known *gauge* symmetry transformation which has been responsible for the precise theoretical description of the *three* out of *four* fundamental interactions of nature. In other words, the *local* gauge theories (based on the Abelian and non-Abelian gauge symmetry groups) describe the electromagnetic, weak and strong interactions of nature (see, e.g. [1]). One of the most theoretically appealing as well as mathematically elegant quantization scheme for these kinds of gauge theories is the well-known Becchi-Rouet-Stora-Tyutin (BRST) formalism [5–8] where the unitarity and quantum gauge (i,e. BRST) invariance are respected *together* at any arbitrary order of perturbative computations for a given physical process that is allowed by the BRST-quantized versions of the interacting gauge theories (see, e.g. [9] for details).

A couple of sacrosanct properties of the BRST formalism is the observation that, for a given local *classical* gauge symmetry transformation for a gauge theory, there are two nilpotent versions of the *quantum* gauge symmetry transformations which are called as the BRST and anti-BRST symmetry transformations for the BRST-quantized gauge theory and these nilpotent symmetry transformation operators are required to absolutely anticommute with each-other. The *former* property encodes the fermionic nature of the (anti-)BRST symmetry transformations under which a fermionic field of a properly BRST-quantized gauge theory transforms to its counterpart bosonic field and vice-versa. On the other hand, the *latter* property implies the linear independence of the BRST and anti-BRST symmetry transformations (which distinguish *them* from the $\mathcal{N} = 2$ SUSY transformations which are *also* nilpotent of order two but they do *not* anticommute with each-other). Physically, the absolute antcommutativity property implies that the BRST and anti-BRST symmetry transformation operators have their own identity and they are important in their own right. This claim becomes quite transparent in the context of the D-dimensional (e.g. D = 2, 3, 4) BRST-quantized versions of the field-theoretic as well as the 1D toy models of Hodge theory (see, e.g. [10–14] and references therein) where the discrete and continuous symmetry transformations (and corresponding conserved charges) provide the physical realization(s) of the de Rham cohomological operators[1] of differential geometry at the *algebraic* level.

In our present investigation, we focus on the four (3 + 1)-dimensional (4D) *combined* field-theoretic system of the free Abelian 3-form and 1-form gauge theories *together* (within the framework of BRST formalism) and discuss the dual-BRST (i.e. co-BRST) and anti-dual (i.e. anti-co-BRST) symmetry transformation operators for the coupled (but equivalent) Lagrangian densities. We establish that the total gauge-fixing terms for the Abelian 3-form and 1-form gauge fields, owing their mathematical origin to the dual-exterior (i.e. co-exterior) derivative of differential geometry (see, e.g. [15–19]), remain invariant under these nilpotent symmetry transformation operators. This observation should be contrasted against the off-shell nilpotent and absolutely anticommuting (anti-)BRST symmetry transformation operators which leave the total kinetic terms for the Abelian 3-form and 1-form gauge fields invariant (see, e.g. [20] for details). In this context, it is worthwhile to mention that the kinetic terms for the above Abelian 3-form and 1-form gauge fields owe their mathematical origin to the exterior derivative of differential geometry [15-19].

In our present endeavor, we have been able to show that the coupled (but equivalent) Lagrangian densities $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)] and $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ [cf. Eqs. (28),(29)] (cf. Secs. 2 and 5) respect the infinitesimal, continuous and off-shell nilpotent co-BRST (i.e. dual-BRST) and anti-co-BRST (i.e. anti-dual-BRST) *symmetry* transformations, respectively, because the corresponding action integrals remain invariant under *them*. We have derived the conserved (but non-nilpotent) Noether dual-BRST and the anti-dual-BRST charges which are found to be the generator for the above off-shell nilpotent dual-BRST and anti-dual-BRST symmetry transformations [cf. Eqs. (3),(30)]. The off-shell nilpotent versions of the co-BRST and anti-co-BRST charges have been systematically derived (cf. Secs 4 and 7) from their counterparts non-nilpotent versions of the Noether conserved charges by exploiting the theoretical proposal made in our earlier work [21]. We have discussed (i) the physicality criterion w.r.t. the conserved and nilpotent version of the co-BRST charge in our Sec. 4, and (ii) the physicality criterion w.r.t. the conserved and nilpotent version of the anti-co-BRST charge has been carried out in our Sec. 7. It has been established that the physicality criteria w.r.t. the off-shell nilpotent versions of the conserved (anti-)dual BRST charges lead to the annihilation of the physical states (existing in the *total* quantum Hilbert space of states) by the operator forms of the *dual* versions of the first-class constraints of our 4D *classical* field-theoretic system.

[1] There exists a set of *three* operators $(d,\ \delta,\ \Delta)$ on a compact spacetime manifold (without a boundary) which are known as the de Rham cohomological operators of differential geometry where the operator $d = \partial_\mu\, dx^\mu$ [with $d^2 = \frac{1}{2!}\,(\partial_\mu\partial_\nu - \partial_\nu\partial_\mu)\,(dx^\mu \wedge dx^\nu) = 0$] is called as the exterior derivative, the operator $\delta = \pm * d *$ (with $\delta^2 = 0$) denotes the co-exterior (i.e. dual-exterior) derivative and the symbol $\Delta = (d + d)^2 = \{d, \delta\}$ stands for the Laplacian operator. In the above relationship (i.e. $\delta = \pm * d *$) between the (co-)exterior derivatives $(\delta)d$, the symbol $*$ is called as the Hodge duality operator (on the given compact spacetime manifold). These operators obey an algebra: $d^2 = 0,\ \delta^2 = 0,\ \Delta = (d + d)^2 = \{d, \delta\},\ [\Delta,\ d] = 0,\ [\Delta,\ \delta] = 0$ where the symbols $\{\ ,\ \}$ and $[\ ,\ ]$ stand for the anticommutator and commutator, respectively. This algebra is popularly known as the Hodge algebra in the realm of differential geometry [15–19].

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Our present investigation is urgent, essential and important on the following counts. First of all, it has been amply indicated in our earlier work (see, e.g. [22] for details) that our present 4D *combined* field-theoretic system is an example for Hodge theory within the framework of BRST formalism where the discrete and continuous symmetry transformations (and corresponding conserved charges) provide the physical realization(s) of the de Rham cohomological operators of differential geometry [15–19] at the *algebraic* level. Second, we have discussed the physicality criteria w.r.t. the conserved and nilpotent versions of the (anti-)BRST charges in our earlier work [20] where we have been able to show that the operator forms of the first-class constraints of our 4D *classical* gauge theory annihilate the physical states of our BRST-quantized 4D field-theoretic system at the *quantum* level (cf. Appendix A). In our present endeavor, we discuss the physicality criteria w.r.t. the nilpotent versions of the conserved (anti-)co-BRST charges and establish that the *dual* versions of the operator forms of the first-class constraints annihilate the physical states (which are identified with the *harmonic* states of the Hodge decomposed quantum states in the quantum Hilbert space of states). Finally, we show that the operator forms of the first-class constraints and their *dual* counterparts are connected with each-other by the *duality* symmetry transformations that are respected by the non-ghost sector of the coupled (but equivalent) Lagrangian densities for our 4D BRST-quantized field-theoretic system.

The theoretical contents of our present endeavor are organized as follows. To set up the useful notations and to explain the key mathematical symbols, in Sec. 2, we recapitulate the bare essentials of our earlier work [11] where the nilpotent co-BRST (i.e. dual-BRST) symmetry transformations have been discussed. Our Sec. 3 is devoted to the derivations of the Noether conserved co-BRST current and corresponding conserved (but non-nilpotent) co-BRST charge. The subject matter of our Sec 4 is connected with the derivation of the off-shell nilpotent conserved co-BRST charge where we also study the physicality criterion w.r.t. to it. Our Sec 5 deals with the discussion on a proper set of anti-co-BRST symmetry transformations that is respected by a coupled (but equivalent) Lagrangian density corresponding to the co-BRST invariant Lagrangian density of our Sec. 2. We derive the Noether conserved current and corresponding conserved (but non-nilpotent) anti-co-BRST charge in our Sec. 6. As far as our Sec. 7 is concerned, we derive the nilpotent version of the conserved anti-co-BRST charge and study the physicality criterion w.r.t. it in a *concise* manner. In our Sec 8, we devote time on the derivation of the *non-trivial* CF-type restrictions by requiring that *both* the coupled Lagrangian densities *must* respect the nilpotent co-BRST as well as the anti-co-BRST transformations. These CF-type restrictions are *also* shown to be responsible for the absolute anticommutativity of the co-BRST and anti-co-BRST transformations. Finally, in Sec. 9, we summarize our key results, mention a few novel observations and point out the future scope of our present investigation.

Our Appendix A deals with a *very* brief summary of our earlier work on the off-shell nilpotent (anti-)BRST symmetries of our *present* 4D field-theoretic system [20] and we establish that the nilpotent versions of the (anti-)BRST charges lead to the annihilation of the physical states by the operator forms of the first-class constraints on our theory when we demand the validity of the physicality criteria w.r.t. *them*. In our Appendix B, we very briefly mention the algebraic structure that is obeyed by the conserved (anti-)co-BRST charges and ghost charge for our 4D BRST-quantized theory.

Conventions and Notations: We adopt the convention of the left derivative w.r.t. *all* the fermionic fields of our theory in the computations of the EL-EoMs, canonical conjugate momenta, Noether conserved current, etc. We choose our 4D Levi-Civita tensor $\varepsilon_{\mu\nu\sigma\rho}$ such that: $\varepsilon_{0123} = +1 = -\,\varepsilon^{0123}$ and, when two of them are contrated together, they satisfy the standard relationships: $\varepsilon_{\mu\nu\eta\kappa}\varepsilon^{\mu\nu\eta\kappa} = -\,4!$, $\varepsilon_{\mu\nu\eta\kappa}\varepsilon^{\mu\nu\eta\rho} = -\,3!\,\delta^{\rho}_{\kappa}$, $\varepsilon_{\mu\nu\eta\kappa}\varepsilon^{\mu\nu\sigma\rho} = -\,2!\,\big(\delta^{\sigma}_{\eta}\delta^{\rho}_{\kappa} - \delta^{\sigma}_{\kappa}\delta^{\rho}_{\eta}\big)$, etc. As far as the Abelian 3-form gauge field $A_{\mu\nu\sigma}$ is concerned, we follow the convention: $(\delta A_{\mu\nu\sigma}/\delta A_{\alpha\beta\gamma}) = \frac{1}{3!}\,\big[\delta^{\alpha}_{\mu}(\delta^{\beta}_{\nu}\,\delta^{\gamma}_{\sigma} - \delta^{\beta}_{\sigma}\,\delta^{\gamma}_{\nu}) + \delta^{\alpha}_{\nu}(\delta^{\beta}_{\sigma}\,\delta^{\gamma}_{\mu} - \delta^{\beta}_{\mu}\,\delta^{\gamma}_{\sigma}) + \delta^{\alpha}_{\sigma}(\delta^{\beta}_{\mu}\,\delta^{\gamma}_{\nu} - \delta^{\beta}_{\nu}\,\delta^{\gamma}_{\mu})\big]$, etc., for the tensorial variation/differentiation for various computational purposes. We take the background 4D flat Minkowskian metric tensor $\eta_{\mu\nu}$ as: $\eta_{\mu\nu}$ = diag $(+1, -1, -1, -1)$ so that the dot product between two *non-null* 4D vectors U_μ and V_μ is defined as: $U \cdot V = \eta_{\mu\nu} U^\mu\, V^\nu \equiv U_0\, V_0 - U_i\, V_i$ where the Greek indices $\mu, \nu, \sigma ... = 0, 1, 2, 3$ denote the time and space directions and Latin indices $i, j, k ... = 1, 2, 3$ stand for the 3D space directions *only*. The off-shell nilpotent (i.e. $s^2_{(a)d} = 0$)

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(anti-)dual BRST [i.e. (anti-)co-BRST] symmetry transformation operators are represented by the symbols $s_{(s)d}$ and the corresponding conserved (but non-nilpotent) Noether charges are denoted by $Q_{(a)d}$. The nilpotent versions of the conserved (anti-)co-BRST charges carry the notations $Q_{(A)D}$. The over dot (i.e. $\dot{\Phi}$) on a generic field Φ denotes the time-derivative (i.e. $\partial\Phi/\partial t$).

2. Preliminary: Co-BRST Symmetry Transformations

We begin with the BRST-invariant Lagrangian density $\mathcal{L}_{(B)}$ which is the *sum* of the non-ghost sector of the Lagrangian density $\mathcal{L}_{(NG)}$ and the FP-ghost sector of the Lagrangian density $\mathcal{L}_{(FP)}$ (see, e.g. [20,22] for details). The explicit form of the *former* is [20,22]

$$\begin{aligned}
\mathcal{L}_{(NG)} &= \frac{1}{2}\,B_2\,(\partial\cdot\phi) - \frac{1}{4}\,B_2^2 + \frac{1}{2}\,B_3\,(\partial\cdot\tilde{\phi}) - \frac{1}{4}\,B_3^2 + \frac{1}{2}\,B^2 - B\,(\partial\cdot A) \\
&+ \frac{1}{2}\,B_1^2 + B_1\left(\frac{1}{3!}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\mu A_{\nu\sigma\rho}\right) - \frac{1}{4}\,(B_{\mu\nu})^2 + \frac{1}{2}\,B_{\mu\nu}\left[\partial_\sigma A^{\sigma\mu\nu} + \frac{1}{2}\,(\partial^\mu\phi^\nu - \partial^\nu\phi^\mu)\right] \\
&- \frac{1}{4}\,(\mathcal{B}_{\preceq\succeq})^2 + \frac{1}{2}\,\mathcal{B}_{\preceq\succeq}\left[\varepsilon^{\mu\nu\sigma\rho}\,\partial_\sigma A_\rho + \frac{1}{2}\,(\partial^\mu\tilde{\phi}^\nu - \partial^\nu\tilde{\phi}^\mu)\right],
\end{aligned} \tag{1}$$

where the fields $B, B_1, B_2, B_3, B_{\mu\nu}, \mathcal{B}_{\mu\nu}$ are the *bosonic* Nakanishi-Lautrup type auxiliary fields that have been invoked to linearize the kinetic as well as the gauge-fixing terms for the fields $A_{\mu\nu\sigma}, A_\mu, \phi_\mu, \tilde{\phi}_\mu$. To be specific, we have invoked the auxiliary fields B and $\mathcal{B}_{\mu\nu}$ to linearize the gauge-fixing and kinetic terms for the Abelian 1-form ($A^{(1)} = A_\mu\,dx^\mu$) gauge field A_μ, respectively. On the other hand, the Nakanishi-Lautrup auxiliary fields B_1 and $B_{\mu\nu}$ have been used to linearize the kinetic and gauge-fixing terms for the Abelian 3-form field $A_{\mu\nu\sigma}$, respectively. The auxiliary fields B_2 and B_3 have been utilized to linearize the gauge-fixing terms for the Lorentz vector and axial-vector fields ϕ_μ and $\tilde{\phi}_\mu$. We would like to clarify that the (axial-)vector fields $(\tilde{\phi}_\mu)\phi_\mu$ have been introduced in the theory due to the reducibility properties of the kinetic term for the Abelian 1-form gauge field and the gauge-fixing term for the Abelian 3-form [i.e. $A^{(3)} = \frac{1}{3!}\,A_{\mu\nu\sigma}\,(dx^\mu\wedge dx^\nu\wedge dx^\sigma)$] gauge field. The FP-ghost part of the BRST-quantized version of our 4D *combined* field theoretic system of the free Abelian 3-form and 1-form gauge theories is (see, e.g. [20,22] for details)

$$\begin{aligned}
\mathcal{L}_{(FP)} &= \frac{1}{2}\Big[(\partial_\mu\bar{C}_{\nu\sigma} + \partial_\nu\bar{C}_{\sigma\mu} + \partial_\sigma\bar{C}_{\mu\nu})\,(\partial^\mu C^{\nu\sigma}) + (\partial_\mu\bar{C}^{\mu\nu} + \partial^\nu\bar{C}_1)\,f_\nu \\
&- (\partial_\mu C^{\mu\nu} + \partial^\nu C_1)\,\bar{F}_\nu + (\partial\cdot\bar{\beta})\,B_4 - (\partial\cdot\beta)\,B_5 - B_4\,B_5 - 2\,\bar{F}^\mu\,f_\mu \\
&- (\partial_\mu\bar{\beta}_\nu - \partial_\mu\bar{\beta}_\nu)\,(\partial^\mu\beta^\nu) - \partial_\mu\bar{C}_2\,\partial^\mu C_2\Big] - \partial_\mu\bar{C}\,\partial^\mu C,
\end{aligned} \tag{2}$$

where the fermionic (i.e. $C^2 = \bar{C}^2 = 0,\ C\,\bar{C} + \bar{C}\,C = 0$) Lorentz scalar *basic* (anti-)ghost fields $(\bar{C})C$, carrying the ghost numbers (-1)+1, are associated with the BRST-quantized version of the Abelian 1-form gauge theory (that is hidden in our *combined* field-theoretic system). On the other hand, the fermionic (i.e. $C_{\mu\nu}^2 = 0,\ \bar{C}_{\mu\nu}^2 = 0,\ C_{\mu\nu}\,\bar{C}_{\sigma\rho} + \bar{C}_{\sigma\rho}\,C_{\mu\nu} = 0$, etc.) *basic* (anti-)ghost fields $(\bar{C}_{\mu\nu})C_{\mu\nu}$, carrying the ghost numebrs (-1)+1, respectively, are associated with the Abelian 3-form gauge field $A_{\mu\nu\sigma}$. We have the ghost-for-ghost *bosonic* (i.e. $\bar{\beta}_\mu^2 \neq 0,\ \beta_\mu^2 \neq 0,\ \beta_\mu\bar{\beta}_\nu = \bar{\beta}_\nu\,\beta_\mu$, etc.) Lorentz vector (anti-)ghost fields $(\bar{\beta}_\mu)\beta_\mu$ in our theory that carry the ghost numbers (-2)+2, respectively. Our BRST-quantized Abelian 3-from gauge theory is endowed with the fermionic (i.e. $C_2^2 = \bar{C}_2^2 = 0,\ C_2\,\bar{C}_2 + \bar{C}_2\,C_2 = 0$) (anti-)ghost fields $(\bar{C}_2)C_2$ which carry the ghost numbers (-3)+3, respectively (because $(\bar{C}_2)C_2$ are the ghost-for-ghost-for-ghost fields in our theory). In addition, we have a pair of *bosonic* auxiliary fields (B_5, B_4) which carry the ghost numbers (-2, +2) and a pair of *fermionic* auxiliary fields ($\bar{F}_\mu$, f_μ) that are endowed with the ghost numbers (-1, +1), respectively. On top of all these basonic/fremionic auxiliary and basic (anti-)ghost fields, we have the additional fermionic (i.e. $C_1^2 = \bar{C}_1^2 = 0,\ C_1\,\bar{C}_1 + \bar{C}_1\,C_1 = 0$) (anti-)ghost fields $(\bar{C}_1)C_1$ that are endowed with the ghost numbers (-1)+1, respectively. All these bosonic and fernionic (anti-)ghost fields are required in our theory to maintain the unitarity at any arbitrary order of perturbative computations for a physical process that is allowed by our 4D BRST-quantized field theoretic system of the Abelian 3-form and 1-form gauge theories.

It is very interesting to point out that the *total* Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1), (2)] respects the following infinitesimal, continuous and off-shell nilpotent (i.e. $s_d^2 = 0$) dual-BRST (i.e. co-BRST) transformations s_d, namely;

$$\begin{aligned}
& s_d A_{\mu\nu\sigma} = \varepsilon_{\mu\nu\sigma\rho}\, \partial^\rho \bar{C}, \quad s_d A_\mu = \frac{1}{2}\, \varepsilon_{\mu\nu\sigma\rho}\, \partial^\nu \bar{C}^{\sigma\rho}, \quad s_d \bar{C}_{\mu\nu} = \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, \quad s_d \bar{f}_\mu = \partial_\mu B_5, \\
& s_d \bar{\mathcal{B}}_{\mu\nu} = \partial_\mu \bar{F}_\nu - \partial_\nu \bar{F}_\mu, \quad s_d \bar{\beta}_\mu = \partial_\mu \bar{C}_2, \quad s_d C_1 = -\,B_3, \quad s_d \beta_\mu = -\,f_\mu, \quad s_d \widetilde{\phi}_\mu = +\,\bar{F}_\mu, \\
& s_d C_{\mu\nu} = -\,\mathcal{B}_{\mu\nu}, \qquad s_d C = +\,B_1, \quad s_d C_2 = B_4, \qquad s_d \bar{C}_1 = B_5, \qquad s_d F_\mu = -\,\partial_\mu B_3, \\
& s_d \Big[\bar{C}_2,\, \bar{C},\, f_\mu,\, \bar{F}_\mu,\, \phi_\mu,\, B,\, B_1,\, B_2,\, B_3,\, B_4,\, B_5,\, B_{\mu\nu},\, \mathcal{B}_{\mu\nu},\, \bar{B}_{\mu\nu},\Big] = 0,
\end{aligned} \tag{3}$$

because we observe that $\mathcal{L}_{(B)}$ transforms to a total spacetime derivative

$$\begin{aligned}
s_d \mathcal{L}_{(B)} \;=\; & \frac{1}{2}\, \partial_\mu \Big[(\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})\, \mathcal{B}_{\nu\sigma} + \mathcal{B}^{\mu\nu} \bar{F}_\nu + B_4\, \partial^\mu \bar{C}_2 \\
+ \; & B_5\, f^\mu + B_3\, \bar{F}^\mu + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)\, f_\nu\Big] + \partial_\mu \Big[B_1\, \partial^\mu \bar{C}\Big],
\end{aligned} \tag{4}$$

thereby rendering the action integral $S = \int d^4x\, \mathcal{L}_{(B)}$ invariant (i.e. $s_d S = 0$) for all the physical fields of our theory which are supposed to vanish off as $x \to \pm\infty$ due to the Gauss divergence theorem. Thus, the infinitesimal, continuous and off-shell nilpotent co-BRST transformations (3) are the *symmetry* transformations for our theory.

We end this section with the following remarks. First of all, the nilpotency property (i.e. $s_d^2 = 0$) of the co-BRST symmetry transformation operator s_d establishes its *fermionic* nature. As a consequence, *this* operator transforms (i) a bosonic field of our theory to a ferminoc field, and (ii) a fermionic field to its counterpart bosonic field. Second, the gauge-fixing terms $(\partial \cdot A)$ and $(\partial_\sigma A^{\sigma\mu\nu})$ owe their origins to the co-exterior (i.e. dual-exterior) derivative of differential geometry because we observe that: $\delta A^{(1)} = (\partial \cdot A)$ and $\delta A^{(3)} = -\,\frac{1}{2!}\,(\partial^\sigma A_{\sigma\mu\nu})\,(dx^\mu \wedge dx^\nu)$. It is interesting to point out that the *total* gauge-fixing terms for the Abelian 3-form and 1-form gauge theories remain invariant under the co-BRST (i.e. dual-BRST) symmetry transformations. Hence the nomenclature (i.e. the co-BRST/dual-BRST) symmetry transformations) for the infinitesimal and continuous transformations in equation (3), is correct. Third, we note that, it is very essential to have *both* the Abelian 1-form and 3-form gauge theories *together* so that our 4D BRST-quantized *combined* field-theoretic system can respect the co-BRST symmetry transformations. Finally, we would like to point out that, in addition to the off-shell nilpotent co-BRST symmetry transformations in (3), the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ *also* respects the off-shell nilpotent BRST symmetry transformations (see, e.g. Appendix A for details).

3. Noether Co-BRST Conserved Current and Charge

According to Noether's theorem, the existence of the infinitesimal continuous and off-shell nilpotent co-BRST *symmetry* transmissions (3) leads to the derivation of the Noether co-BRST current $(J^\mu_{(d)})$ as follows

$$\begin{aligned}
J^\mu_{(d)} \;=\; & s_d \Phi_i\, \frac{\partial \mathcal{L}_{(B)}}{\partial(\partial_\mu \Phi_i)} - \frac{1}{2}\Big[(\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu})\, \mathcal{B}_{\nu\sigma} + \mathcal{B}^{\mu\nu} \bar{F}_\nu + B_4\, \partial^\mu \bar{C}_2 \\
+ \; & B_5\, f^\mu + B_3\, \bar{F}^\mu + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)\, f_\nu\Big] - B_1\, \partial^\mu \bar{C},
\end{aligned} \tag{5}$$

where the generic field Φ_i stands for all the bosonic as well as fermionic *dynamical* fields of our 4D combined system of the BRST-quantized Abelian 3-from and 1-form gauge theories which is described by the *perfectly* (co-)BRST-invariant Lagrangian density $\mathcal{L}_{(B)}$. In other words, we have: $\Phi_i \equiv A_{\mu\nu\sigma}, C_{\mu\nu}, \bar{C}_{\mu\nu}, A_\mu, \phi_\mu, \widetilde{\phi}_\mu, \beta_\mu, \bar{\beta}_\mu, C_1, \bar{C}_1, C, \bar{C}$ in the Lagrangian density $\mathcal{L}_{(B)}$ and the rest of the terms of the above equation (5) are nothing but the terms on the r.h.s. of equation (4) modulo a sign factor which is connected with the standard technique of the Noether theorem. The *first* term of the r.h.s. of (5) can be explicitly computed by taking into account (i) the convention of the *left* derivative w.r.t. the fermionic dynamical fields, (ii) the *usual* convention of the derivative w.r.t. the bosonic dynamical fields, and (iii) the infinitesimal, continuous and off-shell nilpotent co-BRST symmetry transformations (3) of our 4D BRST-quantized theory. The final expression for the conserved Noether co-BRST (i.e. dual-BRST) current ($J^\mu_{(d)}$) turns out to be the following:

$$\begin{aligned} J^\mu_{(d)} &= \frac{1}{2}\Big[\mathcal{B}^{\mu\nu}\,\bar F_\nu - \varepsilon^{\mu\nu\sigma\rho}\,B\,(\partial_\nu \bar C_{\sigma\rho}) + (\partial^\mu \bar C^{\nu\sigma} + \partial^\nu \bar C^{\sigma\mu} + \partial^\sigma \bar C^{\mu\nu})\,\mathcal{B}_{\nu\sigma} \\ &+ \varepsilon^{\mu\nu\sigma\rho}\,(\partial_\nu \bar C)\,B_{\sigma\rho} + B_4\,\partial^\mu \bar C_2 - (\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)\,\partial_\nu \bar C_2 + (\partial^\mu\bar\beta^\nu - \partial^\nu\bar\beta^\mu)\,f_\nu \\ &+ B_3\,\bar F^\mu + B_5\,f^\mu + (\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu})\,(\partial_\nu\bar\beta_\sigma - \partial_\sigma\bar\beta_\nu)\Big] + B_1\,\partial^\mu \bar C. \end{aligned} \tag{6}$$

The conservation law (i.e. $\partial_\mu J^\mu_{(d)} = 0$) can be proven by taking into account the following Euler-Lagrange (EL) equations of motion (EoMs) which emerge out from the (non-)ghost sectors of the Lagrangian densities (1) and (2), namely;

$$\begin{aligned} &\Box\,\bar C = 0, \qquad \Box\,\bar C_2 = 0, \qquad \partial_\mu(\partial^\mu\beta^\nu - \partial^\nu\beta^\mu) = \partial^\nu B_4, \quad (\partial\cdot\bar F) = 0, \\ &\partial_\mu(\partial^\mu\bar\beta^\nu - \partial^\nu\bar\beta^\mu) = -\,\partial^\nu B_5, \qquad \partial_\mu\mathcal{B}^{\mu\nu} + \partial^\nu B_3 = 0, \qquad (\partial\cdot f) = 0, \\ &\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu B_{\sigma\rho} - \partial^\mu B_1 = 0, \qquad \frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \mathcal{B}_{\sigma\rho} + \partial^\mu B = 0, \\ &\partial_\mu(\partial^\mu \bar C^{\nu\sigma} + \partial^\nu \bar C^{\sigma\mu} + \partial^\sigma \bar C^{\mu\nu}) = -\,\frac{1}{2}\,(\partial^\nu \bar F^\sigma - \partial^\sigma \bar F^\nu), \\ &\partial_\mu(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}) = -\,\frac{1}{2}\,(\partial^\nu f^\sigma - \partial^\sigma f^\nu). \end{aligned} \tag{7}$$

The conserved Noether co-BRST current (i.e. $J^\mu_{(d)}$) leads to the definition of the conserved co-BRST charge $Q_d = \int d^3x\, J^0_{(d)}$ whose explicit expression is as follows:

$$\begin{aligned} Q_d &= \int d^3x\,\Big[\frac{1}{2}\Big\{\mathcal{B}^{0i}\,\bar F_i - \varepsilon^{0ijk}\,B\,(\partial_i \bar C_{jk}) + (\partial^0 \bar C^{ij} + \partial^i \bar C^{j0} + \partial^j \bar C^{0i})\,\mathcal{B}_{ij} \\ &+ \varepsilon^{0ijk}\,(\partial_i \bar C)\,B_{jk} + B_4\,\dot{\bar C}_2 - (\partial^0\beta^i - \partial^i\beta^0)\,\partial_i \bar C_2 + (\partial^0\bar\beta^i - \partial^i\bar\beta^0)\,f_i \\ &+ B_3\,\bar F^0 + B_5\,f^0 + (\partial^0 C^{ij} + \partial^i C^{jk} + \partial^j C^{0i})\,(\partial_i\bar\beta_j - \partial_j\bar\beta_i)\Big\} + B_1\,\dot{\bar C}\Big]. \end{aligned} \tag{8}$$

The above conserved charge Q_d is the generator for the infinitesimal, continuous and off-shell nilpotent co-BRST symmetry transformations (3). This can be readily checked by the *standard* relationship between the infinitesimal continuous symmetry transformations and their generator as the Noether conserved charge. In other words, we have the following standard mathematical formula in the case of our dual-BRST transformations, namely;

$$s_d\Phi_i(x) = \pm\, i\,\Big[\Phi_i(x),\; Q_d\Big]_{(\pm)}, \tag{9}$$

where the $(\pm)$ signs, as the subscripts on the square bracket in the above equation, correspond to the square bracket being an (anti)commutator for the generic dynamical field $\Phi_i(x)$ being fermionic/bosonic in nature. The $\pm$ signs, in front of the square bracket on the r.h.s., have to be chosen judiciously as far as the choice of the field Φ_i is concerned.

We wrap-up this section with the following decisive remarks. First of all. we observe that the *total* gauge-fixing terms (owing their *basic* mathematical origin to the co-exterior derivative) for the Abelian 3-form and 1-form gauge theories remain invariant under the nilpotent co-BRST symmetry transformations. This observation is distinctly different from the invariance of the *total* kinetic terms (cf. Appendix A), owing their *basic* origin to the exterior derivative, that remain invariant under the (anti-)BRST symmetry transformations (see, e.g. [20] for details). Second, even though the nilpotent co-BRST symmetry transformations are fermionic in nature, these transformations are *not* like the

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supersymmetric transformations which are also fermionic (cf. the Sec. 5 for more discussions). Thus, the mathematical *basis* for the existence of the co-BRST symmetry transformations lies in the co-exterior derivative of differential geometry. Third, the existence of the co-BRST (and corresponding anti-co-BRST) symmetry transformations are crucial for (i) the consideration of the *combined* 4D BRST-quantized field-theoretic system of the Abelian 3-from and 1-form gauge theories *together*, and (ii) the proof that our present 4D BRST-quantized field-theoretic system is an example for the Hodge theory [22]. Fourth, it is essential to point out that under the (anti-)BRST symmetry transformations (see, e.g. [20]), the BRST-quantized versions of the Abelian 3-form and 1-form gauge theories remain *invariant* separately and independently unlike in the case of the (anti-)co-BRST symmetry transformations where *both* these gauge theories are required to be present *together*. In other words, for the (anti-) co-BRST invariance in our 4D theory, we need to have the free Abelian 3-form and 1-form field-theoretic system *together* because the variation of the kinetic term for the Abelian 3-form gauge field compensates with the variation of the FP-ghost term for the BRST-quantized Abelian 1-form gauge theory. Finally, we note that the relationship, written in equation (9), is a very general expression and it is valid for all kinds of choices for the generic field Φ_i. Choosing the generic field Φ_i to be the co-BRST charge itself (i.e. $\Phi_i = Q_d$), we obtain the following expression from (9), namely;

$$\begin{aligned} s_d\, Q_d = i\,\{Q_d,\; Q_d\} \;&\equiv\; -\frac{1}{2}\int d^3x\,\Big[(\partial^0\mathcal{B}^{ij} + \partial^i\mathcal{B}^{jk} + \partial^j\mathcal{B}^{0i})\,(\partial_i\bar{\beta}_j - \partial_j\bar{\beta}_i) \\ &-\; (\partial^0 f^i - \partial^i f^0)\,\partial_i\bar{C}_2\Big] \neq 0, \end{aligned} \tag{10}$$

where we have explicitly computed the l.h.s. (i.e. $s_d\, Q_d$) by *directly* applying the co-BRST symmetry transformations (3) on the expression for Q_d [cf. Eq. (8)]. In other words, we find that the anticommutator: $i\,\{Q_d,\; Q_d\} = 2\,i\,Q_d^2 \neq 0$ is *not* equal to zero [cf. the r.h.s. of equation (10)] which implies that the Noether conserved co-BRST charge Q_d is *not* nilpotent[2] of order two (i.e. $Q_d^2 \neq 0$). Hence, this Noether conserved charge is *not* suitable for the discussions connected with (i) the BRST cohomology (see. e.g. [23] for details), and (ii) the physicality criterion w.r.t. it (cf. Sec. 4 for more discussions).

4. Nilpotent Co-BRST Charge from Noether Charge

The central theme of this section is to derive the conserved and nilpotent (i.e. $Q_D^2 = 0$) version of the co-BRST charge Q_D from the non-nilpotent (i.e. $Q_d^2 \neq 0$) version of the Noether conserved charge Q_d. In this context, we shall exploit systematically the theoretical method that has been proposed in our earlier work [21]. According to the basic ideas behind our earlier work [21], the emphasis will be laid on the use of (i) the EL-EoMs that are derived from the perfectly co-BRST invariant Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$, (ii) the partial integral followed by the Gauss divergence theorem, and (ii) the infinitesimal, continuous and off-shell nilpotent (i.e. $s_d^2 = 0$) co-BRST symmetry transformations s_d at appropriate places, to make sure that, ultimately, we obtain: $s_d Q_D = i\,\{Q_D,\; Q_D\} = 0$ thereby leading to the derivation of the nilpotent (i.e. $Q_D^2 = 0$) version of the co-BRST charge Q_D from the non-nilpotent (i.e. $Q_d^2 \neq 0$) Noether co-BRST charge Q_d.

Against the backdrop of the key contents of the above paragraph, first of all, we focus on the *fourth* term of the Noether conserved co-BRST charge Q_d [cf. Eq. (8)]. Using the partial integration followed by the Gauss divergence theorem, this term can be written as

$$\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,(\partial_i\bar{C})\,B_{jk} = -\,\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,(\partial_i B_{jk})\,\bar{C}, \tag{11}$$

[2] This happens because of the presence of the *non-trivial* CF-type restrictions (cf. Sec. 8 below) on our theory. The key idea behind it has been discussed, in an elaborate manner, in our earlier work [21].

because we have dropped a total space derivative term. Using the EL-EoM: $\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu B_{\sigma\rho} = \partial^\mu B_1$ [cf. Eq. (7)], we can recast the above integral as:

$$-\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,(\partial_i B_{jk})\,\bar{C} = -\int d^3x\,\dot{B}_1\,\bar{C}. \tag{12}$$

Taking into account the last term of the expression for the Noether co-BRST charge in (8) and the above integral, we have the contribution to a part of the nilpotent (i.e. $Q_D^2 = 0$) version of the co-BRST charge Q_D from the BRST-quantized version of the combined field-theoretic system of the Abelian 3-form and 1-form gauge theories as follows

$$Q_D^{(1)} = \int d^3x\,\big[B_1\,\dot{\bar{C}} - \dot{B}_1\,\bar{C}\big], \tag{13}$$

where the superscript (1) on $Q_D^{(1)}$ denotes the fact that the above contribution incorporates into it the auxiliary field B_1. It is straightforward to note that the above charge $Q_D^{(1)}$ is co-BRST invariant because we find that: $s_d Q_D^{(1)} = 0$ where we have directly applied the off-shell nilpotent (i.e. $s_d^2 = 0$) co-BRST symmetry transformations (3) on (13). Thus, the charge $Q_D^{(1)}$ will be a part of the *full* nilpotent version of the co-BRST charge Q_D. We concentrate now on the *second* term of the Noether co-BRST charge Q_d [cf. Eq. (8)] which can be re-expressed, using the beauty of the partial integration followed by the theoretical strength of the Gauss divergence theorem, as follows

$$-\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,B\,(\partial_i\bar{C}_{jk}) = +\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,(\partial_i B)\,\bar{C}_{jk}, \tag{14}$$

where we have dropped a total space derivative term because all the physical fields vanish off as $x \to \pm\infty$. We can, at this stage, use the EL-EoM: $\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \mathcal{B}_{\sigma\rho} = -\,\partial^\mu B$ [cf. Eq. (7)] which can be recast in the following form

$$\big(\partial^\mu \mathcal{B}^{\nu\sigma} + \partial^\nu \mathcal{B}^{\sigma\mu} + \partial^\sigma \mathcal{B}^{\mu\nu}\big) = -\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\rho B, \tag{15}$$

to obtain the following from the above integral (14), namely;

$$+\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,(\partial_i B)\,\bar{C}_{jk} = -\frac{1}{2}\int d^3x\,\big(\partial^0 \mathcal{B}^{ij} + \partial^i \mathcal{B}^{j0} + \partial^j \mathcal{B}^{0i}\big)\,\bar{C}_{ij}. \tag{16}$$

This term would be incorporated into the nilpotent (i.e. $Q_D^2 = 0$) version of the co-BRST charge Q_D. Toward our main goal of obtaining $s_d Q_D = 0$, we have to apply the co-BRST symmetry transformation operator s_d [cf. Eq. (3)] on the r.h.s. of the above expression which, ultimately, leads to the following explicit result, namely;

$$-\frac{1}{2}\int d^3x\,\big(\partial^0 \mathcal{B}^{ij} + \partial^i \mathcal{B}^{j0} + \partial^j \mathcal{B}^{0i}\big)\,\big(\partial_i\bar{\beta}_j - \partial_j\bar{\beta}_i\big). \tag{17}$$

At this juncture, we have to modify a specific term of the Noether conserved charge Q_d [cf. Eq. (8)] so that, when we apply s_d on a part of *that* term, the outcome should cancel out with our result in (17). Such a term is the last but one term: $\frac{1}{2}\int d^3x\,(\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\,(\partial_i\bar{\beta}_j - \partial_j\bar{\beta}_i)$. This term can be modified by exploiting the simple algebraic trick as:

$$\begin{aligned} &-\frac{1}{2}\int d^3x\,(\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\,(\partial_i\bar{\beta}_j - \partial_j\bar{\beta}_i) \\ &+ \int d^3x\,(\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\,(\partial_i\bar{\beta}_j - \partial_j\bar{\beta}_i). \end{aligned} \tag{18}$$

It is straightforward to note that the outcome of the application of the co-BRST transformations s_d [cf. Eq. (3)] on the *first* term of the above equation cancels out with our result in (17). Hence, in the final expression for the nilpotent version (i.e. $Q_D^2 = 0$) of the co-BRST charge Q_D, the *first* term of the above equation (18) will be present. We would like to point out that, at present stage of our exercise, we have already obtained *three* crucial terms of the nilpotent version of the co-BRST charge Q_D. These very important and useful terms are (i) the full integral in our equation (13), (ii) the complete integral in (16), and (iii) the *first* integral that is present in our equation (18).

Let us now focus on the *second* term of equation (18) which can be re-expressed as: $2 \int d^3x \, (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) \, (\partial_i \bar{\beta}_j)$. Using the partial integral followed by the Gauss divergence theorem, this term turns out to be

$$- 2 \int d^3x \; \partial_i (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) \, \bar{\beta}_j, \tag{19}$$

where we have dropped a total space derivative term. Exploiting the theoretical strength of the appropriate[3] EL-EoMs from (7), this integral can be re-written as:

$$- \int d^3x \; (\partial^0 f^i - \partial^i f^0) \, \bar{\beta}_i. \tag{20}$$

The above integral will be present in the expression for the nilpotent version of the co-BRST charge Q_D. For our central goal of obtaining $s_d Q_D = 0$, we have to apply the co-BRST symmetry transformations (3) on the above integral which leads to the following

$$- s_d \Big[\int d^3x \; (\partial^0 f^i - \partial^i f^0) \, \bar{\beta}_i \Big] = + \int d^3x \; (\partial^0 f^i - \partial^i f^0) \, \partial_i \bar{C}_2, \tag{21}$$

where we have taken into account (i) the fermionic natures of s_d and the auxiliary vector field f_μ, and (ii) the co-BRST symmetry transformations that have been listed in equation (3). As per the method proposed in our earlier work [21], we have to modify an appropriate term of the non-nilpotent co-BRST charge Q_d [cf. Eq. (8)] such that when s_d acts on a part of *that* term, the outcome should cancel out with our result in (21). Such a term is: $- \frac{1}{2} \int d^3x \, (\partial^0 \beta^i - \partial^i \beta^0) \, \partial_i \bar{C}_2$ which happens to be the *sixth* term in the expression for Q_d [cf. Eq. (8)]. Using the simple algebraic trick, this integral can be re-written as follows:

$$+ \int d^3x \, (\partial^0 \beta^i - \partial^i \beta^0) \, \partial_i \bar{C}_2 - \frac{3}{2} \int d^3x \, (\partial^0 \beta^i - \partial^i \beta^0) \, \partial_i \bar{C}_2. \tag{22}$$

It is straightforward to note that the application of the co-BRST symmetry transformations: $s_d \bar{C}_2 = 0, \; s_d \beta_\mu = - f_\mu$ [cf. Eq. (3)] on the *first* integral of the above equation cancels out with our result in (21). Hence, the *first* integral of the above equation will be present in the nilpotent version of the co-BRST charge Q_D. We focus, at this stage, on the *second* integral of (22). Using the partial integration followed by the application of the Gauss divergence theorem, *this* integral can be re-written as

$$+ \frac{3}{2} \int d^3x \; \partial_i (\partial^0 \beta^i - \partial^i \beta^0) \, \bar{C}_2, \tag{23}$$

where we have dropped a total space derivative term because of obvious reason. We are in the position to use the EL-EoM: $\partial_\mu (\partial^\mu \beta^\nu - \partial^\nu \beta^\mu) = \partial^\nu B_4$ which leads to $\partial_i (\partial^0 \beta^i - \partial^i \beta^0) = - \dot{B}_4$ [cf. Eq. (7)] for the choice $\nu = 0$. This specific equation is derived from the FP-ghost part of the Lagrangian density (2). Thus, ultimately, we obtain the following

$$+ \frac{3}{2} \int d^3x \, \partial_i (\partial^0 \beta^i - \partial^i \beta^0) \, \bar{C}_2 = - \frac{3}{2} \int d^3x \, \dot{B}_4 \, \bar{C}_2, \tag{24}$$

[3] To be precise, we have used the EL-EoM: $\partial_\mu (\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}) = - \frac{1}{2} (\partial^\nu f^\sigma - \partial^\sigma f^\nu)$ where we have made the choices: $\nu = 0$ and $\sigma = j$ which leads to: $\partial_i (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}) = \frac{1}{2} (\partial^0 f^j - \partial^j f^0)$.

which turns out to be the co-BRST invariant quantity because we observe that: $s_d(\dot{B}_4\,\bar{C}_2) = 0$ [cf. Eq. (3)]. Assimilating *all* the useful and relevant integrals, we obtain the *final* expression for the nilpotent version of the co-BRST charge Q_D as follows:

$$
\begin{aligned}
Q_D &= \int d^3x \Big[\frac{1}{2}\Big\{\mathcal{B}^{0i}\,\bar{F}_i + B_4\,\dot{\bar{C}}_2 - 3\,\dot{B}_4\,\bar{C}_2 + (\partial^0\bar{C}^{ij} + \partial^i\bar{C}^{j0} + \partial^j\bar{C}^{0i})\,\mathcal{B}_{ij} + B_5\,f^0 \\
&+ (\partial^0\bar{\beta}^i - \partial^i\bar{\beta}^0)\,f_i - (\partial^0 C^{ij} + \partial^i C^{jk} + \partial^j C^{0i})\,(\partial_i\bar{\beta}_j - \partial_j\bar{\beta}_i) + B_3\,\bar{F}^0 \\
&- (\partial^0\mathcal{B}^{ij} + \partial^i\mathcal{B}^{jk} + \partial^j\mathcal{B}^{0i})\,\bar{C}_{ij}\Big\} + (B_1\,\dot{\bar{C}} - \dot{B}_1\,\bar{C}_2) + (\partial^0\beta^i - \partial^i\beta^0)\,\partial_i\bar{C}_2 \\
&- (\partial^0 f^i - \partial^i f^0)\,\bar{\beta}_i\Big]. \qquad (25)
\end{aligned}
$$

A few comments are in order now. First of all, it can be checked that $s_d Q_D = 0$ where, for this proof, we have to apply the co-BRST symmetry transformations (3) *directly* on the expression for the modified version of the co-BRST charge Q_D [cf. Eq. (25)]. This observation implies that we have obtained the nilpotent (i.e. $Q_D^2 = 0$) version of the co-BRST charge Q_D from the non-nilpotent (i.e. $Q_d^2 \neq 0$) version of the Noether co-BRST charge Q_d. Second, toward our main goal of obtaining Q_D from Q_d, first of all, we have performed the partial integration followed by the Gauss divergence theorem on the terms of the Noether co-BRST charge Q_d [cf. Eq. (8)] that contain the gauge-fixing terms for the basic gauge fields A_μ and $A_{\mu\nu\sigma}$. For instance, we have started off with the terms like: $-\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,B\,(\partial_i\bar{C}_{jk})$ and $+\frac{1}{2}\int d^3x\,\varepsilon^{0ijk}\,(\partial_i C)\,B_{jk}$ because we know that: $B = (\partial\cdot A)$ and $B_{\mu\nu} = \partial^\sigma A_{\sigma\mu\nu} + \frac{1}{2}(\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)$ which are nothing but the gauge-fixing terms for the Abelian 1-form and 3-form gauge fields, respectively. Third, in our derivation of Q_D from Q_d, we have exploited (i) the appropriate EL-EoMs, and (ii) the partial integration followed by Gauss's divergence theorem. Hence, both forms of the charges (i.e. Q_d and Q_D) are *conserved* due to the basic tenets behind Noether's theorem. Finally, it worthwhile to point out that many of the terms that are present in Q_D [cf. Eq. (25)] are same as the ones that are present in Q_d [cf. Eq. (8)] because such terms are *trivially* co-BRST invariant.

Right from the beginning [cf. Eqs. (1),(2)], we observe that the basic as well as the auxiliary (anti-)ghost fields (present in the FP-ghost part of the Lagrangian density $\mathcal{L}_{(FP)}$) are decoupled from the rest of the *physical* theory (described by the Lagrangian density $\mathcal{L}_{(NG)}$). As a consequence, a specific state in the quantum Hilbert space of states is the direct product [24] of the physical states (i.e. $|phys>$) and the ghost states (i.e $|ghost>$). Within the framework of BRST formalism, the physicality criterion, w.r.t. to a nilpotent and conserved charge, requires that the physical states (i.e. $|phys>$) are *those* that are annihilated by *this* conserved and nilpotent charge[4]. A close and careful look at the nilpotent version of the co-BRST charge Q_D [cf. Eq. (25)] demonstrates that every individual term of this charge carries a ghost number equal to -1 which is made up of (i) the basic/auxiliary physical fields with ghost number equal to zero [cf. Eq. (1)] and the basic/auxiliary ghost field(s) [cf. Eq. (2)] with ghost number equal to -1, and (ii) the basic/auxiliary (anti-)ghost fields present in $\mathcal{L}_{(FP)}$. In the physicality criterion w.r.t. the nilpotent version of the co-BRST charge, every individual term of *this* charge acts on a specifically chosen quantum state. When the ghost fields of Q_D act on the ghost states (i.e $|ghost>$), they always produce *non-zero* result. Hence, the physicality criterion (i.e. $Q_D\,|phys> = 0$) w.r.t. the conserved and nilpotent version of the co-BRST charge Q_D requires that the physical fields (carrying the ghost number equal to zero) must annihilate the true physical states (i.e. $|phys>$) which should be consistent with the Dirac quantization conditions for systems that are endowed with constraints. As far as our 4D BRST-quantized field theoretic system is concerned, the physicality criterion w.r.t.

[4] On a compact spacetime manifold without a boundary, any arbitrary n-form f_n (with $n = 1, 2, 3.....$) can be written as a *unique* sum of the harmonic form h_n (with $\Delta h_n = 0,\ dh_n = 0,\ \delta h_n = 0$), an exact form ($d\,e_{n-1}$) and a co-exact form ($\delta\,c_{n+1}$) as: $f_n = h_n + d\,e_{n-1} + d\,c_{n+1}$ where, as pointed out earlier, the set of three operators ($d,\ \delta,\ \Delta$) are the de Rham cohomological operators of differential geometry [15-19]. This observation is what is famously known as the Hodge decomposition theorem in the domain of differential geometry. This theorem can be utilized in the quantum Hilbert space of states for the BRST-quantized field-theoretic systems that are examples for Hodge theory where the physical states (i.e. $|phys>$) can be chosen to be the *harmonic states* that will be annihilated (i.e. $Q_{(A)B}\,|phys> = 0,\ Q_{(A)D}\,|phys> = 0$) by the conserved and nilpotent versions of the (anti-)BRST ($Q_{(A)B}$) and (anti-)co-BRST ($Q_{(A)D}$) charges.

the co-BRST charge must lead to the annihilation of the physical states by the *dual* version of the first-class constraints of the 4D classical field-theoretic system because the physicality criterion (i.e. $Q_B\,|phys>=0$) w.r.t. the conserved and nilpotent version of the BRST charge leads to the annihilation of the physical states, at the quantum level, by the operator forms of *purely* the first-class constraints of our 4D classical field-theoretic system (see, e.g. Appendix A).

Against the backdrop of the above paragraph, we note that the physicality criterion (i.e. $Q_D\,|phys>=0$) w.r.t. the conserved and nilpotent version of the co-BRST charge Q_D, leads to the following conditions on the physical states, namely;

$$\begin{aligned}
&B_1\,|phys>=0, \qquad \dot{B}_1\,|phys>=0, \qquad \frac{1}{2}\,\mathcal{B}_{ij}\,|phys>=0,\\
&\frac{1}{2}\,(\partial^0\mathcal{B}^{ij}+\partial^i\mathcal{B}^{j0}+\partial^j\mathcal{B}^{0i})\,|phys>=0.
\end{aligned} \tag{26}$$

It is worthwhile to point out a few subtle issues that have been taken care of in the above derivation from the requirement: $Q_D\,|phys>=0$. As mentioned earlier, we have focused on the operation of the physical fields (with ghost number equal to zero) which are associated with the *basic* anti-ghost fields of our 4D BRST-quantized theory. We lay emphasis on this issue because there are a couple of terms: $\frac{1}{2}\int d^3x\,(\mathcal{B}^{0i}\,\bar{F}_i+B_3\,\bar{F}^0)$ in the expression for Q_D where the auxiliary fields $\mathcal{B}^{0i}$ and B_3 carry the ghost number equal to zero. However, these fields have *not* contributed anything in the conditions (26) because they are *not* associated with the *basic* anti-ghost fields of our theory. Rather, they are associated with the components of the auxiliary anti-ghost fermionic vector field $\bar{F}_\mu$. A close look at the above conditions on the physical states (i.e. $|phys>$) w.r.t. the nilpotent and conserved version of the co-BRST charge Q_D and the *ones* that have been obtained in equation (A.5) w.r.t. the nilpotent and conserved version of the BRST charge Q_B (cf. Appendix A) establishes that these conditions are *dual* to one-another. This is due to the fact that under the following discrete *duality* symmetry transformations (see. e.g. [22])

$$\begin{aligned}
&A_\mu \longrightarrow \mp\frac{1}{3!}\,\varepsilon_{\mu\nu\sigma\rho}\,A^{\nu\sigma\rho}, \quad A_{\mu\nu\sigma} \longrightarrow \pm\,\varepsilon_{\mu\nu\sigma\rho}\,A^{\rho}, \quad B_{\mu\nu}\to\pm\,\mathcal{B}_{\mu\nu}, \quad \mathcal{B}_{\mu\nu}\to\mp\,B_{\mu\nu},\\
&B\to\pm\,B_1, \quad B_1\to\mp\,B, \quad B_2\to\pm\,B_3, \quad B_3\to\mp\,B_2, \; \phi_\mu\to\pm\,\widetilde{\phi}_\mu, \; \widetilde{\phi}_\mu\to\mp\,\phi_\mu,
\end{aligned} \tag{27}$$

the non-ghost part of the Lagrangian density $\mathcal{L}_{(NG)}$ [cf. Eq. (1)] remains invariant. There exists a set of *duality* symmetry transformations in the ghost sector, too [22]. However, for our immediate purpose, we do *not* need to mention them urgently here. In other words we note that the conditions (26) on the physical states (i.e. $|phys>$) w.r.t. the conserved and nilpotent co-BRST charge Q_D are deeply *connected* with the conditions: $B\,|phys>=0,\ \dot{B}\,|phys>=0,\ \frac{1}{2}\,B_{ij}\,|phys>=0,\ \frac{1}{2}\,(\partial^0 B^{ij}+\partial^i B^{j0}+\partial^j B^{0i})\,|phys>=0$ [cf. Eq. (A.5)] that emerge out from the physicality criterion (i.e. $Q_B\,|phys>=0$) w.r.t. the conserved and nilpotent version of the BRST charge due to the presence of the discrete *duality* symmetry transformations in (27) in the non-ghost sector of our theory.

5. Nilpotent Anti-Co-BRST Transformations

Corresponding to the co-BRST invariant Lagrangian density $\mathcal{L}_{(B)}=\mathcal{L}_{(NG)}+\mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)], we have the anti-co-BRST invariant Lagrangian density $\mathcal{L}_{(\bar{B})}=\mathcal{L}_{(ng)}+\mathcal{L}_{(fp)}$ which incorporates into it the non-ghost part [i.e. $\mathcal{L}_{(ng)}$] and the FP-ghost part [i.e. $\mathcal{L}_{(fp)}$]. These Lagrangian densities (i.e. $\mathcal{L}_{(ng)}$ and $\mathcal{L}_{(fp)}$) are the analogues of the non-ghost part of the Lagrangian density $\mathcal{L}_{(NG)}$ [cf. Eq. (1)] and FP-ghost part of the Lagrangian density $\mathcal{L}_{(FP)}$ [cf. Eq. (2)] of $\mathcal{L}_{(B)}$. First of all, let us focus on the non-ghost sector $\mathcal{L}_{(ng)}$ of the Lagrangian density $\mathcal{L}_{(\bar{B})}$ which is described by the following

$$
\begin{aligned}
\mathcal{L}_{(ng)} &= \frac{1}{2}\, B^2 - B\,(\partial \cdot A) + \frac{1}{2}\, B_2\,(\partial \cdot \phi) - \frac{1}{4}\, B_2^2 + \frac{1}{2}\, B_3\,(\partial \cdot \tilde{\phi}) - \frac{1}{4}\, B_3^2 \\
&+ \frac{1}{2}\, B_1^2 + B_1 \Big(\frac{1}{3!}\, \varepsilon^{\mu\nu\sigma\rho}\, \partial_\mu A_{\nu\sigma\rho}\Big) - \frac{1}{4}\, (\bar{B}_{\mu\nu})^2 - \frac{1}{2}\, \bar{B}_{\mu\nu} \Big[\partial_\sigma A^{\sigma\mu\nu} - \frac{1}{2}\,(\partial^\mu \phi^\nu - \partial^\nu \phi^\mu)\Big] \\
&- \frac{1}{4}\, (\bar{\mathcal{B}}_{\mu\nu})^2 - \frac{1}{2}\, \bar{\mathcal{B}}_{\mu\nu} \Big[\varepsilon^{\mu\nu\sigma\rho}\, \partial_\sigma A_\rho - \frac{1}{2}\,(\partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu)\Big], \qquad (28)
\end{aligned}
$$

where a couple of *bosonic* Nakanishi-Lautrup type auxiliary fields $\bar{B}_{\mu\nu}$ and $\bar{\mathcal{B}}_{\mu\nu}$ (with *zero* ghost numbers) have been invoked to linearize the gauge-fixing term for the free Abelian 3-form gauge field and kinetic term for the free Abelian 1-form field, respectively. A comparison between the Lagrangian density (1) and (28) establishes that the *last four* terms have a few decisive differences in the signs. The FP-ghost sector of the Lagrangian density $\mathcal{L}_{(\bar{B})}$ for our 4D field-theoretic system is described by the following Lagrangian density [22]

$$
\begin{aligned}
\mathcal{L}_{(fp)} &= \frac{1}{2} \Big[(\partial_\mu \bar{C}_{\nu\sigma} + \partial_\nu \bar{C}_{\sigma\mu} + \partial_\sigma \bar{C}_{\mu\nu})\,(\partial^\mu C^{\nu\sigma}) - (\partial_\mu \bar{C}^{\mu\nu} - \partial^\nu \bar{C}_1)\, F_\nu \\
&+ (\partial_\mu C^{\mu\nu} - \partial^\nu C_1)\, \bar{f}_\nu + (\partial \cdot \bar{\beta})\, B_4 - (\partial \cdot \beta)\, B_5 - B_4\, B_5 - 2\, \bar{f}^\mu\, F_\mu \\
&- (\partial_\mu \bar{\beta}_\nu - \partial_\mu \bar{\beta}_\nu)\,(\partial^\mu \beta^\nu) - \partial_\mu \bar{C}_2\, \partial^\mu C_2\Big] - \partial_\mu \bar{C}\, \partial^\mu C, \qquad (29)
\end{aligned}
$$

which is *different* from our earlier FP-ghost Lagrangian density [cf. Eq. (2)] that has been taken into account in the co-BRST invariant Lagrangian density $\mathcal{L}_{(B)}$. It should be pointed out that we have (i) two *new* fermionic (anti-)ghost [i.e. $(\bar{f}_\mu)F_\mu$] auxiliary fields in (29) that carry the ghost numbers (-1)+1, respectively, and (ii) the *second* and *third* terms of the FP-ghost parts of the Lagrangian densities in (2) and (29) are *quite* different.

As per the sacrosanct requirements of the BRST formalism, it is essential that we should have the (i) off-shell *nilpotent* (i.e. $s_{(a)d}^2 = 0$), and (ii) absolutely anticommuting (i.e. $s_d\, s_{ad} + s_{ad}\, s_d = 0$) infinitesimal co-BRST (i.e. s_d) and anti-co-BRST (i.e. s_{ad}) transformations at the *quantum* level in a *properly* BRST-quantized theory. In view of these *two* requirements, corresponding to the infinitesimal, continuous and off-shell nilpotent ($s_d^2 = 0$) co-BRST symmetry transformations (3), we have the following infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-BRST transformations (s_{ad}), namely;

$$
\begin{aligned}
& s_{ad} A_{\mu\nu\sigma} = \varepsilon_{\mu\nu\sigma\rho}\, \partial^\rho C, \qquad s_{ad} A_\mu = \frac{1}{2}\, \varepsilon_{\mu\nu\sigma\rho}\, \partial^\nu C^{\sigma\rho}, \qquad s_{ad} C_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \\
& s_{ad} \mathcal{B}_{\mu\nu} = \partial_\mu F_\nu - \partial_\nu F_\mu, \qquad s_{ad} \beta_\mu = \partial_\mu C_2, \qquad s_{ad} \bar{\beta}_\mu = -\, \bar{f}_\mu, \quad s_d \tilde{\phi}_\mu = +\, F_\mu, \\
& s_{ad} \bar{C}_{\mu\nu} = -\, \bar{\mathcal{B}}_{\mu\nu}, \qquad s_{ad} \bar{C} = -\, B_1, \qquad s_{ad} \bar{C}_2 = B_5, \qquad s_{ad} \bar{C}_1 = B_3, \\
& s_{ad} C_1 = +\, B_4, \qquad s_{ad} f_\mu = +\, \partial_\mu B_4, \qquad s_{ad} \bar{F}_\mu = +\, \partial_\mu B_3, \\
& s_{ad} \Big[C,\, C_2,\, B,\, B_1,\, B_2,\, B_3,\, B_4,\, B_5,\, \bar{f}_\mu,\, F_\mu,\, \phi_\mu,\, B_{\mu\nu},\, \bar{B}_{\mu\nu},\, \bar{\mathcal{B}}_{\mu\nu}\Big] = 0, \qquad (30)
\end{aligned}
$$

which transform the Lagrangian density $\mathcal{L}_{(\bar{B})}$ to the total spacetime derivative:

$$
\begin{aligned}
s_{ad}\, \mathcal{L}_{(\bar{B})} &= \frac{1}{2}\, \partial_\mu \Big[\bar{\mathcal{B}}^{\mu\nu}\, F_\nu - (\partial^\mu\, C^{\nu\sigma} + \partial^\nu\, C^{\sigma\mu} + \partial^\sigma\, C^{\mu\nu})\, \bar{\mathcal{B}}_{\nu\sigma} - B_5\, \partial^\mu\, C_2 \\
&+ B_3\, F^\mu - B_4\, \bar{f}^\mu + (\partial^\mu\, \beta^\nu - \partial^\nu\, \beta^\mu)\, \bar{f}_\nu\Big] + \partial_\mu \Big[B_1\, \partial^\mu C\Big]. \qquad (31)
\end{aligned}
$$

The above observation renders the action integral (i.e. $S = \int d^4x\, \mathcal{L}_{(\bar{B})}$), corresponding to the Lagrangian density $\mathcal{L}_{(\bar{B})}$, invariant (i.e. $s_{ad} S = 0$) due to Gauss's divergence theorem where *all* the physical fields vanish off as $x \to \pm\infty$. Hence the infinitesimal, continuous and nilpotent anti-co-BRST transformations (30) are the *symmetry* transformations for our 4D BRST-quantized field-theoretic system of the Abelian 3-form and 1-form gauge theories.

We wrap-up this short (but quite important) section with the following crucial remarks. First of all, we would like to point out that the Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ are *coupled* because some of *their* fields are *not* independent (in the true sense of the word) because of the presence of the *non-trivial* CF-type restrictions on our 4D BRST -quantized theory (cf. Sec. 8 below for details). These coupled Lagrangian densities are also *equivalent* from the points of view of (i) the symmetry considerations (cf. Sec. 8 below), and (ii) the direct equality (see, e.g. [20] for details). Second, the nilpotency property establishes that the anti-co-BRST symmetry transformation operator s_{ad} is fermionic in nature which transforms a bosonic field into its counterpart fermionic field and vice-versa. Third, the operation of the anti-co-BRST symmetry transformation operator on a field leads to the increment in the ghost number as well as the mass dimension (in the natural units: $\hbar = c = 1$) by *one* [cf. Eq. (30)]. Fourth, the absolute anticommutativity property (i.e. $s_d s_{ad} + s_{as} s_d = 0$) between the co-BRST and anti-co-BRST symmetry transformation operators encodes (i) the linear independence of these two transformation operators, and (ii) the validity of the (anti-)BRST and (anti-)co-BRST invariant CF-type restrictions (cf. Sec. 8 for details). Finally, the nilpotent co-BRST and anti-co-BRST symmetry transformation operators are quite *different* from the nilpotent $\mathcal{N} = 2$ SUSY transformation operators because of their absolute anticommuting property. Hence, the *latter* property is the distinguishing (and very crucial) feature of the co-BRST and anti-co-BRST symmetry transformation operators.

6. Conserved Noether Anti-Co-BRST Charge: Comment on Its Nilpotency Property

In our previous section, we have seen that the action integral (corresponding to the Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$) remains invariant because of (i) our observation in (31), and (ii) the application of the partial integration followed by the Gauss divergence theorem due to which *all* the physical fields vanish off as $x \to \pm\infty$. This specific observation, according to the Noether theorem, leads to the derivation of the Noether anti-co-BRST current (i.e. J^{μ}_{ad}) by exploiting the following formula, namely;

$$
\begin{aligned}
J^{\mu}_{(ad)} \;&=\; s_{ad}\Phi_i \,\frac{\partial \mathcal{L}_{(\bar{B})}}{\partial(\partial_\mu \Phi_i)} - \frac{1}{2}\Big[\bar{\mathcal{B}}^{\mu\nu}\, F_\nu - (\partial^\mu\, C^{\nu\sigma} + \partial^\nu\, C^{\sigma\mu} + \partial^\sigma\, C^{\mu\nu})\,\bar{\mathcal{B}}_{\nu\sigma} - B_5\,\partial^\mu\, C_2 \\
&+\; B_3\, F^\mu - B_4\,\bar{f}^\mu + (\partial^\mu\,\beta^\nu - \partial^\nu\,\beta^\mu)\,\bar{f}_\nu\Big] - B_1\,\partial^\mu C,
\end{aligned} \tag{32}
$$

where the generic field $\Phi_i \equiv A_{\mu\nu\sigma}, C_{\mu\nu}, \bar{C}_{\mu\nu}, A_\mu, \phi_\mu, \widetilde{\phi}_\mu, \beta_\mu, \bar{\beta}_\mu, C_1, \bar{C}_1, C, \bar{C}$ stands for all the bosonic/ fermionic dynamical fields of the Lagrangian density $\mathcal{L}_{(\bar{B})}$. These fields are dynamical because of the presence of their derivative terms in $\mathcal{L}_{(\bar{B})}$. The rest of the terms (ROTs), in the above equation, are the ones that are present in the square brackets of equation (31) modulo a sign factor which is consistent with the basic tenets of Noether's theorem. The *first* term, in the above equation (32), can be precisely computed by using (i) the proper rules for the derivatives w.r.t. the fermionic and bosonic fields of our 4D BRST-quantized theory, and (ii) the infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-co-BRST symmetry transformations s_{ad} [cf. Eq. (30)]. The explicit expression for the Noether anti-co-BRST current $J^{\mu}_{(ad)}(fp)$, from the FP-ghost part (i.e. $\mathcal{L}_{(fp)}$) of the Lagrangian density $\mathcal{L}_{(\bar{B})}$ *plus* the ROTs, turns out to be the following

$$
\begin{aligned}
J^{\mu}_{(ad)}(fp) \;&=\; -\frac{1}{2}\Big[(\partial^\mu\,\bar{C}^{\nu\sigma} + \partial^\nu\,\bar{C}^{\sigma\mu} + \partial^\sigma\,\bar{C}^{\mu\nu})\,(\partial_\nu\beta_\sigma - \partial_\sigma\beta_\nu) + B_5\,\partial^\mu\, C_2 \\
&+\; B_4\,\bar{f}^\mu - (\partial^\mu\,\beta^\nu - \partial^\nu\,\beta^\mu)\,\bar{f}_\nu + (\partial^\mu\,\bar{\beta}^\nu - \partial^\nu\,\bar{\beta}^\mu)\,\partial_\nu C_2\Big],
\end{aligned} \tag{33}
$$

where the argument (fp) in the above current [i.e. $J^{\mu}_{(ad)}(fp)$] denotes the fact that (i) we have taken into account *only* the Lagrangian density $\mathcal{L}_{(fp)}$, and (ii) the rest of the terms (ROTs) of equation (31) which have been taken into account from the terms that are present in the square brackets in the transformation $s_{ad}\mathcal{L}_{(\bar{B})}$ [cf. Eq. (32)]. It is very interesting to point out that the above part [i.e. $J^{\mu}_{(ad)}(fp)$]

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of the total anti-co-BRST current $J^\mu_{(ad)}$ is conserved [i.e. $\partial_\mu J^\mu_{(ad)}(fp) = 0$] provided we use the following EL-EoMs

$$\Box C_2 = 0, \qquad \partial_\mu\big(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu\big) = \partial^\nu B_4, \qquad \partial_\mu\big(\partial^\mu \bar\beta^\nu - \partial^\nu \bar\beta^\mu\big) = -\,\partial^\nu B_5,$$
$$\partial_\mu\big(\partial^\mu \bar C^{\nu\sigma} + \partial^\nu \bar C^{\sigma\mu} + \partial^\sigma \bar C^{\mu\nu}\big) = \frac{1}{2}\big(\partial^\nu \bar f^\sigma - \partial^\sigma \bar f^\nu\big), \qquad (\partial\cdot\bar f) = 0, \tag{34}$$

which emerge out from the Lagrangian density $\mathcal{L}_{(fp)}$. The contribution [i.e. $J^\mu_{(ad)}(ng)$] to the total anti-co-BRST current $J^\mu_{(ad)}$, that emerges from the non-ghost part of the Lagrangian density (i.e. $\mathcal{L}_{(ng)}$) of the *total* Lagrangian density $\mathcal{L}_{(\bar B)}$, is as follows:

$$\begin{aligned} J^\mu_{(ad)}(ng) &= B_1\,\partial^\mu C - \frac{1}{2}\Big[\big(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}\big)\,\bar{\mathcal{B}}_{\nu\sigma} - B_3\,F^\mu - \bar{\mathcal{B}}^{\mu\nu}\,F_\nu \\ &+ \varepsilon^{\mu\nu\sigma\rho}\,(\partial_\nu C)\,\bar B_{\sigma\rho} + \varepsilon^{\mu\nu\sigma\rho}\,B\,(\partial_\nu C_{\sigma\rho})\Big]. \end{aligned} \tag{35}$$

It is quite interesting to point out that the above Noether current $J^\mu_{(ad)}(ng)$ has been derived from (i) the use of the Lagrangian density $\mathcal{L}_{(ng)}$ [cf. Eq. (28)], and (ii) the infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-co-BRST symmetry transformations s_{ad}. Just like the current $J^\mu_{(ad)}(fp)$, *this* current [i.e. $J^\mu_{(ad)}(ng)$], emerging out purely from the non-ghost sector of the Lagrangian density $\mathcal{L}_{(ng)}$ [cf. Eq. (28)], is also conserved [i.e. $\partial_\mu J^\mu_{(ad)}(ng) = 0$]. To substantiate this statement, we have to use the following EL-EoMs

$$\partial_\mu \bar{\mathcal{B}}^{\mu\nu} + \partial^\nu B_3 = 0, \quad \frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \bar B_{\sigma\rho} + \partial^\mu B_1 = 0, \quad \frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \bar{\mathcal{B}}_{\sigma\rho} - \partial^\mu B = 0,$$
$$\partial_\mu\big(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}\big) = \frac{1}{2}\big(\partial^\nu F^\sigma - \partial^\sigma F^\nu\big), \quad (\partial\cdot F) = 0, \quad \Box C = 0, \tag{36}$$

which are derived from the Lagrangian density $\mathcal{L}_{(ng)}$. We would like to lay emphasis on the following *two* inputs, namely;

$$\big(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}\big)\,\partial_\mu \bar{\mathcal{B}}_{\nu\sigma} = \big(\partial^\mu \bar{\mathcal{B}}^{\nu\sigma} + \partial^\nu \bar{\mathcal{B}}^{\sigma\mu} + \partial^\sigma \bar{\mathcal{B}}^{\mu\nu}\big)\,\partial_\mu C_{\nu\sigma},$$
$$\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \bar{\mathcal{B}}_{\sigma\rho} - \partial^\mu B = 0 \Longleftrightarrow \big(\partial^\mu \bar{\mathcal{B}}^{\nu\sigma} + \partial^\nu \bar{\mathcal{B}}^{\sigma\mu} + \partial^\sigma \bar{\mathcal{B}}^{\mu\nu}\big) - \varepsilon^{\mu\nu\sigma\rho}\,\partial_\rho B = 0, \tag{37}$$

which are found to be very useful in the proof of the conservation law [i.e. $\partial_\mu J^\mu_{(ad)}(ng) = 0$]. It is worthwhile to mention that the *latter* (i.e. bottom) entry in the *above* equation is nothing but the *two* different ways of writing the EL-EoMs w.r.t the Abelian 1-form vector field A_μ that is derived from non-ghost sector of the Lagrangian density $\mathcal{L}_{(ng)}$.

At this juncture, we would like to point out that the *total* anti-co-BRST current $J^\mu_{(ad)} = J^\mu_{(ad)}(ng) + J^\mu_{(ad)}(fp)$ is the sum of the currents in equations (35) and (33). It is crystal clear that this total current is *also* conserved whose explicit expression is as follows:

$$\begin{aligned} J^\mu_{(ad)} &= -\frac{1}{2}\Big[\big(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}\big)\,\bar{\mathcal{B}}_{\nu\sigma} - B_3\,F^\mu - \bar{\mathcal{B}}^{\mu\nu}\,F_\nu \\ &+ \varepsilon^{\mu\nu\sigma\rho}\,(\partial_\nu C)\,\bar B_{\sigma\rho} + \varepsilon^{\mu\nu\sigma\rho}\,B\,(\partial_\nu C_{\sigma\rho}) + B_4\,\bar f^\mu + B_5\,\partial^\mu C_2 \\ &+ \big(\partial^\mu \bar C^{\nu\sigma} + \partial^\nu \bar C^{\sigma\mu} + \partial^\sigma \bar C^{\mu\nu}\big)\,\big(\partial_\nu \beta_\sigma - \partial_\sigma \beta_\nu\big) \\ &- \big(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu\big)\,\bar f_\nu + \big(\partial^\mu \bar\beta^\nu - \partial^\nu \bar\beta^\mu\big)\,\partial_\nu C_2\Big] + B_1\,\partial^\mu C. \end{aligned} \tag{38}$$

From the above conserved current, it is straightforward to compute the expression for the conserved Noether anti-co-BRST charge (i.e. $Q_{ad} = \int d^3x \, J^0_{(ad)}$) as follows:

$$
\begin{aligned}
Q_{ad} \;&=\; \int d^3x \Big[B_1\,\dot{C} - \frac{1}{2}\Big\{(\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\,\bar{\mathcal{B}}_{ij} - B_3\,F^0 - \bar{\mathcal{B}}^{0i}\,F_i \\
&+\; \varepsilon^{0ijk}\,(\partial_i C)\,\bar{B}_{jk} + \varepsilon^{0ijk}\,B\,(\partial_i C_{jk}) + B_4\,\bar{f}^0 + B_5\,\dot{C}_2 \\
&+\; (\partial^0\,\bar{C}^{ij} + \partial^i\,\bar{C}^{j0} + \partial^j\,\bar{C}^{0i})\,(\partial_i\beta_j - \partial_j\beta_i) \\
&-\; (\partial^0\,\beta^i - \partial^i\,\beta^0)\,\bar{f}_i + (\partial^0\,\bar{\beta}^i - \partial^i\,\bar{\beta}^0)\,\partial_i C_2\Big\}\Big].
\end{aligned}
\tag{39}
$$

The above conserved Noether anti-co-BRST charge is the generator for the infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-co-BRST symmetry transformations s_{ad} [cf. Eq. (30)] which can be verified in a straightforward manner by taking into account equation (9) with the following replacements: $s_d \to s_{ad},\; Q_d \to Q_{ad}$.

We end this section with the following clinching remarks. First of all, we find that using the basic tenets and techniques of Noether's theorem leads to the derivation of the conserved Noether anti-co-BRST charge Q_{ad} [cf. Eq. (39)] from the Noether conserved anti-co-BRST current $J^\mu_{(ad)}$. Second, we observe that the Noether current $J^\mu_{(ad)}(ng)$ [cf. Eq. (38)], that is derived from the non-ghost sector of the Lagrangian density $\mathcal{L}_{(ng)}$ [cf. Eq. (28)], is found to be conserved [i.e. $\partial_\mu J^\mu_{(ad)}(ng) = 0$] provided we use the EL-EoMs [cf. Eq. (36)] that are derived from the Lagrangian density $\mathcal{L}_{(ng)}$. Third, we note that the the sum of the Noether current (that is derived from the ghost sector of the Lagrangian density $\mathcal{L}_{(fp)}$ [cf. Eq. (33)]) and the rest of the terms (ROTs) [that have been taken into account in equation (32) due to Noether's theorem because of our observation in (31)] also turn out to be conserved [i.e. $\partial_\mu J^\mu_{(ad)}(fp) = 0$] provided we use the EL-EoMs [cf. Eq. (34)] that have been derived from the Lagrangian density $\mathcal{L}_{(fp)}$. Fourth, it is interesting to point out that the ROTS in equation (32) are *not* required for the proof of the conservation law $\partial_\mu J^\mu_{(ad)}(ng) = 0$ (for the Noether current $J^\mu_{(ad)}(ng)$ that emerges out from the non-ghost sector of Lagrangian density $\mathcal{L}_{(ng)}$). Finally, we find that the conserved Noether anti-co-BRST charge Q_{ad} is *not* nilpotent of order two (i.e. $Q_{ad}^2 \neq 0$). This can be verified by using the standard relationship between the continuous symmetry transformations and their generator as the Noether conserved charge. For instance, we note the following relationship, namely;

$$
\begin{aligned}
s_{ad}\,Q_{ad} = i\,\{Q_{ad},\; Q_{ad}\} \;&\equiv\; +\frac{1}{2}\int d^3x \Big[(\partial^0\bar{\mathcal{B}}^{ij} + \partial^i\bar{\mathcal{B}}^{jk} + \partial^j\bar{\mathcal{B}}^{0i})\,(\partial_i\beta_j - \partial_j\beta_i) \\
&+\; (\partial^0\bar{f}^i - \partial^i\bar{f}^0)\,\partial_i C_2\Big] \neq 0,
\end{aligned}
\tag{40}
$$

which, ultimately, implies: $i\,\{Q_{ad},\; Q_{ad}\} = 2\,i\,Q_{ad}^2 \neq 0$. In other words, we find that the explicit computation of the quantity $s_{ad}Q_{ad}$ [by using equations (30) and (39)] leads to our conclusion that the Noether anti-co-BRST charge Q_{ad} is *not* nilpotent (i.e. $Q_{ad}^2 \neq 0$) of order two. As a consequence, this conserved (but non-nilpotent) Noether charge is *not* suitable for the discussion on the cohomological aspects of BRST formalism (see, e.g. [23] for explicit examples) and the physicality criterion w.r.t. *it* in the BRST-quantized version of the quantum Hilbert space of states (cf. Sec. 7 below for details).

7. Nilpotent and Conserved Anti-Co-BRST Charge from Non-nilpotent Noether Anti-co-BRST Charge

The key purpose of our present section is to derive explicitly the conserved and off-shell nilpotent (i.e. $Q_{AD}^2 = 0$) version of the anti-co-BRST charge Q_{AD} from the non-nilpotent (i.e. $Q_{ad}^2 \neq 0$) version [cf. Eq. (39)] of the Noether conserved anti-co-BRST charge Q_{ad} [cf. Eq. (40)]. In this context, we shall exploit the theoretical techniques that have been proposed in our earlier work [21] where the emphasis have been laid on the use of (i) the appropriate EL-EoMs which are derived from the perfectly anti-co-BRST invariant Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$, (ii) the partial integration followed

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by the application of the celebrated Gauss divergence theorem, and (iii) the infinitesimal, continuous and off-shell nilpotent anti-co-BRST symmetry transformations s_{ad} [cf. Eq. (30)] at appropriate places, to ensure that we finally obtain: $s_{ad} Q_{AD} = i\,\{Q_{AD},\, Q_{AD}\} = 0 \;\Rightarrow\; Q^2_{AD} = 0$.

In view of our elaborate discussion on the derivation of the conserved and nilpotent version of the co-BRST charge Q_D from the conserved (but non-nilpotent) version of the Noether co-BRST charge Q_d, we shall be quite *brief* in our present section as far as the derivation of the nilpotent version of the anti-co-BRST charge Q_{AD} from its counterpart non-nilpotent version [cf. Eq. (40)] of the Noether anti-co-BRST charge Q_{ad} is concerned. In this context, let us focus, first of all, on the *fifth* and *sixth* terms of the conserved (but non-nilpotent) anti-co-BRST charge Q_{ad} [cf. Eq. (39)]. Application of the partial integration followed by the Gauss divergence theorem leads to the following

$$\frac{1}{2}\int d^3x\, \varepsilon^{0ijk}\Big[(\partial_i C)\,\bar{B}_{jk} + B\,(\partial_i C_{jk})\Big] = -\frac{1}{2}\int d^3x\, \varepsilon^{0ijk}\Big[C\,(\partial_i \bar{B}_{jk}) + (\partial_i B)\,C_{jk}\Big], \tag{41}$$

where we have dropped the total space derivative terms for obvious reasons. Using the following EL-EoMs (that have been derived from $\mathcal{L}_{(ng)}$ [cf. Eq. (28)]), namely;

$$\begin{aligned} &\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \bar{B}_{\sigma\rho} = -\partial^\mu B_1 \Longrightarrow \frac{1}{2}\,\varepsilon^{oijk}\,\partial_i \bar{B}_{jk} = -\dot{B}_1, \\ &\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu \bar{\mathcal{B}}_{\sigma\rho} = \partial^\mu B = 0 \Longleftrightarrow \left(\partial^\mu \bar{\mathcal{B}}^{\nu\sigma} + \partial^\nu \bar{\mathcal{B}}^{\sigma\mu} + \partial^\sigma \bar{\mathcal{B}}^{\mu\nu}\right) = \varepsilon^{\mu\nu\sigma\rho}\,\partial_\rho B, \end{aligned} \tag{42}$$

we find the following form of equation (41), namely;

$$-\int d^3x\, \dot{B}_1\, C + \frac{1}{2}\int d^3x\, \left(\partial^0 \bar{\mathcal{B}}^{ij} + \partial^i \bar{\mathcal{B}}^{j0} + \partial^j \bar{\mathcal{B}}^{i0}\right) C_{ij}, \tag{43}$$

where we have used: $\varepsilon^{0ijk}\,\partial_k B = (\partial^0 \bar{\mathcal{B}}^{ij} + \partial^i \bar{\mathcal{B}}^{jk} + \partial^j \bar{\mathcal{B}}^{i0})$. Both the above inetgrals will be present in the nilpotent version of the anti-co-BRST charge Q_{AD}. The first *integrals* of (i) the Noether non-nilpotent charge Q_{ad} [cf. Eq. (39)], and (ii) the above equation can be combined together to obtain the analogue of the expression (13) (cf. Sec. 4) as follows

$$Q^{(1)}_{AD} = \int d^3x\, \left[B_1\, \dot{C} - \dot{B}_1\, C\right], \tag{44}$$

which is an anti-co-BRST invariant quantity (i.e. $s_{ad} Q^{(1)}_{AD} = 0$). The superscript (1) on $Q^{(1)}_{AD}$ denotes the fact that the auxiliary field B_1 is present in this expression. Toward our main goal of obtaining $s_{ad} Q_{AD} = 0$, we have to apply the anti-co-BRST symmetry operator s_{ad} on the *second* integral of equation (43) which leads to the following integral:

$$+\frac{1}{2}\int d^3x\, \left(\partial^0 \bar{\mathcal{B}}^{ij} + \partial^i \bar{\mathcal{B}}^{j0} + \partial^j \bar{\mathcal{B}}^{i0}\right)\left(\partial_i \beta_j - \partial_j \beta_i\right). \tag{45}$$

We have to modify an appropriate integral in the expression for Q_{ad} [cf. Eq. (39)] so that when the transformations s_{ad} act on a part of *that* term, the outcome should cancel out with our result in (45). Such an integral is: $-\frac{1}{2}\int d^3x\, (\partial^0\, \bar{C}^{ij} + \partial^i\, \bar{C}^{j0} + \partial^j\, \bar{C}^{0i})\,(\partial_i \beta_j - \partial_j \beta_i)$ which can be modified in the following form:

$$\begin{aligned} &+\frac{1}{2}\int d^3x\, \left(\partial^0\, \bar{C}^{ij} + \partial^i\, \bar{C}^{j0} + \partial^j\, \bar{C}^{0i}\right)\left(\partial_i \beta_j - \partial_j \beta_i\right) \\ &-\int d^3x\, \left(\partial^0\, \bar{C}^{ij} + \partial^i\, \bar{C}^{j0} + \partial^j\, \bar{C}^{0i}\right)\left(\partial_i \beta_j - \partial_j \beta_i\right). \end{aligned} \tag{46}$$

It is obvious that if we apply s_{ad} [cf. Eq. (30)] on the *first* integral of the above equation, the outcome will cancel out with our result in (45). Hence, the *first* integral will be present in the nilpotent version

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of the anti-co-BRST charge Q_{AD}. Toward ou main goal of obtaining $s_{ad}Q_{AD} = 0$, we now concentrate on the *second* integral of (46) which is equivalent to

$$-2\int d^3x\,(\partial^0\,\bar{C}^{ij} + \partial^i\,\bar{C}^{j0} + \partial^j\,\bar{C}^{0i})\,(\partial_i\beta_j) \equiv +2\int d^3x\,\partial_i(\partial^0\,\bar{C}^{ij} + \partial^i\,\bar{C}^{j0} + \partial^j\,\bar{C}^{0i})\,\beta_j, \tag{47}$$

where the r.h.s. of the above equation has been obtained after the applications of (i) the partial integration, and (ii) the Gauss divergence theorem. Using the EL-EoM: $\partial_\mu(\partial^\mu\bar{C}^{\nu\sigma} + \partial^\nu\bar{C}^{\sigma\mu} + \partial^\sigma\bar{C}^{\mu\nu}) = \frac{1}{2}\,(\partial^\nu\bar{f}^\sigma - \partial^\sigma\bar{f}^\nu)$ (with the choices: $\nu = 0,\ \sigma = j$), it can be checked that the r.h.s. of equation (47) can be re-written as

$$+2\int d^3x\,\partial_i(\partial^0\,\bar{C}^{ij} + \partial^i\,\bar{C}^{j0} + \partial^j\,\bar{C}^{0i})\,\beta_j = -\int d^3x\,(\partial^0\bar{f}^i - \partial^0\bar{f}^i)\,\beta_i, \tag{48}$$

which will be present in the *final* expression of the nilpotent Q_{AD}. Toward our central goal of achieving $s_{ad}Q_{AD} = 0$, we have to apply s_{ad} on the r.h.s. of (48) which yields:

$$-\,s_{ad}\Big[\int d^3x\,(\partial^0\bar{f}^i - \partial^i\bar{f}^0)\,\beta_i\Big] = \int d^3x\,(\partial^0\bar{f}^i - \partial^0\bar{f}^i)\,\partial_i C_2, \tag{49}$$

where we have taken into account the fermionic (i.e. $s_{ad}\,\bar{f}_\mu + \bar{f}_\mu\,s_{ad} = 0$) natures of s_{ad} and $\bar{f}_\mu$. To cancel out the above result, an appropriate integral in the expression for Q_{ad} [cf. Eq. (39)] has to be modified. Such a term is: $-\frac{1}{2}\int d^3x\,(\partial^0\bar{\beta}^i - \partial^i\bar{\beta}^0)\,\partial_i C_2$ which can be re-written in the following form:

$$\int d^3x\;(\partial^0\bar{\beta}^i - \partial^i\bar{\beta}^0)\,\partial_i C_2 - \frac{3}{2}\int d^3x\;(\partial^0\bar{\beta}^i - \partial^i\bar{\beta}^0)\,\partial_i C_2. \tag{50}$$

It is clear that the application of s_{ad} [cf. Eq. (30)] on the *first* integral would yield the result that would cancel out with the r.h.s. of (49). Hence the *first* integral of (50) will be present in the final nilpotent version of Q_{AD}. At this stage, we focus on the *second* integral of equation (50) and apply (i) the partial integration, and (ii) the Gauss divergence theorem which, taken together, lead to the following result:

$$+\,\frac{3}{2}\int d^3x\,\partial_i(\partial^0\bar{\beta}^i - \partial^i\bar{\beta}^0)\,C_2. \tag{51}$$

Application of the EL-EoM: $\partial_\mu(\partial^\mu\bar{\beta}^\nu - \partial^\nu\bar{\beta}^\mu) = -\,\partial^\nu B_5$ (with the choice $\nu = 0$) leads to the above integral to take the following explicit form

$$+\,\frac{3}{2}\int d^3x\,\dot{B}_5\,C_2, \tag{52}$$

which turns out to be an anti-co-BRST invariant quality [i.e. $s_{ad}(\dot{B}_5\,C_2) = 0$]. Collecting (i) all the useful and relevant terms of our exercise, and (ii) the anti-co-BRST invariant integrands of Q_{ad} [cf. Eq. (39)], we obtain the off-shell nilpotent (i.e. $Q_{AD}^2 = 0$) version of the conserved anti-co-BRST charge Q_{AD} as follows:

$$\begin{aligned}
Q_{AD} \;&=\; \int d^3x\,\Big[(B_1\,\dot{C} - \dot{B}_1\,C) + (\partial^0\bar{\beta}^i - \partial^i\bar{\beta}^0)\,\partial_i C_2 - (\partial^0\bar{f}^i - \partial^i\bar{f}^0)\,\beta_i \\
&-\; \frac{1}{2}\Big\{(\partial^0\,C^{ij} + \partial^i\,C^{j0} + \partial^j\,C^{0i})\,\bar{\mathcal{B}}_{ij} - B_3\,F^0 - \bar{\mathcal{B}}^{0i}\,F_i + B_4\,\bar{f}^0 \\
&-\; (\partial^0\,\mathcal{B}^{ij} + \partial^i\,\bar{\mathcal{B}}^{j0} + \partial^j\,\bar{\mathcal{B}}^{0i})\,C_{ij} - (\partial^0\,\bar{C}^{ij} + \partial^i\,\bar{C}^{j0} + \partial^j\,\bar{C}^{0i})\,(\partial_i\beta_j - \partial_j\beta_i) \\
&-\; (\partial^0\,\beta^i - \partial^i\,\beta^0)\,\bar{f}_i + B_5\,\dot{C}_2 - 3\,\dot{B}_5\,C_2\Big\}\Big].
\end{aligned} \tag{53}$$

It is an elementary exercise to check that $s_{ad}Q_{AD} = i\,\{Q_{AD},\,Q_{AD}\} = 0\ \Rightarrow\ Q_{AD}^2 = 0$ by computing the l.h.s (i.e. $s_{ad}Q_{AD}$) by the direct application of the co-BRST symmetry transformation operator s_{ad} [cf. Eq. (30)] on the above explicit expression for Q_{AD}. We would like to lay emphasis on the fact that the non-nilpotent Noether charge Q_{ad} and the nilpotent version of the anti-co-BRST charge Q_{AD}

both are conserved quantities because we have used *only* the appropriate EL-EoMs and the partial integration followed by the Gauss divergence theorem which are consistent with the basic tenets of the Noether theorem.

We would like to dwell a bit on the physicality criterion w.r.t. the above nilpotent version of the anti-co-BRST charge Q_{AD} and establish that the conditions on the physical states (i.e. $|phys>$) are *dual* to the conditions [cf. Eq. (A.12)] that have been obtained on the physical states by the nilpotent version of the anti-BRST charge Q_{AB} (cf. Appendix A). A careful look at the Lagrangian densities ((28) and (29) show that the basic/auxiliary (anti-)ghost fields are decoupled from the basic/auxiliary physical fields of our theory. As a consequence, a quantum state in the Hilbert space of states is the direct product [24] of the physical states (i.e. $|phys>$) and the ghost states (i.e. $|ghost>$). Before pointing out the consequences coming out from the physicality criterion: $Q_{AD}\,|phys> 0$, we would like to lay emphasis on the fact that every individual term in the integrand of Q_{AD} [cf. Eq.(53)] (i) carry the ghost number equal to $+1$, (ii) made up of the physical fields (with ghost number equal to zero) and the basic/auxiliary (anti-)ghost fields (carrying the ghost number equal to $+1$), and (iii) constituted by only the basic/auxiliary (anti-)ghost fields. When we apply the anti-co-BRST charge operator acts on a quantum state, the ghost fields act on the ghost states (i.e. $|ghost>$) leading to the non-zero result. Thus, to satisfy the sacrosanct requirement that the physical states are *those* that are annihilated the conserved and nilpotent version of the anti-co-BRST charge Q_{AD}, it is clear that *only* the physical fields (i) carrying the ghost number equal to zero, and (ii) associated with the basic ghost fields (carrying the ghost number equal to $+1$) would act on the *true* physical states[5]. Thus, the requirement: $Q_{AD}\,|phys> 0$ leads to the following conditions on the physical states (i.e. the *harmonic* states in the Hodge decomposed quantum states)

$$
\begin{aligned}
& B_1\,|phys>=0, \qquad \dot{B}_1\,|phys>=0, \qquad \frac{1}{2}\,\bar{\mathcal{B}}_{ij}\,|phys>=0,\\
& \frac{1}{2}\,(\partial^0\bar{\mathcal{B}}^{ij}+\partial^i\bar{\mathcal{B}}^{j0}+\partial^j\bar{\mathcal{B}}^{0i})\,|phys>=0,
\end{aligned}
\tag{54}
$$

which are *dual* to what we have obtained from the physicality criterion: $Q_{AB}\,|phys> 0$ w.r.t. the conserved and nilpotent version of the anti-BRST charge Q_{AB} (see Appendix A for details). In other words, our results (i.e. $B\,|phys>=0$, $\dot{B}\,|phys>=0$, $\frac{1}{2}\,\bar{B}_{ij}\,|phys>=0$ and $\frac{1}{2}\,(\partial^0\bar{B}^{ij}+\partial^i B^{j0}+\partial^j\bar{B}^{0i})\,|phys>=0$) that have been listed in (A.12) due to the physicality criterion: $Q_{AB}\,|phys> 0$ w.r.t. the conserved and nilpotent version of the anti-BRST charge Q_{AB} are deeply connected with conditions in equation (54). To corroborate and substantiate this claim, we would like to point out that the non-ghost sector of the Lagrangian density $\mathcal{L}_{(ng)}$ [cf. Eq. (28)] respects (i.e. $\mathcal{L}_{(ng)}\to\mathcal{L}_{(ng)}$) the following discrete *duality* symmetry transformations:

$$
\begin{aligned}
& A_\mu \longrightarrow \mp\,\frac{1}{3!}\,\varepsilon_{\mu\nu\sigma\rho}\,A^{\nu\sigma\rho}, \quad A_{\mu\nu\sigma}\longrightarrow \pm\,\varepsilon_{\mu\nu\sigma\rho}\,A^{\rho}, \quad \bar{B}_{\mu\nu}\to\mp\,\bar{\mathcal{B}}_{\mu\nu}, \quad \bar{\mathcal{B}}_{\mu\nu}\to\pm\,\bar{B}_{\mu\nu},\\
& B\to\pm\,B_1, \quad B_1\to\mp\,B, \quad B_2\to\pm\,B_3, \quad B_3\to\mp\,B_2,\ \phi_\mu\to\pm\,\tilde{\phi}_\mu,\ \tilde{\phi}_\mu\to\mp\,\phi_\mu.
\end{aligned}
\tag{55}
$$

Thus, we note that if we choose the *harmonic* states (in the Hodge decomposed quantum states in the Hilbert space) to be the *true* physical states (i.e. $|phys>$), the conditions that emerge out from the conserved and nilpotent versions of the (anti-)BRST Q_{AB} and (anti-)co-BRST Q_{AD} charges are found to be deeply connected with one-another due to the presence of the discrete duality symmetry transformations in equation (55).

We conclude this section with the *final* remark that the conserved and off-shell nilpotent versions of the (anti-)BRST and (anti-)co-BRST charges lead to the annihilation of the physical states by the

---

[5] It is worthwhile to point out that the auxiliary fields $\bar{\mathcal{B}}^{0i}$ and B_3 (that are present in the expression for Q_{AD} and even though carry the ghost number equal to zero) do *not* lead to any conditions on the physical states (i.e. $|phys>$) because they are associated with the specific components of the fermionic *auxiliary* vector field $F_\mu=-\frac{1}{2}\,(\partial^\rho C_{\rho\mu}-\partial_\mu C_1)$. The *latter* relationship has been derived from the FP-ghost sector of the Lagrangian density (29) which proves that the auxiliary field F_μ is *not* the *basic* ghost field.

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operator forms of the (i) the first-class constraints (see, e.g. [20] and our Appendix A), and (ii) the *dual* versions of the first-class constraints (see, e.g. Secs. 4 and 7 for details), respectively, provided we choose the physical states to be the *harmonic* state in a given Hodge decomposed quantum state. The *latter* state, as is self-evident, is chosen from the *total* quantum Hilbert space of states of our BRST-quantized field-theoretic system. Our present crucial and decisive statements, connected with the emergence of the type of constraints from the physicality criteria, are *true* only for the field-theoretic systems which happen to be (i) the gauge theories at the *classical* level, and (ii) the examples for Hodge theory at the *quantum* level within the framework of BRST formalism.

8. Curci-Ferrari Type Restrictions: Derivations

In our present section, we derive the *non-trivial* (anti-)co-BRST invariant Curci-Ferrari (CF) type restrictions from two different theoretical angles. Let us, first of all, focus on the symmetry considerations where we demand that the *coupled* Lagrangian densities must respect the nilpotent co-BRST as well as the anti-co-BRST symmetry transformations so that their *equivalence* could be established. In this context, we would like to point out that the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)] transforms to the total spacetime derivative [cf. Eq. (4)] under the off-shell nilpotent co-BRST symmetry transformations (3) thereby rendering the action integral (corresponding to this Lagrangian density) invariant under the co-BRST symmetry transformations (3). We *also* demand, at this stage, the symmetry invariance of the *coupled* Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ [cf. Eqs. (28),(29)] under the co-BRST symmetry transformations (3). This requirement, as stated earlier, leads to the derivation of the CF-type restrictions: $\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = (\partial_\mu \tilde{\phi}_\nu - \partial_\nu \tilde{\phi}_\mu)$, $f_\mu + F_\mu = \partial_\mu C_1$, $\bar{f}_\mu + \bar{F}_\mu = \partial_\mu \bar{C}_1$. To corroborate this statement, as a first step, we note that the non-ghost part (i.e. $\mathcal{L}_{(ng)}$) of the coupled Lagrangian density $\mathcal{L}_{(\bar{B})}$ transforms, under the off-shell nilpotent co-BRST symmetry transformations (3), as follows:

$$
\begin{aligned}
s_d\, \mathcal{L}_{(ng)} &= -\,\partial_\mu \Big[\varepsilon^{\mu\nu\sigma\rho}\, \bar{F}_\nu\, (\partial_\sigma A_\rho) + \frac{1}{2}\, (\partial^\mu\, \bar{C}^{\nu\sigma} + \partial^\nu\, \bar{C}^{\sigma\mu} + \partial^\sigma\, \bar{C}^{\mu\nu})\, \mathcal{B}_{\nu\sigma} - \frac{1}{2}\, B_3\, \bar{F}^\mu\Big] \\
&+ B_1\, \Box \bar{C} + \frac{1}{2}\Big[(\partial^\mu\, \bar{C}^{\nu\sigma} + \partial^\nu\, \bar{C}^{\sigma\mu} + \partial^\sigma\, \bar{C}^{\mu\nu})\, \partial_\mu \bar{\mathcal{B}}_{\nu\sigma} - \bar{\mathcal{B}}^{\mu\nu}\, (\partial_\mu \bar{F}_\nu) \\
&+ (\partial^\mu \tilde{\phi}^\nu - \partial^\nu \tilde{\phi}^\mu)\, (\partial_\mu \bar{F}_\nu) - \bar{F}^\mu\, \partial_\mu B_3\Big]. \qquad (56)
\end{aligned}
$$

In the next step, we apply the co-BRST symmetry transformations (3) on the FP-ghost part (i.e. $\mathcal{L}_{(fp)}$) of the Lagrangian density $\mathcal{L}_{(\bar{B})}$ which leads to the following:

$$
\begin{aligned}
s_d\, \mathcal{L}_{(fp)} &= \frac{1}{2}\Big[(\partial^\mu\, \bar{C}^{\nu\sigma} + \partial^\nu\, \bar{C}^{\sigma\mu} + \partial^\sigma\, \bar{C}^{\mu\nu})\, \partial_\mu \mathcal{B}_{\nu\sigma} + \mathcal{B}^{\mu\nu}\, (\partial_\mu \bar{f}_\nu) + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)\, (\partial_\mu F_\nu) \\
&+ (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)\, (\partial_\mu f_\nu) + (\partial^\mu \bar{C}_1 - \bar{f}^\mu)\, \partial_\mu B_3 - (f^\mu + F^\mu - \partial^\mu C_1)\, \partial_\mu B_5\Big] \\
&+ \partial_\mu B_1\, \partial^\mu \bar{C} - \frac{1}{2}\, \partial_\mu \Big[\bar{C}^{\mu\nu}\, \partial_\nu B_3 + C^{\mu\nu}\, \partial_\nu B_5 + (\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu)\, F_\nu \\
&+ \mathcal{B}^{\mu\nu}\, \bar{f}_\nu - B_5\, f^\mu - B_4\, \partial^\mu \bar{C}_2\Big]. \qquad (57)
\end{aligned}
$$

To observe the full consequence of the application of the co-BRST symmetry transformations (3) on the *total* coupled Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$, we have to add the results that have been obtained in (56) and (57) which leads to the following:

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$$
\begin{aligned}
s_d\,\mathcal{L}_{(\bar B)} &= \partial_\mu\Big[B_1\partial^\mu \bar C - \varepsilon^{\mu\nu\sigma\rho}\,\bar F_\nu\,(\partial_\sigma A_\rho)\Big] - \frac{1}{2}\,\partial_\mu\,\Big[\bar C^{\mu\nu}\,\partial_\nu B_3 + C^{\mu\nu}\,\partial_\nu B_5 + \mathcal{B}^{\mu\nu}\,\bar f_\nu \\
&+ (\partial^\mu\bar\beta^\nu - \partial^\nu\bar\beta^\mu)\,F_\nu - B_3\,\bar F^\mu - B_5\,f^\mu + (\partial^\mu\,\bar C^{\nu\sigma} + \partial^\nu\,\bar C^{\sigma\mu} + \partial^\sigma\,\bar C^{\mu\nu})\,\mathcal{B}_{\nu\sigma} - B_4\,\partial^\mu\bar C_2\Big] \\
&+ \frac{1}{2}\,\Big\{\mathcal{B}^{\mu\nu}\,\partial_\mu\,(\bar f_\nu + \bar F_\nu - \partial_\nu\bar C_1) - (\bar f^\mu + \bar F^\mu - \partial^\mu\bar C_1)\,\partial_\mu B_3 - (f^\mu + F^\mu - \partial^\mu\bar C_1)\,\partial_\mu B_5 \\
&- \big[\mathcal{B}^{\mu\nu} + \bar{\mathcal{B}}^{\mu\nu} - (\partial^\mu\tilde\phi^\nu - \partial^\nu\tilde\phi^\mu)\big]\,(\partial_\mu\bar F_\nu) + (\partial^\mu\bar\beta^\nu - \partial^\nu\bar\beta^\mu)\,\partial_\mu\big[f_\nu + F_\nu - \partial_\nu C_1\big] \\
&+ (\partial^\mu\,\bar C^{\nu\sigma} + \partial^\nu\,\bar C^{\sigma\mu} + \partial^\sigma\,\bar C^{\mu\nu})\,\partial_\mu\big[\mathcal{B}_{\nu\sigma} + \bar{\mathcal{B}}_{\nu\sigma} - (\partial_\nu\tilde\phi_\sigma - \partial_\sigma\tilde\phi_\nu)\big]\Big\}. \qquad (58)
\end{aligned}
$$

Thus, we note that the application of the nilpotent co-BRST symmetry transformation operator s_d [cf. Eq. (3)] on the *coupled* Lagrangian density $\mathcal{L}_{(\bar B)}$ leads to the sum of (i) the total spacetime derivative terms, and (ii) the terms that contain the CF-type restrictions: $\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = (\partial_\mu\tilde\phi_\nu - \partial_\nu\tilde\phi_\mu),\ f_\mu + F_\mu = \partial_\mu C_1,\ \bar f_\mu + \bar F_\mu = \partial_\mu\bar C_1$. In other words, the requirement of the co-BRST invariance of the *perfectly* anti-co-BRST invariant *coupled* Lagrangian density $\mathcal{L}_{(\bar B)}$ [cf. Eq. (31)] leads to the derivations of the (anti-)co-BRST and (anti-)BRST invariant CF-type restrictions. To be more precise, the *coupled* Lagrangian density $\mathcal{L}_{(\bar B)}$ respects *both* the off-shell nilpotent (i.e. co-BRST and anti-co-BRST) symmetry transformations provided the whole 4D BRST-quantized field-theoretic system is considered on the submanifold of the quantum fields where the above *three* CF-type restrictions[6] are satisfied.

Against the backdrop of the above discussions, we would like to point out that the action integral, corresponding to the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)], remains invariant under the infinitesimal, continuous and off-shell nilpotent (i.e. $s_d^2 = 0$) co-BRST symmetry transformations s_d [cf. Eqs. (3),(4)]. We demand, at this juncture, the invariance of *this* Lagrangian density $\mathcal{L}_{(B)}$ under the infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-co-BRST symmetry transformations s_{ad} [cf. Eq. (30)], too. Under this requirement, as stated earlier, we envisage to derive of the non-trivial CF-type restrictions (i.e. $\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = (\partial_\mu\tilde\phi_\nu - \partial_\nu\tilde\phi_\mu),\ f_\mu + F_\mu = \partial_\mu C_1,\ \bar f_\mu + \bar F_\mu = \partial_\mu\bar C_1$). Toward this goal in mind, first of all, we note that the non-ghost part (i.e. $\mathcal{L}_{(NG)}$) of the Lagrangian density $\mathcal{L}_{(B)}$ transforms, under the infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-co-BRST symmetry transformations s_{ad}, as:

$$
\begin{aligned}
s_{ad}\,\mathcal{L}_{(NG)} \;&=\; \partial_\mu\,\Big[\varepsilon^{\mu\nu\sigma\rho}\,F_\nu\,(\partial_\sigma A_\rho) + \frac{1}{2}\,(\partial^\mu\,C^{\nu\sigma} + \partial^\nu\,C^{\sigma\mu} + \partial^\sigma\,C^{\mu\nu})\,\mathcal{B}_{\nu\sigma} + \frac{1}{2}\,B_3\,F^\mu\Big] \\
&-\; \frac{1}{2}\,\Big[(\partial^\mu\,C^{\nu\sigma} + \partial^\nu\,C^{\sigma\mu} + \partial^\sigma\,C^{\mu\nu})\,\partial_\mu\mathcal{B}_{\nu\sigma} + B^{\mu\nu}\,(\partial_\mu F_\nu) + F^\mu\,\partial_\mu B_3 \\
&-\; (\partial^\mu\tilde\phi^\nu - \partial^\nu\tilde\phi^\mu)\,(\partial_\mu F_\nu)\Big] + B_1\,\Box C. \qquad (59)
\end{aligned}
$$

On the other hand, we note that the FP-ghost part (i.e. $\mathcal{L}_{(FP)}$) of the Lagrangian density $\mathcal{L}_{(B)}$ transforms, under the infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ad}^2 = 0$) anti-co-BRST symmetry transformations s_{ad} [cf. Eq. (30)], as:

$$
\begin{aligned}
s_{ad}\,\mathcal{L}_{(FP)} &= \frac{1}{2}\,\Big[\bar{\mathcal{B}}^{\mu\nu}\,(\partial_\mu f_\nu) - (\partial^\mu\,C^{\nu\sigma} + \partial^\nu\,C^{\sigma\mu} + \partial^\sigma\,C^{\mu\nu})\,\partial_\mu\bar{\mathcal{B}}_{\nu\sigma} + +(\partial^\mu C_1 - f^\mu)\,\partial_\mu B_3 \\
&+ (\partial^\mu\,\beta^\nu - \partial^\nu\beta^\mu)\,\partial_\mu[\bar f_\nu + \bar F_\nu - \partial_\nu\bar C_1] + (\bar f^\mu + \bar F^\mu - \partial^\mu\bar C_1)\,\partial_\mu B_4\Big] + \partial_\mu B_1\,\partial^\mu C \\
&- \frac{1}{2}\,\partial_\mu\,\Big[\bar{\mathcal{B}}^{\mu\nu}\,f_\nu - \bar C^{\mu\nu}\,\partial_\nu B_4 - C^{\mu\nu}\,\partial_\nu B_3 + (\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)\,\bar F_\nu + B_4\,\bar f^\mu + B_5\,\partial^\mu C_2\Big]. \qquad (60)
\end{aligned}
$$

[6] It should be noted that there is another CF-type restriction: $B_{\mu\nu} + \bar B_{\mu\nu} = (\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)$ on our 4D BRST-quantized field-theoretic system [20]. However, this specific CF-type restriction is useful in the proof of the absolute anticommutativity between the BRST and anti-BRST symmetry transformation operators. We lay emphasis on the fact that *this* restriction does *not* play any role in the context of our present discussions on the off-shell nilpotent (anti-)co-BRST symmetry transformation operators ($s_{(a)d}$).

To obtain the full outcome of the application of the anti-co-BRST symmetry transformation operator on the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)], we have to take the sum of our observations in equations (59) and (60) which leads to the following:

$$\begin{aligned}
s_{ad}\,\mathcal{L}_{(B)} &= \partial_\mu\Big[B_1\partial^\mu C + \varepsilon^{\mu\nu\sigma\rho}\,F_\nu\,(\partial_\sigma A_\rho)\Big] + \frac{1}{2}\,\partial_\mu\,\Big[C^{\mu\nu}\,\partial_\nu B_3 + \bar{C}^{\mu\nu}\,\partial_\nu B_4 - \bar{\mathcal{B}}^{\mu\nu}\,f_\nu \\
&- (\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)\,\bar{F}_\nu + B_3\,F^\mu - B_4\,\bar{f}^\mu + (\partial^\mu\,C^{\nu\sigma} + \partial^\nu\,C^{\sigma\mu} + \partial^\sigma\,C^{\mu\nu})\,B_{\nu\sigma} - B_5\,\partial^\mu C_2\Big] \\
&+ \frac{1}{2}\Big\{\bar{\mathcal{B}}^{\mu\nu}\,\partial_\mu(f_\nu + F_\nu - \partial_\nu C_1) - (f^\mu + F^\mu - \partial^\mu C_1)\,\partial_\mu B_3 + (\bar{f}^\mu + \bar{F}^\mu - \partial^\mu\bar{C}_1)\,\partial_\mu B_4 \\
&- \big[\mathcal{B}^{\mu\nu} + \bar{\mathcal{B}}^{\mu\nu} - (\partial^\mu\tilde{\phi}^\nu - \partial^\nu\tilde{\phi}^\mu\big]\,(\partial_\mu F_\nu) + (\partial^\mu\beta^\nu - \partial^\nu\beta^\mu)\,\partial_\mu\big[\bar{f}_\nu + \bar{F}_\nu - \partial_\nu\bar{C}_1\big] \\
&- (\partial^\mu\,C^{\nu\sigma} + \partial^\nu\,C^{\sigma\mu} + \partial^\sigma\,C^{\mu\nu})\,\partial_\mu\big[\mathcal{B}_{\nu\sigma} + \bar{\mathcal{B}}_{\nu\sigma} - (\partial_\nu\tilde{\phi}_\sigma - \partial_\sigma\tilde{\phi}_\nu)\big]\Big\}. \qquad (61)
\end{aligned}$$

The above observation establishes that the Lagrangian density $\mathcal{L}_{(B)}$ respects (i) the co-BRST transformations [cf. Eqs. (3),(4)], and (ii) the anti-co-BRST transformations (30) provided we consider the entire 4D BRST-quantized theory on the submanifold of the quantum fields where the (anti-)BRST and (anti-)co-BRST invariant CF-type restrictions (i.e. $\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = (\partial_\mu\tilde{\phi}_\nu - \partial_\nu\tilde{\phi}_\mu),\ f_\mu + F_\mu = \partial_\mu C_1,\ \bar{f}_\mu + \bar{F}_\mu = \partial_\mu\bar{C}_1$) are satified. In other words, from the above equation, we note that the Lagrangina density $\mathcal{L}_{(B)}$ transforms to a total spacetime derivative under the anti-co-BRST transformations s_{ad} if we take into account the validity of the above *three* CF-type restrictions. Thus, the requirements of the symmetry invariance of the Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$, under the nilpotent co-BRST as well as the nilpotent anti-co-BRST transformations *together*, lead to the derivations of the (anti-)BRST as well as the (anti-)co-BRST invariant CF-type restrictions.

We discuss concisely the absolute anticommutativity property of the off-shell nilpotent co-BRST and anti-co-BRST transformation operators and lay emphasis on the (anti-) BRST and (anti-)co-BRST invariant CF-type restrictions. It turns out that the absolute anticommutativity property (i.e. $\{s_d, s_{ad}\} = 0$) is satisfied for *all* the fields of our 4D BRST-quantized theory *except* fields $A_\mu, C_{\mu\nu}$ and $\bar{C}_{\mu\nu}$ because we observe the following:

$$\begin{aligned}
\{s_d,\,s_{ad}\}\,A_\mu &= -\,\frac{1}{2}\,\varepsilon_{\mu\nu\sigma\rho}\,\partial^\nu\,(\mathcal{B}^{\sigma\rho} + \bar{\mathcal{B}}^{\sigma\rho}), \\
\{s_d,\,s_{ad}\}\,C_{\mu\nu} &= -\partial_\mu(f_\nu + F_\nu) + \partial_\nu(f_\mu + F_\mu), \\
\{s_b,\,s_{ad}\}\,\bar{C}_{\mu\nu} &= -\partial_\mu(\bar{f}_\nu + \bar{F}_\nu) + \partial_\nu(\bar{f}_\mu + \bar{F}_\mu). \qquad (62)
\end{aligned}$$

However, we have the sanctity of the absolute anticommutativity property (i.e. $\{s_d, s_{ad}\} = 0$) of the (anti-)co-BRST transformations for the *above* fields, too, provided we use the following (anti-)co-BRST as well as the (anti-)BRST invariant CF-type restrictions:

$$\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = \partial_\mu\tilde{\phi}_\nu - \partial_\nu\tilde{\phi}_\mu, \qquad f_\mu + F_\mu = \partial_\mu C_1, \qquad \bar{f}_\mu + \bar{F}_\mu = \partial_\mu\bar{C}_1. \qquad (63)$$

We lay emphasis on the fact that the above crucial CF-type restrictions have appeared in our earlier equations (58) and (61) where we have demanded that the *coupled* Lagrangian densities of our theory must respect the co-BRST as well as the anti-co-BRST symmetry transformations simultaneously. Using the off-shell nilpotent (anti-)BRST transformations [cf. Eqs. (A.8),(A.1)] and off-shell nilpotent (anti-)co-BRST symmetry transformations [cf. Eqs. (30),(3)], it can be checked that all the *four* CF-type restrictions on our 4D BRST-quantized field-theoretic system are (anti-)BRST as well as (anti-)co-BRST invariant quantities, namely;

$$\begin{aligned}
&s_{(a)b}[B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)] = 0, \qquad s_{(a)b}[f_\mu + F_\mu - \partial_\mu C_1] = 0, \\
&s_{(a)b}[\bar{f}_\mu + \bar{F}_\mu - \partial_\mu\bar{C}_1] = 0, \qquad s_{(a)b}[\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} - (\partial_\mu\tilde{\phi}_\nu - \partial_\nu\tilde{\phi}_\mu)] = 0, \\
&s_{(a)d}[B_{\mu\nu} + \bar{B}_{\mu\nu} - (\partial_\mu\phi_\nu - \partial_\nu\phi_\mu)] = 0, \qquad s_{(a)d}[f_\mu + F_\mu - \partial_\mu C_1] = 0, \\
&s_{(a)d}[\bar{f}_\mu + \bar{F}_\mu - \partial_\mu\bar{C}_1] = 0, \qquad s_{(a)d}[\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} - (\partial_\mu\tilde{\phi}_\nu - \partial_\nu\tilde{\phi}_\mu)] = 0, \qquad (64)
\end{aligned}$$

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which corroborate our earlier claim [after equation (62)]. Hence, these CF-type restrictions on our theory are *physical* and they are connected with the geometrical objects called gerbes (see, e.g. [25,26]). The *coupled* nature of the Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ has been demonstrated in our earlier work [20] by using the EL-EoMs that are derived from the *above* Lagrangian densities w.r.t the bosonic/fermionic auxiliary fields (of our 4D BRST-quantized field-theoretic system) which lead to the derivations of all the *four* CF-type restrictions on our theory. In other words, it is the existence of the *non-trivial* CF-type restrictions that are responsible for (i) the *coupled* nature of *these* Lagrangian densities, and (ii) the *equivalent* nature of the Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ from the point of view of the symmetry considerations. In fact, the direct *equality* of the BRST-quantized versions of the coupled Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ also leads to the derivations of *all* the four CF-type restrictions on our 4D BRST-quantized field-theoretic system (see, e.g. [20]).

9. Conclusions

In our present endeavor, we have been able to demonstrate that the coupled (but equivalent) Lagrangian densities $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ and $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ respect the infinitesimal, continuous and off-shell nilpotent co-BRST (i.e. dual-BRST) and anti-co-BRST (i.e. anti-dual-BRST) symmetry transformations under which the *total* gauge-fixing terms for the Abelian 3-form and 1-form gauge fields remain invariant in the description of our present 4D BRST-quantized *combined* field-theoretic system within the framework of the BRST invariant Lagrangian formulation. It has been clearly explained that the gauge-fixing terms for the Abelian 3-form and 1-form gauge fields owe their mathematical origin to the co-exterior (i.e. dual-exterior) derivative of differential geometry [15-19]. The vector field ϕ_μ appears in the gauge-fixing term for the Abelian 3-form gauge field because of its reducibility property. This field *also* remains invariant (i.e. $s_{(a)d}\phi_\mu = 0$) under the (anti-)co-BRST symmetry transformations $s_{(a)d}$ [cf. Eqs. (30),(3)]. Hence, the nomenclatures (that are associated with the infinitesimal, continuous and off-shell nilpotent (anti-)dual-BRST [i.e. (anti-)co-BRST] symmetry transformations $s_{(a)d}$) are absolutely correct.

We have laid a great deal of importance and emphasis on the discussions and derivations of the (anti-)BRST and (anti-)co-BRST invariant *non-trivial* CF-type restrictions [cf. Eqs. (63),(64)] because their existence is one of the decisive features of a properly BRST-quantized theory. In our earlier works (see, e.g. [25,26]), we have been able to establish that these *physical* restrictions, on a properly BRST-quantized theory, are connected with the geometrical objects called gerbes. We have been able to derive these non-trivial *physical* restrictions on our 4D BRST-quantized theory from *two* different theoretical angles. First of all, we have derived these CF-type restrictions from the requirements that *both* the Lagrangian densities $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ and $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ *must* respect the off-shell nilpotent co-BRST as well as the anti-co-BRST symmetry transformations [cf. Eqs. (3),(30)]. On the other hand, the requirement of the absolute anticmmutativity property (i.e. $\{s_d,\ s_{ad}\} = 0$) between the off-shell nilpotent co-BRST and anti-co-BRST symmetry transformation operators *also* indicates [cf. Eq. (62)] the validity of the CF-type restrictions on our 4D BRST-quantized theory. In other words, the absolute anticommutativity property (i.e. $\{s_d,\ s_{ad}\} = 0$) is satisfied on the submanifold of the quantum fields where the celebrated CF-type restrictions are valid.

As far as the continuous symmetry considerations are concerned, we find that (i) the anti-co-BRST insurance (i.e. $s_{ad}\mathcal{L}_{(B)} = 0$) of the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eq. (61)], and (ii) the co-BRST invariance (i.e. $s_d\mathcal{L}_{(\bar{B})} = 0$) of the Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ [cf. Eq. (57)] are valid if and only if we invoke the validity of the CF-type restrictions (i.e. $\mathcal{B}_{\mu\nu} + \bar{\mathcal{B}}_{\mu\nu} = \partial_\mu\widetilde{\phi}_\nu - \partial_\nu\widetilde{\phi}_\mu,\ f_\mu + F_\mu = \partial_\mu C_1,\ \bar{f}_\mu + \bar{F}_\mu = \partial_\mu\bar{C}_1$). The *coupled* nature of the Lagrangian desities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ *also* owe their origin to the existence of the *physically* important (anti-)BRST as well as the (anti-)co-BRST invariant CF-type restrictions which can be verified in the language of the EL-EoMs w.r.t. the bosonic/fermionic auxiliary fields which are derived from the above *coupled* Lagrangian densities $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ and $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ (see, e.g. [20] for details). In other words, we note that the EL-EoMs from the *coupled* Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ are such that they imply the validity

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of the CF-type restrictions on our theory. The *direct* equality of the coupled Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ also leads to the derivation of the above *three* CF-type restrictions along with the *fourth* one as: $B_{\mu\nu} + \bar{B}_{\mu\nu} = \partial_\mu \phi_\nu - \partial_\nu \phi_\mu$ which plays a crucial role in the discussion on the (anti-)BRST symmetries (see, e.g. [20]).

One of the highlights of our present investigation is the observation that the non-ghost parts [cf. Eqs. (1),(28)] of the coupled Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ remain invariant under a set of discrete duality symmetry transformations [cf. Eqs. (27),(59)] which provide a deep connection between the *conditions* on the physical states (i.e. $|phys>$) that emerge out from the physicality criteria $Q_{(A)B}\,|phys>=0$ and $Q_{(A)D}\,|phys>=0$ (cf. Secs. 4 and 7 or details). To be utmost precise, we find that (i) the physicality criteria w.r.t. the conserved and nilpotent versions of the (anti-)BRST charges lead to the annihilation of the physical states (i.e. $|phys>$) by the operator forms of the first-class constraints existing on our 4D *classical* theory, and (ii) the physicality criteria w.r.t. the nilpotent versions of the (anti-)co-BRST charges lead to the annihilation of the physical states (i.e. $|phys>$) by the *dual* versions of the first-class constraints[7]. The mathematical origin for such a crucial and decisive observation is hidden in the discrete *duality* symmetry transformations [cf. Eqs. (27),(59)] that exist in our 4D BRST-quantized theory.

In our present endeavor, we have focused *only* on (i) the infinitesimal, continuous and off-shell nilpotent (anti-)co-BRST symmetry transformations, (ii) the derivations of the non-nilpotent Noether (anti-)co-BRST conserved charges, (iii) the deductions of the off-shell nilpotent versions of the (anti-)co-BRST charges from the non-nilpotent Noether (anti-)co-BRST charges, (iv) the discussions and deliberations on the existence of the (anti-)BRST as well as the (anti-)co-BRST invariant CF-type restrictions, and (v) the physicality criteria w.r.t. the conserved and nilpotent versions of the (anti-)co-BRST charges. In our earlier work (see, e.g. [20] for details) such kinds of discussions have been performed in the context of the infinitesimal, continuous and off-shell nilpotent (anti-)BRST symmetry transformations. In our future endeavor, we envisage to discuss the (anti-)BRST and (anti-)co-BRST symmetry transformation operators, a *unique* bosonic symmetry transformation operator that is derived from the appropriate anticommutator(s) between the nilpotent (anti-)BRST and (anti-)co-BRST symmetry operators, the ghost-scale symmetry transformation operator along with a couple of discrete duality symmetry transformation operators. We plan to derive the extended algebra among all these discrete as well as the continuous symmetry transformation operators. The derivation of (i) the conserved Noether currents and corresponding conserved charges, (ii) the deductions of the *extended* BRST algebra amongst *all* the appropriate conserved charges, and (iii) the derivations of the mapping between the *appropriate* set of conserved charges and the cohomological operators of differential geometry, are yet another set of future directions which we envisage to pursue so that we can prove, in an elaborate fashion, that our present 4D BRST-quantized field-theoretic system[8] is a tractable example for Hodge theory.

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[7] For the BRST-quantized field-theoretic examples for Hodge theories, the physical states (excising in the total quantum Hilbert space of states) are the *harmonic* states that are annihilated by the nilpotent versions of the conserved (anti-)BRST as well as the (anti-)co-BRST charges. These harmonic states appear in the Hodge decomposed versions of the quantum states in the Hilbert space of states [27,28].

[8] The study of the BRST-quantized field-theoretic examples for Hodge theory (see, e.g. [10-12] and references therein) have become quite interesting because they lead to the existence of fields with *negative* kinetic terms which obey Klein-Gordon equation. Hence, they are a set of possible candidates for (i) the "phantom" and/or "ghost" fields of the cyclic, bouncing and self-accelerated cosmological models of the Universe (see, e.g. [29-31] and references therein), and (ii) the dark matter/dark energy (see, e.g. [32,33] and references therein). In our earlier work [34], we have been able to establish that the 2D BRST-quantized it free (non-)Abelian gauge theories (without any interaction with the matter fields) are the tractable field-theoretic examples for (i) the Hodge theory, and (ii) a new type of topological field theory (TFT) that captures a few key properties of the Witten-type TFTs [35], and (ii) some salient features of the Schwarz-type TFTs [36].

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Appendix A. On (Anti-)BRST Symmetries and Physicality Criteria

The central theme of our present Appendix is discuss the bare essentials of our earlier work [20] in a nut-shell and establish that the physicality criteria (i.e. $Q_{(A)B}\,|phys>= 0$) w.r.t the off-shell nilpotent (i.e. $Q^2_{(A)B} = 0$) versions of the conserved (anti-)BRST charges $Q_{(A)B}$ lead to the annihilation of the physical states (i.e. $|phys>$) by the operator forms of the first-class constraints [37-40] of our 4D *classical* combined field-theoretic system of the free Abelian 3-form and 1-form gauge theories at the *quantum* level. In this context, first of all, we note that, in addition to the infinitesimal, continuous and off-shell nilpotent (i.e. $s_d^2 = 0$) co-BRST symmetry transformations s_d in equation (3), the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)] *also* respects the following infinitesimal, continuous and off-shell nilpotent (i.e. $s_b^2 = 0$) BRST transformations s_b, namely;

$$
\begin{aligned}
& s_b A_{\mu\nu\sigma} = \partial_\mu C_{\nu\sigma} + \partial_\nu C_{\sigma\mu} + \partial_\sigma C_{\mu\nu}, \qquad s_b C_{\mu\nu} = \partial_\mu \beta_\nu - \partial_\nu \beta_\mu, \qquad s_b \bar{C}_{\mu\nu} = B_{\mu\nu}, \\
& s_b A_\mu = \partial_\mu C, \quad s_b \bar{C} = B, \quad s_b \bar{\beta}_\mu = \bar{F}_\mu, \quad s_b \beta_\mu = \partial_\mu C_2, \quad s_b \bar{B}_{\mu\nu} = \partial_\mu f_\nu - \partial_\nu f_\mu, \\
& s_b \bar{C}_2 = B_5, \qquad s_b C_1 = -B_4, \qquad s_b \bar{C}_1 = B_2, \qquad s_b \phi_\mu = f_\mu, \qquad s_b F_\mu = -\,\partial_\mu B_4, \\
& s_b \bar{f}_\mu = \partial_\mu B_2, \; s_b \Big[C,\, C_2,\, B,\, B_1,\, B_2,\, B_3,\, B_4,\, B_5,\, f_\mu,\, \bar{F}_\mu,\, \tilde{\phi}_\mu, B_{\mu\nu},\, \mathcal{B}_{\preceq\succeq},\, \bar{\mathcal{B}}_{\mu\nu}\Big] = 0,
\end{aligned}
\tag{A.1}
$$

because we observe that the co-BRST invariant [cf. Eq. (4)] Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ transforms to the total spacetime derivative as

$$
\begin{aligned}
s_b\, \mathcal{L}_{(B)} = \frac{1}{2}\, \partial_\mu \Big[& (\partial^\mu\, C^{\nu\sigma} + \partial^\nu\, C^{\sigma\mu} + \partial^\sigma\, C^{\mu\nu})\, B_{\nu\sigma} + B^{\mu\nu}\, f_\nu - B_5\, \partial^\mu\, C_2 \\
& B_2\, f^\mu + B_4\, \bar{F}^\mu - (\partial^\mu\, \beta^\nu - \partial^\nu\, \beta^\mu)\, \bar{F}_\nu \Big] - \partial_\mu \Big[B\, \partial^\mu C \Big],
\end{aligned}
\tag{A.2}
$$

thereby rendering the action integral, corresponding to the Lagrangian density $\mathcal{L}_{(B)}$, invariant due to the celebrated Gauss divergence theorem where *all* the physical fields vanish off as $x \to \pm\infty$. Hence, the infinitesimal, continuous and off-shell nilpotent BRST transformations (A.1) are the *symmetry* transformations for the action integral which lead to the derivation of the Noether conserved charge Q_b as (see, e.g. [20] for details)

$$
\begin{aligned}
Q_b = \int d^3x \Big[\frac{1}{2} \Big\{ & \varepsilon^{0ijk}\, B_1\, (\partial_i C_{jk}) + (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\, B_{ij} + B^{0i}\, f_i \\
& + \varepsilon^{0ijk}\, (\partial_i C)\, \mathcal{B}_{jk} - B_5\, \partial^0 C_2 - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)\, \partial_i C_2 - (\partial^0 \beta^i - \partial^i \beta^0)\, \bar{F}_i \\
& + B_2\, f^0 + B_4\, \bar{F}^0 - (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i})\, (\partial_i \beta_j - \partial_j \beta_i) \Big\} - B\, \partial^0 C \Big],
\end{aligned}
\tag{A.3}
$$

which is found to be non-nilpotent (i.e. $Q_b^2 \neq 0$) (see, e.g. [20] for details). The conserved and nilpotent (i.e. $Q_B^2 = 0$) version of the BRST charge Q_B has been derived systematically in our earlier work [20] from the non-nilpotent (i.e. $Q_b^2 \neq 0$) Noether BRST charge Q_b where the theoretical strength of the partial integration followed by the Gauss divergence theorem and the appropriate EL-EoMs have been exploited. The explicit expression for this nilpotent and conserved BRST charge Q_B (which emerges out from these exercises) is:

$$
\begin{aligned}
Q_B = \int d^3x \Big[\frac{1}{2} \Big\{ & (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\, B_{ij} - (\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i})\, C_{ij} + B^{0i}\, f_i \\
& + 3\, \dot{B}_5\, C_2 - B_5\, \dot{C}_2 - (\partial^0 \beta^i - \partial^i \beta^0)\, \bar{F}_i + (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i})\, (\partial_i \beta_j - \partial_j \beta_i) \\
& + B_4\, \bar{F}^0 + B_2\, f^0 \Big\} + (\dot{B}\, C - B\, \dot{C}) + (\partial^0 \bar{F}^i - \partial^i \bar{F}^0)\, \beta_i + (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)\, \partial_i C_2 \Big].
\end{aligned}
\tag{A.4}
$$

A close look at the every individual terms of the above expression for the nilpotent version of the BRST charge Q_B shows that they are made up of either by the basic/auxiliary ghost fields *alone* or these are

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product of the physical fields (with the ghost number equal to zero) and basic/auxiliary ghost fields (with the non-zero ghost numbers).

A keen and careful observation of the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)] shows that the auxiliary as well as the basic (anti-)ghost fields are decoupled from the rest of the theory. In other words, the physical fields (with the ghost number equal to zero) and the auxiliary as well as the basic FP-ghost fields (with the non-zero ghost numbers) have *no* interaction with each-other, right from the beginning. Hence, a quantum state (in the *total* quantum Hilbert space of states) will be the direct product (see, e.g. [24]) of the physical states (i.e. $|phys>$) and the ghost states (i.e. $|ghost>$). When the auxiliary and/or basic ghost fields of the conserved and nilpotent BRST charge Q_B act on the ghost states (i.e. $|ghost>$), they lead to the *non-zero* result. Thus, the physicality criterion w.r.t. the nilpotent BRST charge Q_B is the requirement that the physical fields (with the zero ghost numbers) *must* annihilate the physical states in the physicality criterion: $Q_B\,|phys>= 0$. In other words, the physical states (i.e. $|phys>$), exiting in the *total* quantum Hilbert space of states, are *those* that are annihilated by the physical fields that are associated with the *basic* (anti-)ghost fields in the expression for the nilpotent and conserved BRST charge Q_B [cf. Eq. (A.4)] of our theory. Taking all these inputs into account, we observe that the requirement: $Q_B\,|phys>= 0$ leads [20] to the following conditions on the physical states (i.e. $|phys>$) of our theory, namely;

$$\begin{aligned}
&B\,|phys> = 0 \quad\Longrightarrow\quad (\partial\cdot A)\,|phys> = 0 \quad\Longrightarrow\quad \Pi^0_{(A1)}|phys> = 0,\\
&\dot{B}\,|phys> = 0 \quad\Longrightarrow\quad \frac{1}{2}\,\varepsilon^{0ijk}\,\partial_i\mathcal{B}_{jk}\,|phys> = 0 \quad\Longrightarrow\quad \partial_i\Pi^i_{(A1)}\,|phys> = 0,\\
&\frac{1}{2}\,B_{ij}\,|phys> = 0 \quad\Longrightarrow\quad 3\,\Pi^{0ij}_{(A3)}|phys> = 0,\\
&\frac{1}{2}\,\big(\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}\big)\,|phys> = 0 \quad\Longrightarrow\quad \frac{1}{2}\,\varepsilon^{0ijk}\,\partial_k B_1\,|phys>= 0\\
&\quad\Longrightarrow\quad 3\,\partial_k\Pi^{kij}_{(A3)}\,|phys>= 0,
\end{aligned} \tag{A.5}$$

where $\Pi^\mu_{(A1)}$ and $\Pi^{\mu\nu\sigma}_{(A3)}$ are the canonical conjugate momenta w.r.t. the free Abelian 1-form gauge field A_μ and Abelian 3-form gauge field $A_{\mu\nu\sigma}$, respectively. The precise expression for these canonical conjugate momenta are as follows (see, e.g. [20] for details)

$$\begin{aligned}
&\Pi^\mu_{(A1)} = \frac{\partial\mathcal{L}_{(B)}}{\partial(\partial_0 A_\mu)} = \frac{1}{2}\,\varepsilon^{0\mu\nu\sigma}\,\mathcal{B}_{\nu\sigma} - \eta^{0\mu}\,B \;\Longrightarrow\; \Pi^0_{(A1)} = -\,B, \qquad \Pi^i_{(A1)} = \frac{1}{2}\,\varepsilon^{0ijk}\,\mathcal{B}_{jk},\\
&\Pi^{\mu\nu\sigma}_{(A3)} = \frac{\partial\mathcal{L}_{(B)}}{\partial(\partial_0 A_{\mu\nu\sigma})} = \frac{1}{3!}\Big[\varepsilon^{0\mu\nu\sigma}\,B_1 + \big(\eta^{0\mu}B^{\nu\sigma} + \eta^{0\nu}B^{\sigma\mu} + \eta^{0\sigma}B^{\mu\nu}\big)\Big] \quad\Longrightarrow\\
&\Pi^{0ij}_{(A3)} = \frac{1}{3!}\,B^{ij} \equiv \frac{1}{3!}\,B_{ij}, \qquad \Pi^{ijk}_{(A3)} = \frac{1}{3!}\,\varepsilon^{0ijk}\,B_1,
\end{aligned} \tag{A.6}$$

where the subscripts $(A1)$ and $(A3)$, associated with the canonical conjugate momenta, in the above equation, represent the fact that these momenta have been derived w.r.t. the Abelian 1-form and 3-form gauge fields, respectively. In the derivation of the conditions on the physical states [cf. Eq. (A.5)], it is worthwhile to mention that the following EL-EoMs from the Lagrangian density $\mathcal{L}_{(B)}$ w.r.t the gauge fields $A_{\mu\nu\sigma}$ and A_μ, namely;

$$\begin{aligned}
&\frac{1}{3!}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\rho B_1 = \frac{1}{3!}\,\big(\partial^\mu B^{\nu\sigma} + \partial^\nu B^{\sigma\mu} + \partial^\sigma B^{\mu\nu}\big)\\
&\Longrightarrow\; \partial_k\Pi^{kij}_{(A3)} = \frac{1}{3!}\,\big(\partial^0 B^{ij} + \partial^i B^{j0} + \partial^j B^{0i}\big),\\
&\frac{1}{2}\,\varepsilon^{\mu\nu\sigma\rho}\,\partial_\nu\mathcal{B}_{\sigma\rho} = \partial^\mu B \;\Longrightarrow\; \partial_i\Pi^i_{(A1)} = \dot{B},
\end{aligned} \tag{A.7}$$

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have been used (see, e.g. [20]). Thus, we note that the operator forms of the *first-class* primary (i.e. $\Pi^0_{(A1)} \approx 0,\ \Pi^{0ij}_{(A3)} \approx 0$) as well as the secondary constraints (i.e. $\partial_i \Pi^i_{(A1)} \approx 0,\ \partial_i \Pi^{ijk}_{(A3)} \approx 0$) of our 4D *classical* field-theoretic system annihilate the physical states (i.e. $|phys>$) that exist in the *total* quantum Hilbert space of states. This observation is consistent with the Dirac quantization conditions (see, e.g. [38] for details) that exist for a gauge theory that is endowed with a set of first-class constraints (in the language of Dirac's prescription for the classification scheme of the constraints [37-40]).

Against the backdrop of the above discussions, we *very briefly* mention that exactly the same kinds of conditions on the physical states (i.e. $|phys>$) emerge out from the physicality criterion $Q_{AB}\,|phys> = 0$ w.r.t the off-shell nilpotent version of the anti-BRST charge Q_{AB}. Toward this goal in mind, first of all, we note that the Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{ng)} + \mathcal{L}_{(fp)}$ [cf. Eqs. (28),(29)] respects the following infinitesimal, continuous and off-shell nilpotent (i.e. $s_{ab}^2 = 0$) anti-BRST symmetry transformations s_{ab}, namely;

$$
\begin{aligned}
&s_{ab} A_{\mu\nu\sigma} = \partial_\mu \bar{C}_{\nu\sigma} + \partial_\nu \bar{C}_{\sigma\mu} + \partial_\sigma \bar{C}_{\mu\nu}, \quad s_{ab} \bar{C}_{\mu\nu} = \partial_\mu \bar{\beta}_\nu - \partial_\nu \bar{\beta}_\mu, \quad s_{ab} C_{\mu\nu} = \bar{B}_{\mu\nu},\\
&s_{ab} A_\mu = \partial_\mu \bar{C}, \quad s_{ab} C = -\,B, \quad s_{ab} \beta_\mu = F_\mu, \quad s_{ab} \bar{\beta}_\mu = \partial_\mu \bar{C}_2, \quad s_{ab} B_{\mu\nu} = \partial_\mu \bar{f}_\nu - \partial_\nu \bar{f}_\mu,\\
&s_{ab} C_2 = B_4, \quad s_{ab} C_1 = -B_2, \quad s_{ab} \bar{C}_1 = -\,B_5, \quad s_{ab} \phi_\mu = \bar{f}_\mu, \quad s_{ab} \bar{F}_\mu = -\,\partial_\mu B_5,\\
&s_{ab} f_\mu = -\,\partial_\mu B_2,\ s_{ab} \Big[\bar{C},\, \bar{C}_2,\, B,\, B_1,\, B_2,\, B_3,\, B_4,\, B_5,\, \bar{f}_\mu,\, F_\mu,\, \widetilde{\phi}_\mu,\, \bar{B}_{\mu\nu},\, \mathcal{B}_{\mu\nu},\, \bar{\mathcal{B}}_{\mu\nu}\Big] = 0, \qquad (A.8)
\end{aligned}
$$

in addition to the anti-co-BRST symmetry transformations [cf. Eq. (30)], because we observe that the above nilpotent (i.e. $s_{ab}^2 = 0$) anti-BRST symmetry transformations s_{ab} transform the Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{ng)} + \mathcal{L}_{(fp)}$ to the total spacetime derivative

$$
\begin{aligned}
s_{ab}\, \mathcal{L}_{(\bar{B})} = \frac{1}{2}\, \partial_\mu \Big[&\bar{B}^{\mu\nu}\, \bar{f}_\nu - (\partial^\mu\, \bar{C}^{\nu\sigma} + \partial^\nu\, \bar{C}^{\sigma\mu} + \partial^\sigma\, \bar{C}^{\mu\nu})\, \bar{\mathcal{B}}_{\nu\sigma} + B_4\, \partial^\mu\, \bar{C}_2\\
&- B_5\, F^\mu + B_2\, \bar{f}^\mu - (\partial^\mu\, \bar{\beta}^\nu - \partial^\nu\, \bar{\beta}^\mu)\, F_\nu\Big] - \partial_\mu \Big[B\, \partial^\mu \bar{C}\Big], \qquad (A.9)
\end{aligned}
$$

thereby rendering the action integral $S = \int d^4x\, \mathcal{L}_{(\bar{B})}$ (corresponding to the Lagrangian density $\mathcal{L}_{(\bar{B})}$) invariant (i.e. $s_{ab}\, S = 0$) due to Gauss's divergence theorem. This observation, according to the celebrated Noether theorem, leads to the derivation of the conserved anti-BRST charge Q_{ab} as follows (see, e.g.. [20] for details):

$$
\begin{aligned}
Q_{ab} = \int d^3x\, \Big[\frac{1}{2}\, \Big\{&\varepsilon^{0ijk}\, B_1 (\partial_i \bar{C}_{jk}) - (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i})\, \bar{B}_{ij} + \bar{B}^{0i}\, \bar{f}_i\\
&-\varepsilon^{0ijk}\, (\partial_i \bar{C})\, \bar{\mathcal{B}}_{jk} + B_4\, \partial^0 \bar{C}_2 - (\partial^0 \beta^i - \partial^i \beta^0)\, \partial_i \bar{C}_2 - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)\, F_i\\
&- B_5\, F^0 + B_2\, \bar{f}^0 + (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\, (\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i)\Big\} - B\, \partial^0 \bar{C}\Big]. \qquad (A.10)
\end{aligned}
$$

The above Noether conserved charge turns out to be the generator for the infinitesimal, continuous and nilpotent anti-BRST symmetry transformations (A.8). It has been explicitly shown in our earlier work [20] that the charge Q_{ab} is *not* nilpotent of order two (i.e. $Q_{ab}^2 \neq 0$). The off-shell nillptent (i.e. $Q_{AB}^2 = 0$) version of the anti-BRST charge Q_{AB} can be systematically derived from the Noether non-nilpotent i.e. $Q_{ab}^2 \neq 0$) conserved charge Q_{ab} by following the theoretical proposal of our work [21] where the emphasis has been laid on (i) the partial integration followed by the use of the Gauss divergence theorem, and (ii) appropriate use of the EL-EoMs that are derived from the Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{ng)} + \mathcal{L}_{(fp)}$ [cf. Eqs. (28),(29)]. As a consequence, *both* forms of the anti-BRST charges Q_{ab} and Q_{AB} are conserved. The precise expression for the nilpotent anti-BRST charge Q_{AB}, derived from the non-nilpotent Noether anti-BRST charge Q_{ab}, is:

$$
Q_{AB} = \int d^3x\, \Big[\frac{1}{2}\, \Big\{B_4\, \dot{\bar{C}}_2 - 3\, \dot{B}_4\, \bar{C}_2 - (\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i})\, \bar{B}_{ij} - (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)\, F_i
$$

$$+(\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{j0} + \partial^j \bar{B}^{0i})\,\bar{C}_{ij} - (\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i})\,(\partial_i \bar{\beta}_j - \partial_j \bar{\beta}_i) - B_2\,\bar{f}^0$$
$$+\bar{B}^{0i}\,\bar{f}_i - B_5\,F^0\Big\} + (\dot{B}\,\bar{C} - B\,\dot{\bar{C}}) + (\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0)\,\partial_i \bar{C}_2 + (\partial^0 F^i - \partial^i F^0)\,\bar{\beta}_i\Big]. \qquad (A.11)$$

It is worthwhile to mention that *both* the conserved anti-BRST charges (i.e. Q_{ab} and Q_{AB}) have their own identity and importance within the framework of BRST formalism [20].

As far as the physicality criterion w.r.t. the anti-BRST charge is concerned, it is the nilpotent (i.e. $Q_{AB}^2 = 0$) version of the conserved anti-BRST charge Q_{AB} that plays a decisive role. To be utmost precise, it can be readily checked that the following conditions on the physical states (i.e. $|phys>$) emerge out when we demand that the physical states (existing in the *total* quantum Hilbert space of states) are *those* that are annihilated (i.e. $Q_{AB}\,|phys>= 0$) by the conserved and nilpotent version of the anti-BRST charge Q_{AB}. In fact, the physicality criterion: $Q_{AB}\,|phys>= 0$ (w.r.t. the nilpotent version of Q_{AB}) leads to the following conditions on the physical states, namely;

$$B\,|phys>= 0 \implies \Pi^0_{(A1)}|phys>= 0, \quad \dot{B}\,|phys>= 0 \implies \partial_i \Pi^i_{(A1)}\,|phys>= 0,$$
$$\frac{1}{2}\,(\partial^0 \bar{B}^{ij} + \partial^i \bar{B}^{j0} + \partial^j \bar{B}^{0i})\,|phys>= 0 \implies -3\,\partial_k \Pi^{kij}_{(A3)}\,|phys>= 0,$$
$$-\frac{1}{2}\,\bar{B}_{ij}\,|phys>= 0 \implies 3\,\Pi^{0ij}_{(A3)}|phys>= 0, \qquad (A.12)$$

where the expressions for the canonical conjugate momenta $\Pi^{\mu}_{(A1)}$ and $\Pi^{\mu\nu\sigma}_{(A3)}$ w.r.t. the Abelian 1-form and 3-form gauge fields have been listed in (A.6). The above observations [cf. Eq. (A.12)] establish that the operator forms of the primary and secondary constraints (modulo some constant numeral factors) of our 4D *combined* field-thermionic system of the *classical* theory [20] annihilate the physical states of the 4D BRST-quantized version of our present field-theoretic system at the *quantum* level (through the physicality criterion w.r.t. the nilpotent version of the anti-BRST charge Q_{AB}). Thus, we conclude that the physicality criteria w.r.t. the nilpotent (i.e. $Q^2_{(A)B} = 0$) versions of the (anti-)BRST charges $Q_{(A)B}$, ultimately, lead to the *same* consequences which are consistent with the Dirac quantization conditions for the gauge theories that are endowed with the *first-class* constraints.

Appendix B. On (Anti-)co-BRST and Ghost Charges: Algebraic Strcuture

The main objective of our present Appendix is to derive and discuss the algebraic structure that is satisfied by the conserved (anti-)co-BRST charges and the conserved ghost charge of our present 4D BRST-quantized theory. In this connection, first of all, we note that the FP-ghost parts ($\mathcal{L}_{(FP)}$ and $\mathcal{L}_{(fp)}$ of the Lagrangian densities [cf. Eqs. (28),(29)] remain invariant under the following explicit ghost-scale symmetry transformations, namely;

$$C_{\mu\nu} \to e^{+\Omega} C_{\mu\nu}, \quad \bar{C}_{\mu\nu} \to e^{-\Omega} \bar{C}_{\mu\nu}, \quad \beta_\mu \to e^{+2\Omega} \beta_\mu, \quad \bar{\beta}_\mu \to e^{-2\Omega} \bar{\beta}_\mu,$$
$$f_\mu \to e^{+\Omega} f_\mu, \quad \bar{f}_\mu \to e^{-\Omega} \bar{f}_\mu, \quad F_\mu \to e^{+\Omega} F_\mu, \quad \bar{F}_\mu \to e^{-\Omega} \bar{F}_\mu,$$
$$C_2 \to e^{+3\Omega} C_2, \quad \bar{C}_2 \to e^{-3\Omega} \bar{C}_2, \quad C \to e^{+\Omega} C, \quad \bar{C} \to e^{-\Omega} \bar{C},$$
$$C_1 \to e^{+\Omega} C_1, \quad \bar{C}_1 \to e^{-\Omega} \bar{C}_1, \quad B_4 \to e^{+2\Omega} B_4, \quad B_5 \to e^{-2\Omega} B_5, \qquad (B.1)$$

where the symbol Ω stands for the spacetime-independent (i.e. global) scale transformation parameter and the numerals (± 3, ± 2, ± 1) in the exponents correspond to the ghost numbers for the FP-ghost fields. In view of the above statement, it is interesting to point out that the non-ghost parts (i.e. $\mathcal{L}_{(NG)}$ and $\mathcal{L}_{(ng)}$) of the Lagrangian densities [cf. Eqs. (1),(28)] with the generic field $\Phi_i = A_{\mu\nu\sigma}, B_{\mu\nu}, \mathcal{B}_{\mu\nu}, \bar{B}_{\mu\nu}, \bar{\mathcal{B}}_{\mu\nu}, A_\mu, B, B_1, B_2, B_3, \phi_\mu, \tilde{\phi}_\mu$ carry the ghost number equal to *zero*. Hence, these fields (present in the Lagrangian densities $\mathcal{L}_{(NG)}$ and $\mathcal{L}_{(ng)}$) do *not* transform *at all* under the above ghost-scale symmetry transformations. For the sake of brevity, we set the global ghost-scale

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transformation parameter $\Omega = 1$ so that the infinitesimal versions (s_g) of the above ghost-scale transformations are

$$s_g C_{\mu\nu} = + C_{\mu\nu}, \quad s_g \bar{C}_{\mu\nu} = - \bar{C}_{\mu\nu}, \quad s_g \beta_\mu = +2\, \beta_\mu, \quad s_g \bar{\beta}_\mu = -2\, \bar{\beta}_\mu,$$
$$s_g f_\mu = + f_\mu, \quad s_g \bar{f}_\mu = - \bar{f}_\mu, \quad s_g F_\mu = + F_\mu, \quad s_g \bar{F}_\mu = - \bar{F}_\mu,$$
$$s_g C_2 = +3\, C_2, \quad s_g \bar{C}_2 = -3\, \bar{C}_2, \quad s_g C = + C, \quad s_g \bar{C} = - \bar{C},$$
$$s_g C_1 = + C_1, \quad s_g \bar{C}_1 = - \bar{C}_1, \quad s_g B_4 = +2\, B_4, \quad s_g B_5 = -2\, B_5, \quad s_g \Phi_i = 0, \qquad (B.2)$$

where it can be readily checked that s_g is bosonic (i.e. $s_g^2 \neq 0$) in nature. As a consequence, we observe that the bosonic fields of the coupled (but equivalent) Lagrangian densities $\mathcal{L}_{(B)}$ and $\mathcal{L}_{(\bar{B})}$ for our 4D field-theoretic system transform to the bosonic fields and the fermionic fields transform to the fermionic fields *without* any change in the mass and ghost numbers. Under these infinitesimal transformations, the *perfectly* (co-)BRST invariant Lagrangian density $\mathcal{L}_{(B)}$ remains invariant (i.e. $s_g \mathcal{L}_{(B)} = 0$). As a consequence, according to Noether's theorem, we have the following expression for the ghost current $J^\mu_{(g)}$, namely;

$$J^\mu_{(g)} = \frac{1}{2}\Big[\left(\partial^\mu C^{\nu\sigma} + \partial^\nu C^{\sigma\mu} + \partial^\sigma C^{\mu\nu}\right) \bar{C}_{\nu\sigma} + \left(\partial^\mu \bar{C}^{\nu\sigma} + \partial^\nu \bar{C}^{\sigma\mu} + \partial^\sigma \bar{C}^{\mu\nu}\right) C_{\nu\sigma}$$
$$-2\left(\partial^\mu \bar{\beta}^\nu - \partial^\nu \bar{\beta}^\mu\right) \beta_\nu + 2\left(\partial^\mu \beta^\nu - \partial^\nu \beta^\mu\right) \bar{\beta}_\nu - C^{\mu\nu}\, \bar{F}_\nu - \bar{C}^{\mu\nu}\, f_\nu - C_1\, \bar{F}^\mu$$
$$- \bar{C}_1\, f^\mu + 3\, C_2\, \partial^\mu \bar{C}_2 + 3\, \bar{C}_2\, \partial^\mu C_2 - 2\, \beta^\mu\, B_5 - 2\, \bar{\beta}^\mu\, B_4\Big] + C\, \partial^\mu \bar{C} + \bar{C}\, \partial^\mu C, \qquad (B.3)$$

derived from the Lagrangian density $\mathcal{L}_{(B)} = \mathcal{L}_{(NG)} + \mathcal{L}_{(FP)}$ [cf. Eqs. (1),(2)]. The expression for the ghost charges $Q_{(g)} = \int d^3x J^0_{(g)}$ is as follows:

$$Q_{(g)} = \int d^3x\, \Big[\frac{1}{2}\Big\{\left(\partial^0 \bar{C}^{ij} + \partial^i \bar{C}^{j0} + \partial^j \bar{C}^{0i}\right) C_{ij} + \left(\partial^0 C^{ij} + \partial^i C^{j0} + \partial^j C^{0i}\right) \bar{C}_{ij}$$
$$-2\left(\partial^0 \bar{\beta}^i - \partial^i \bar{\beta}^0\right) \beta_i + 2\left(\partial^0 \beta^i - \partial^i \beta^0\right) \bar{\beta}_i - \frac{1}{2}\, C^{0i}\, \bar{F}_i + \frac{1}{2}\, C^{i0}\, \bar{F}_i - \frac{1}{2}\, \bar{C}^{0i}\, f_i + \frac{1}{2}\, \bar{C}^{i0}\, f_i$$
$$-C_1\, \bar{F}^0 - \bar{C}_1\, f^0 + 3\, C_2\, \dot{\bar{C}}_2 + 3\, \bar{C}_2\, \dot{C}_2 - 2\, \beta^0\, B_5 - 2\, \bar{\beta}^0\, B_4\Big\} + C\, \dot{\bar{C}} + \bar{C}\, \dot{C}\Big]. \qquad (B.4)$$

The above conserved charges $Q_{(g)}$ is the generator for the infinitesimal ghost-scale symmetry transformations (B.2) if we express the ghost charge in terms of the canonical conjugate momenta w.r.t. the *basic* dynamical (anti-)ghost fields of the ghost-sectors of the Lagrangian densities $\mathcal{L}_{(FP)}$ [cf. Eq. (2)]. It is worthwhile to mention, in passing, that the Lagrangian density $\mathcal{L}_{(\bar{B})} = \mathcal{L}_{(ng)} + \mathcal{L}_{(fp)}$ also leads to the derivation of the conserved ghost charge which *also* turns out to be the generator [20] for the infinitesimal versions of the ghost-scale transformations (B.2) provided we express this conserved charge in terms of the canonical conjugate momenta w.r.t. the dynamical fields that are present in $\mathcal{L}_{(fp)}$.

At this stage, we treat the conserved ghost charge Q_g as the generator [cf. Eq. (9)] for the infinitesimal ghost-scale transformations s_g and observe that the following relationships between the ghost charge and the non-nilpotent (i.e. $Q^2_{(a)d} \neq 0$) versions and nilpotent (i.e. $Q^2_{(A)D} = 0$) versions of the (anti-)co-BRST charges (i.e. $Q_{(a)d}, Q_{(A)D}$) are true, namely;

$$s_g Q_d = -i\left[Q_d,\, Q_g\right] = - Q_d \quad\Longrightarrow\quad i\left[Q_g,\, Q_d\right] = - Q_d,$$
$$s_g Q_{ad} = -i\left[Q_{ad},\, Q_g\right] = + Q_{ad} \quad\Longrightarrow\quad i\left[Q_g,\, Q_{ad}\right] = + Q_{ad},$$
$$s_g Q_D = -i\left[Q_D,\, Q_g\right] = - Q_D \quad\Longrightarrow\quad i\left[Q_g,\, Q_D\right] = - Q_D,$$
$$s_g Q_{AD} = -i\left[Q_{AD},\, Q_g\right] = + Q_{AD} \quad\Longrightarrow\quad i\left[Q_g,\, Q_{AD}\right] = + Q_{AD}, \qquad (B.5)$$

which establish that the Noether non-nilpotent (anti-)co-BRST charges $Q_{(a)d}$ and the nilpotent versions of the (anti-)co-BRST charges $Q_{(A)D}$ obey exactly the *same* kinds of algebras with the conserved ghost charge Q_g. However, we know that the nilpotency property of the (anti-)co-BRST charges is very important from the point of view of (i) the BRST cohomology (see, e.g. [23]), and (ii) the physicality criteria [9] and their consistency with the Dirac quantization conditions for the systems with constraints [37-40]. In particular, for the BRST-quantized field-theoretic models of Hodge theory, the physical states (i.e. $|phys>$) are the *harmonic* states (in the Hodge decomposed quantum states of the Hilbert space) which are annihilated by the conserved and nilpotent versions of the (anti-)BRST as well as the (anti-)co-BRST charges. These charges are expected to satisfy a very specific kinds of algebraic structures with the conserved ghost charge. In our present case, the standard kind of algebra is obeyed by the nilpotent (i.e. $Q^2_{(A)D} = 0$) versions of the (anti-)co-BRST charges $Q_{(A)D}$ and the conserved ghost charge Q_g. This algebra is as follows:

$$Q_D^2 = 0, \qquad Q_{AD}^2 = 0, \qquad i\,[Q_g,\; Q_D] = -\,Q_D, \qquad i\,[Q_g,\; Q_{AD}] = +\,Q_{AD}. \qquad (B.6)$$

The above algebra encodes the fact that the ghost numbers of the nilpotent (anti-)co-BRST charges are $(+1) - 1$, respectively. In other words, the co-BRST symmetry transformation operator s_d lowers the ghost number of a field by one [cf. Eq. (3)]. On the other hand, the ghost number is raised by one for a field which is acted upon by the anti-co-BRST symmetry transformation operator s_{ad} [cf. Eq. (30)].

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