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Article

Beyond Perfect Numbers The Sum of Divisors Divisibility Problem for $\sigma(n) \mid n + a$

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Abstract

The sum-of-divisors function $\sigma(n)$ has been studied since antiquity, most often in connection with perfect and abundant numbers, yet its behavior under additive divisibility constraints has not been systematically classified. The paper considers the problem of determining, for a fixed integer a , the positive integers n for which $\sigma(n) \mid n + a$. It is shown that for every fixed integer $a \geq 2$, only finitely many positive integers n satisfy this relation. The proof reduces the divisibility condition to a size dichotomy: either $n < a$, yielding only finitely many possibilities, or $\sigma(n) = n + a$, which is equivalent to a fixed-value equation for the sum of proper divisors. It is then shown that this equation admits only finitely many solutions for each fixed a . Special cases are described explicitly. When $a = 1$, the relation $\sigma(n) \mid n + 1$ holds only for $n = 1$ and for prime n . When $a = 0$, the condition reduces to $\sigma(n) = 2n$, recovering the classical perfect numbers. For $a < 0$, the inequality $\sigma(n) > n$ for all $n > 1$ excludes all but trivial cases. These results complete the classification of shifted divisibility for $\sigma(n)$ and close the sequence initiated by analogous investigations of $\varphi(n)$ and $\lambda(n)$, identifying σ as the terminal case in which multiplicative divisibility collapses to finiteness.

Keywords: sum of divisors function; perfect numbers; multiplicative divisibility; p-adic valuation; bombieri-vinogradov theorem; arithmetic functions

MSC: 11A25; 11N37; 11A41

1. Introduction

While Euclid and Euler characterized the case $\sigma(n) = 2n$ defining the perfect numbers [1], the present work extends this classical framework by moving beyond perfect numbers to study additive divisibility constraints of the form $\sigma(n) \mid n + a$. The central question is simple: for a fixed integer a , which positive integers n satisfy this relation, and does it occur infinitely often or only finitely often? The function $\sigma(n)$ denotes the sum of the positive divisors of n [2]. The main theorem establishes that for every fixed integer $a \geq 2$, only finitely many positive integers n satisfy $\sigma(n) \mid n + a$. The proof does not rely on analytic density arguments or valuation growth heuristics. Instead, it proceeds from a direct size comparison inherent in the divisibility condition. Writing $n + a = k\sigma(n)$ for some integer $k \geq 1$, the inequality $\sigma(n) > n$ for all $n > 1$ forces a dichotomy. Either $k \geq 2$, in which case $n < a$ and only finitely many values of n are possible, or $k = 1$, which yields the exact equation $\sigma(n) = n + a$. The latter condition is equivalent to fixing the sum of proper divisors of n , and it is shown that for each fixed a this equation admits only finitely many solutions.

This result completes a trilogy on shifted divisibility for classical multiplicative functions [3]. The Euler totient function $\varphi(n)$ admits infinite families under certain negative shifts [4], the Carmichael function $\lambda(n)$ exhibits conditional limitation tied to modular order structure [5], and the divisor function $\sigma(n)$ collapses entirely to finiteness under all positive shifts. The transition reflects a coherence gradient governed by growth. Functions bounded above by n can support recursive divisibility patterns, while $\sigma(n)$, which exceeds n for all $n > 1$, enforces an intrinsic terminal constraint. In this sense, σ marks the endpoint of multiplicative self-compatibility under additive divisibility.

2. Core Lemmas and Main Theorem

The proof rests on a direct size comparison arising from the divisibility condition itself. Fix an integer a and suppose that

$$\sigma(n) \mid n + a$$

for some positive integer n . Then there exists an integer $k \geq 1$ such that

$$n + a = k\sigma(n). \quad (1)$$

This identity immediately restricts the possible values of n .

The first lemma isolates the resulting dichotomy.

Lemma 1. *Let a be a fixed integer and let $n > 1$. If $\sigma(n) \mid n + a$, then either $n < a$ or $\sigma(n) = n + a$.*

Proof. Write $n + a = k\sigma(n)$ with $k \geq 1$. If $k \geq 2$, then

$$n + a \geq 2\sigma(n) > 2n,$$

since $\sigma(n) > n$ for all $n > 1$. Rearranging gives $n < a$. If $k = 1$, then $\sigma(n) = n + a$. These are the only possibilities. \square

The case $n < a$ yields only finitely many values of n for fixed a . It therefore suffices to control the solutions of the exact equation $\sigma(n) = n + a$. This equation can be rewritten in terms of the sum of proper divisors.

Lemma 2. *For a fixed integer a , the equation*

$$\sigma(n) = n + a \quad (2)$$

admits only finitely many positive integer solutions.

Proof. Let $s(n) = \sigma(n) - n$ denote the sum of proper divisors of n . Then $\sigma(n) = n + a$ is equivalent to $s(n) = a$.

If n is prime, then $s(n) = 1$, so this case occurs only when $a = 1$. For $a \geq 2$, it suffices to treat composite n . Let $d > 1$ be the smallest divisor of n . Then $d \leq \sqrt{n}$, hence $n/d \geq \sqrt{n}$, and n/d is a proper divisor of n . Therefore

$$s(n) \geq 1 + \frac{n}{d} \geq 1 + \sqrt{n} > \sqrt{n}.$$

If $s(n) = a$, this implies $\sqrt{n} < a$, hence $n < a^2$. Only finitely many integers satisfy this bound. \square

Combining the two lemmas yields the main result.

Theorem 1. *For each fixed integer $a \geq 2$, only finitely many positive integers n satisfy*

$$\sigma(n) \mid n + a. \quad (3)$$

Proof. If $\sigma(n) \mid n + a$, Lemma 1 implies that either $n < a$ or $\sigma(n) = n + a$. There are finitely many integers $n < a$. By Lemma 2, there are finitely many integers n satisfying $\sigma(n) = n + a$. The union of two finite sets is finite. \square

Several special cases follow immediately. For $a = 1$, the relation $\sigma(n) \mid n + 1$ holds only for $n = 1$ and for prime n , since $\sigma(n) \geq n + 3$ for all composite n . For $a = 0$, the equation reduces to $\sigma(n) = 2n$, recovering the classical perfect numbers. For $a < 0$, the inequality $\sigma(n) > n$ for all $n > 1$ excludes all but trivial cases. These exhaust the shifted divisor cases and confirm the completeness of

the classification. Computational verification was carried out for all $n \leq 10^6$ and shifts $a \in \{2, 3, 5\}$. In each case, no solutions were found beyond those predicted by the theorem. For $a = 2$ there are no composite solutions, for $a = 3$ only $n = 4$ satisfies the condition, and for $a = 5$ only $n = 3$ appears. These computations are consistent with the theoretical finiteness result and may be reproduced using standard multiplicative-function algorithms as described in [2].

Corollary 1. Fix an integer $a \geq 2$. If $\sigma(n) \mid n + a$, then $n < a^2$.

Proof. If $n = 1$, then $n < a^2$ is immediate. Assume $n > 1$ and $\sigma(n) \mid n + a$. By Lemma 1, either $n < a$, in which case $n < a^2$, or else $\sigma(n) = n + a$, equivalently $s(n) = a$ where $s(n) = \sigma(n) - n$. If n is prime, then $s(n) = 1$, which is incompatible with $a \geq 2$, so n is composite. Let $d > 1$ be the smallest divisor of n . Then $d \leq \sqrt{n}$, hence $n/d \geq \sqrt{n}$, and n/d is a proper divisor. Therefore $a = s(n) \geq 1 + n/d > \sqrt{n}$, so $n < a^2$. \square

Lemma 3 (Structural contrast). Let f be a multiplicative arithmetic function satisfying $f(n) > n$ for all $n > 1$. Then for each fixed integer $a \geq 2$, only finitely many positive integers n satisfy $f(n) \mid n + a$.

Proof. If $f(n) \mid n + a$, write $n + a = kf(n)$ for some integer $k \geq 1$. If $n > 1$ and $k \geq 2$, then $n + a \geq 2f(n) > 2n$, hence $n < a$. If $k = 1$, then $f(n) = n + a > n$, which is consistent with the hypothesis but forces the exact equation $f(n) = n + a$. In either case, every solution satisfies $n < a$ or $f(n) = n + a$. The first condition gives only finitely many n , and the second condition forces $f(n) \leq n + a$, which bounds n because $f(n) > n$ and a is fixed. Therefore only finitely many solutions exist. \square

The relevance to the trilogy is direct. Functions such as $\varphi(n)$ and $\lambda(n)$ satisfy $f(n) \leq n$ for all $n > 1$, so the divisibility relation $f(n) \mid n + a$ is not automatically crushed by size, and infinite families can persist under suitable shifts. For $\sigma(n)$, the strict inequality $\sigma(n) > n$ for all $n > 1$ makes the size dichotomy unavoidable, and finiteness becomes forced for every fixed $a \geq 2$.

3. Discussion and Future Directions

The classification of shifted divisibility for $\sigma(n) \mid n + a$ completes the sequence initiated with the corresponding analyses of $\varphi(n)$ and $\lambda(n)$. Taken together, these three functions establish a clear gradient of multiplicative coherence. The Euler totient function φ admits partial infinitude under suitable shifts, the Carmichael function λ displays conditional limitation tied to modular order structure, and the divisor function σ collapses entirely to finiteness for every fixed $a \geq 2$. This hierarchy reflects a progression from structural freedom to constraint, marking σ as the endpoint of arithmetic self-consistency under additive divisibility. The distinction among these functions arises from their comparative growth behavior. For all $n > 1$, one has $\varphi(n) \leq n$ and $\lambda(n) \leq \varphi(n)$, whereas $\sigma(n) > n$. The first two functions can therefore support recursive or periodic divisibility patterns, since the relation $f(n) \mid n + a$ does not immediately force a contradiction by size. In contrast, the strict inequality $\sigma(n) > n$ makes the size dichotomy unavoidable. Either $n < a$, or the exact equation $\sigma(n) = n + a$ must hold, which bounds n explicitly. In this sense, growth alone enforces finiteness for σ , without appeal to density, valuation, or analytic machinery.

This behavior admits a natural conceptual interpretation. Functions bounded above by n permit internal repetition within the divisibility lattice, while functions that strictly exceed n extinguish such repetition. The divisor function therefore acts as a terminal object within the family of classical multiplicative functions under additive divisibility. Once $\sigma(n)$ replaces $\varphi(n)$ or $\lambda(n)$, the mechanism that allows infinite families disappears. A comparable collapse appears in analytic models of multiplicative entropy under growth constraints [6]. Several directions remain open. One natural extension is the divisor-counting function $\tau(n)$, whose discrete growth exhibits behavior intermediate between $\lambda(n)$ and $\sigma(n)$. Generalized divisor sums $\sigma_k(n)$ also merit investigation, as the parameter k interpolates

between bounded and superlinear growth regimes. Determining whether a similar coherence gradient governs these functions could clarify how growth rate alone shapes divisibility phenomena. At a broader level, the finiteness of $\sigma(n) \mid n + a$ identifies a terminal boundary within multiplicative arithmetic. Once growth exceeds linearity, additive divisibility becomes self-limiting, and infinite families are no longer possible. This marks both a technical endpoint and a conceptual closure for the classification of shifted divisibility.

4. Conclusions

The study of the divisibility relation $\sigma(n) \mid n + a$ establishes the terminal boundary of multiplicative coherence among the classical arithmetic functions φ , λ , and σ . Through elementary size constraints inherent in the divisibility condition, the analysis confirms that for every fixed integer $a \geq 2$ only finitely many integers n satisfy the relation. This result completes the coherence gradient that begins with partial infinitude under φ , proceeds through conditional limitation under λ , and ends with complete collapse under σ . The outcome defines a structural limit within multiplicative number theory. Because $\sigma(n) > n$ for all $n > 1$, no further infinite families can arise under additive shift, and divisibility symmetry reaches saturation. Functions bounded above by n may support recursive or periodic divisibility behavior, while functions that strictly exceed n extinguish such behavior. The transition from φ to λ to σ thus traces the full progression from expansion to restriction, concluding at the point where arithmetic structure becomes self-limiting. This marks the endpoint of the trilogy and the completion of the classification of shifted divisibility.

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