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Article

Beyond Perfect Numbers the Sum of Divisors Divisibility Problem for $\sigma(n) \mid n + a$

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Abstract

The sum-of-divisors function $\sigma(n)$ has been studied since antiquity, most often in connection with perfect and abundant numbers, yet its divisibility behavior under integer shifts has never been classified. The paper asks a simple question. For which integers a does $\sigma(n)$ divide $n + a$? The result completes the trilogy on multiplicative divisibility, following the earlier analyses of $\varphi(n)$ and $\lambda(n)$. The proof establishes that for every fixed integer $a \geq 2$, only finitely many positive integers n satisfy $\sigma(n) \mid n + a$. The argument combines a 2-adic wall limiting the number of odd prime factors, an overflow lemma that forces q-adic excess for large $\omega(n)$, and a finiteness schema constraining the remaining primes to a fixed modulus set. Special cases include $a = 1$, where $\sigma(n) \mid n + 1$ holds only for $n = 1$ and for prime n, $a = 0$, which yields the classical perfect numbers satisfying $\sigma(n) = 2n$, and $a < 0$, where no infinite families exist since $\sigma(m) \nmid m$ for all $m > 1$. The result marks the terminus of multiplicative coherence, the point at which arithmetic structure collapses into absolute finiteness.

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1. Introduction

While Euclid and Euler characterized the case $\sigma(n) = 2n$ defining the perfect numbers, the present work extends this classical framework by moving beyond perfect numbers to classify all integer shifts a for which $\sigma(n) \mid n + a$ holds infinitely or finitely often. Notation follows standard analytic number theory. The function $\sigma(n)$ denotes the sum of the positive divisors of n. For any prime q, $v_q(m)$ denotes the q-adic valuation of an integer m, that is, the largest exponent e such that q^e divides m. The symbol $\omega(n)$ counts the number of distinct prime factors of n, and $\mathrm{rad}(n)$ denotes the product of those distinct primes.

The main theorem asserts that for every fixed integer $a \geq 2$, only finitely many positive integers n satisfy $\sigma(n) \mid n + a$. The proof proceeds through three central lemmas forming a valuation overflow schema chain. First, a 2-adic wall bounds the number of odd prime factors of n in relation to $v_2(n + a)$. Second, an overflow lemma shows that when $\omega(n)$ exceeds a threshold depending on a, some odd prime q satisfies $v_q(\sigma(n)) > v_q(n + a)$, forcing non-divisibility. Third, a finiteness schema proves that if overflow never occurs, the remaining primes lie in a finite modulus set determined by a, implying that only finitely many n remain. Together these results yield the complete classification of $\sigma(n) \mid n + a$ across integer shifts. This result completes a trilogy establishing a coherence gradient: $\varphi(n)$ admits infinite families under negative shifts, $\lambda(n)$ displays conditional infinitude, and $\sigma(n)$ collapses entirely to finiteness, marking the terminus of multiplicative self-compatibility.

2. Core Lemmas and Main Theorem

The argument proceeds through three structural components that together establish the finiteness of all integers n satisfying $\sigma(n) \mid n + a$ for fixed $a \geq 2$. Each component restricts divisibility through valuation, density, or radical constraints, converging in the main theorem. Consider first the 2-adic

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valuation. For any odd prime p, the local sum $\sigma(p^k) = 1 + p + \cdots + p^k$ has $k+1$ terms, each odd, so $\sigma(p^k) \equiv k+1 \pmod 2$. Every distinct odd prime factor therefore contributes at least one factor of two to $\sigma(n)$, and multiplicativity across coprime prime powers gives $2^t \mid \sigma(n)$, where t counts the odd prime divisors of n [9], giving

$$2^t \mid \sigma(n). \tag{1}$$

Whenever $\sigma(n) \mid n + a$, one must therefore have

$$t \leq v_2(n+a), \tag{2}$$

which bounds $\omega(n)$ for any fixed a. This restriction, referred to as the 2-adic wall, marks the first obstruction to infinite divisibility. Once the number of odd prime factors exceeds the 2-adic valuation of $n + a$, the divisibility relation collapses.

A second restriction arises from the behavior of odd primes. Let $Q = \{3, 5, 7\}$ and fix $a \geq 2$. There exists a threshold $r_0(a, Q)$ such that if n is squarefree and $\omega(n) \geq r_0(a, Q)$, then for some $q \in Q$ the inequality

$$v_q(\sigma(n)) > v_q(n+a) \tag{3}$$

must hold. For squarefree n, the product formula $\sigma(n) = \prod_{p|n}(p+1)$ implies that $q^2 \mid (p+1)$ whenever $p \equiv -1 \pmod{q^2}$. By the Bombieri-Vinogradov theorem [7,8], primes in this residue class have natural density $1/\varphi(q^2) = 1/(q(q-1))$. Hence, for sufficiently large $\omega(n)$, at least one divisor p of n must satisfy $p \equiv -1 \pmod{q^2}$ for some $q \in Q$, yielding $v_q(p+1) \geq 2$, which contributes to $v_q(\sigma(n))$. Since $v_q(n+a)$ remains bounded independently of $\omega(n)$, this produces $v_q(\sigma(n)) > v_q(n+a)$ and hence a contradiction to $\sigma(n) \mid n + a$. This overflow mechanism eliminates all sufficiently composite configurations. The threshold $r_0(a, Q)$ is defined as the smallest integer such that any squarefree n with $\omega(n) \geq r_0(a, Q)$ must contain a prime factor $p \equiv -1 \pmod{q^2}$ for some $q \in Q$. This follows from the average equidistribution of primes in arithmetic progressions under the Bombieri-Vinogradov theorem [5,7]. Hence existence of $r_0(a, Q)$ is ensured without requiring its explicit computation. Explicitly, since the set of primes avoiding all congruences $p \equiv -1 \pmod{q^2}$ for $q \in Q$ has positive but bounded density, there exists a minimal $r_0(a, Q)$ beyond which such avoidance is impossible for $\omega(n)$ primes, ensuring the stated overflow.

If overflow fails for all $q \in Q$, the remaining possibilities obey an exact structural condition. Matching valuations $v_q(\sigma(n)) = v_q(n+a)$ for all odd primes q forces every $(p+1)$ to contain no new prime divisors beyond those of $(a+1)$. Let M denote the squarefree part of $(a+1)$. Then

$$\operatorname{rad}(p+1) \mid M \qquad \text{for all } p \mid n. \tag{4}$$

When valuations satisfy $v_q(\sigma(n)) = v_q(n+a)$ for all odd q, any new factor from $(p+1)$ that fails to divide $(a+1)$ would increase $v_q(\sigma(n))$ beyond $v_q(n+a)$. Therefore $\operatorname{rad}(p+1) \mid \operatorname{rad}(a+1)$ for each $p \mid n$, which confines p to finitely many possibilities [4]. This confines p to the finite set $\{\, d-1 : d \mid M \,\}$ and produces only finitely many possible n. This configuration is referred to as the exceptional schema.

Combining these restrictions yields the main result.

Theorem 1. *For each fixed integer $a \geq 2$, only finitely many positive integers n satisfy*

$$\sigma(n) \mid n + a. \tag{5}$$

Proof. If $\omega(n) > v_2(n+a)$, the 2-adic wall gives $2^{\omega(n)} \mid \sigma(n)$ but $2^{\omega(n)} \nmid (n+a)$, contradicting divisibility. If $\omega(n) \leq v_2(n+a)$ yet $\omega(n) \geq r_0(a, Q)$, the overflow condition ensures that some $q \in Q$ satisfies $v_q(\sigma(n)) > v_q(n+a)$, again impossible. Finally, if $\omega(n) < r_0(a, Q)$ and overflow never occurs, all primes $p \mid n$ are confined to the finite schema determined by $M \mid (a+1)$, so only finitely many n remain. For non-squarefree integers $n = \prod p^{k_p}$, each local factor satisfies

$$\sigma(p^{k_p}) = \frac{p^{k_p+1} - 1}{p - 1} > p^{k_p},$$

so $\sigma(p^{k_p})$ grows strictly faster than p^{k_p}. Since $\sigma(n) \mid n + a$ for fixed a, each exponent k_p is therefore bounded by a constant depending only on a, yielding only finitely many exponent patterns. □

Three corollaries follow immediately. For $a = 1$, the relation $\sigma(n) \mid n + 1$ holds only for $n = 1$ or for prime n, since $\sigma(n) \geq n + 3$ for all composite n. For $a = 0$, divisibility reduces to $\sigma(n) = 2n$, identifying the perfect numbers. For $a < 0$, the inequality $\sigma(m) > m$ for all $m > 1$ precludes any further solutions. These exhaust the shifted divisor cases and confirm the completeness of the classification. The multiplicative functions $\varphi(n)$, $\lambda(n)$, and $\sigma(n)$ form a structural hierarchy under the relation $f(n) \mid n + a$ for a fixed integer a. The behavior of φ was examined systematically by Lehmer in 1932, the function λ was investigated in connection with Carmichael numbers and modular orders, and σ completes the progression through its total collapse of divisibility. This framework accords with the analytic development presented in [7] and the classical arithmetic foundations of [2,6]. The hierarchy summarized below represents the final gradient of multiplicative coherence.

Function	$a = 1$	$a \geq 2$	$a < 0$	Behavior
$\varphi(n)$	finite (Lehmer-type)	finite	infinite families if $\varphi(m) \mid m$	partial coherence
$\lambda(n)$	finite (Carmichael)	finite	infinite families if $\lambda(m) \mid m$	conditional coherence
$\sigma(n)$	$n = 1$, primes	finite	finite ($\sigma(m) \nmid m$ for $m > 1$)	complete collapse

Computational tests of the relation $\sigma(n) \mid n + a$ were performed for all $n \leq 10^6$ and for shifts $a \in \{2, 3, 5\}$. In every case, the results matched the theoretical predictions and no additional families appeared beyond the trivial or prime configurations. For $a = 2$ there are no composite solutions, for $a = 3$ only $n = 4$ satisfies the condition, and for $a = 5$ only $n = 3$ appears. The explicit small-shift solutions are: $a = 2$ gives $n \in \{1\}$, $a = 3$ gives $n \in \{1, 4\}$, and $a = 5$ gives $n \in \{1, 3\}$. These observations confirm the 2-adic wall and overflow constraints described in the lemmas, while supporting the finiteness condition established by the exceptional schema. All computations employed standard multiplicative-function algorithms following the methods outlined in [2]. Code reproducing these results may be included as a supplementary appendix for independent verification.

3. Discussion and Future Directions

The classification of shifted divisibility for $\sigma(n) \mid n + a$ completes the sequence initiated with the corresponding analyses of $\varphi(n)$ and $\lambda(n)$. Together, these three functions establish a clear gradient of multiplicative coherence. The totient function φ admits partial infinitude, the Carmichael function λ displays conditional limitation, and the divisor function σ collapses entirely to finiteness. This hierarchy mirrors a progression from structural freedom to constraint, marking σ as the endpoint of arithmetic self-consistency under additive shifts. A comparable coherence gradient appears in analytic models of multiplicative entropy, where growth constraints produce analogous collapses in divisor systems [11]. The distinction among these functions arises from their comparative growth rates. For all $n > 1$, $\varphi(n) \leq n$ and $\lambda(n) \leq \varphi(n)$, whereas $\sigma(n) > n$. The first two allow periodic or recursive behavior through divisibility of suborders, while the third exceeds its argument and therefore saturates the divisibility condition. In this sense, the σ-function represents an entropy analogue within arithmetic, its superlinear expansion extinguishes internal repetition, leaving only isolated solutions. This observation provides a conceptual closure for the trilogy, framing divisibility as a function of growth rather than symmetry.

The comparison table presented earlier encapsulates this entire spectrum. The transition from partial to conditional to complete coherence corresponds to the collapse of infinite divisibility families

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into finite residue structures. Within this view, the arithmetic hierarchy forms a continuum of stability, culminating in $\sigma(n)$, where coherence becomes absolute and motion within the divisibility lattice ceases. Future investigations may extend these methods to other multiplicative functions. A natural candidate is the divisor-counting function $\tau(n)$, whose discrete jumps create valuation structures similar to those found in σ. Generalized divisor sums $\sigma_k(n)$ also offer potential insight, where the parameter k modulates the rate of growth and may interpolate between partial and complete collapse. Such analyses could determine whether the observed coherence gradient extends beyond the classical trio of φ, λ, and σ. At a broader level, the finiteness of $\sigma(n) \mid n + a$ signifies a terminal boundary within multiplicative arithmetic. The absence of infinite families under any shift a implies that the divisor function enforces an intrinsic upper limit on arithmetic self-compatibility. The conclusion therefore defines both a technical and conceptual terminus. The point at which multiplicative structure, once free to replicate under φ and λ, becomes self-limiting under σ. This marks the saturation point of multiplicative coherence and completes the intended classification of shifted divisibility.

4. Conclusions

The study of the divisibility relation $\sigma(n) \mid n + a$ establishes the terminal boundary of multiplicative coherence among the classical arithmetic functions φ, λ, and σ. Through valuation bounds, density exhaustion, and radical constraints, the analysis confirms that for every fixed $a \geq 2$ only finitely many integers n satisfy the relation. This result completes the coherence gradient that begins with partial infinitude under φ, proceeds through conditional limitation under λ, and ends with complete collapse under σ. The outcome defines a structural limit within multiplicative number theory. Because $\sigma(n) > n$ for all $n > 1$, no further infinite families can arise under additive shift, and divisibility symmetry reaches saturation. The transition from φ to λ to σ thus traces the full progression from expansion to restriction, concluding at the point where arithmetic structure becomes self-limiting. This marks the endpoint of the trilogy and the completion of the classification of shifted divisibility.

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