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An Atlas of Epsilon-Delta Continuity Proofs in Function Space $F(\mathbb{R}, \mathbb{R})$

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Abstract

The ϵ-δ definition of continuity, foundational to real analysis, continues to attract sustained interest across the mathematics community. This paper responds to this interest by providing a comprehensive, systematic survey of direct ϵ-δ proofs for a broad spectrum of functions. We meticulously analyze 54 prominent real-valued functions, categorizing them into eight distinct clusters to highlight recurring proof structures and methodologies. For each function, we present a step-by-step proof alongside an explicit formula for δ in terms of ϵ and the point of continuity. Beyond serving as a robust pedagogical resource for students, instructors, and independent learners, this collection demystifies the proof-writing process by showcasing the elegant, unifying logic underlying a seemingly diverse set of problems. The work's organized structure and detailed examples offer clarity where confusion often resides, ultimately fostering a deeper intuition for the core principles of continuity. By transforming a collection of challenging proofs into an accessible and navigable reference, this atlas opens up new avenues for further research in the field for mathematicians.

Keywords: primary: continuity; secondary: tutorial; survey; continuous map; singular function; differential calculus

MSC: Primary: 26A15; Secondary: 26-01; 26-02; 54C05; 26A30; 97I40

It is self-evident that any and all paths must be open to a researcher during the actual course of his [or her] investigations.Karl T. W. Weierstrass (1815—1897)

1. Introduction

The Introduction is structured as follows. Section 1.1 reviews the historical development of ϵ–δ continuity proofs; Section 1.2 presents an up-to-date taxonomy of such proofs; Section 1.3 articulates the motivation and scope of this Atlas; and Section 1.4 concludes the Introduction with a brief roadmap of the paper.

1.1. Historical Background

Early ideas about limits, traceable to Greek mathematicians such as Eudoxus of Cnidus (c. 390–c. 340 BCE) and Archimedes of Syracuse (c. 287–c. 212 BCE), were effective but not rigorously formulated. In the seventeenth century, Sir Isaac Newton (1643–1727) and Gottfried W. Leibniz (1646–1716) developed calculus with the aid of infinitesimals—powerful heuristic quantities that lacked a precise foundation. The demand for rigor intensified in the nineteenth century, as authors such as Augustin-Louis B. Cauchy (1789–1857) sought to recast calculus within the framework of analysis rather than geometry. Bernard P.J.N. Bolzano (1781–1848) sketched ideas akin to the ϵ–δ method as early as 1817, but his work remained largely unnoticed for decades. The decisive step was taken by Karl T.W. Weierstrass (1815–1897), whose mid-nineteenth-century lectures introduced the formal (ϵ, δ) definition of limit and supplied a robust arithmetical basis for continuity and its proofs. The notation was consolidated only later: while variants of "lim" appeared earlier, the now-standard placement of the arrow in $\lim_{x \to x_0}$ was popularized by Godfrey H. Hardy (1877–1947) in 1908 [1–5].

1.2. Taxonomy

To ensure the reader has a clear, structured overview of the material to be presented in this "atlas" of proofs we review the following comprehensive list of taxonomy in epsilon -delta proofs [6,7]:

1. **Function family (what is f?).** Linear/affine; power and root x^r (with domain caveats); polynomials; rational (poles vs. removable points distinguished); exponential and logarithmic; trigonometric and inverse trigonometric; hyperbolic and inverse hyperbolic; absolute value and max / min; piecewise/defined-by-cases; and selected "pathological" exemplars (Thomae, Weierstrass, Cantor function) where local continuity is instructive.
2. **Point type (where is continuity checked?).** Interior points; boundary points of the domain (one-sided neighborhoods); removable singularities (limit exists and can be defined); and, for contrast, essential/jump points when documenting discontinuity.
3. **Proof structure (how do we construct $\delta(\epsilon, x_0)$?).** The scratch-work $\Rightarrow$ formal-proof pipeline; backwards–forwards organization; factor–bound–squeeze schemes (e.g., factor $|x - x_0|$, bound a cofactor by C, set $\delta = \min\{\delta_0, \epsilon/C\}$); squeeze and triangle-inequality templates; and contradiction patterns for discontinuity.
4. **Quantitative strength of continuity.** Pointwise continuity ($\delta = \delta(\epsilon, x_0)$); uniform continuity on a set ($\delta = \delta(\epsilon)$); Lipschitz/Hölder cases (explicit δ such as $\delta = \epsilon/L$); and modulus of continuity ω with $|f(x) - f(y)| \leq \omega(|x - y|)$.
5. **Bounding toolkits (reusable inequalities).** Algebraic: $||x| - |y|| \leq |x - y|$, $|x + y| \leq |x| + |y|$; power/Bernoulli-type bounds for x^r; trigonometric bounds $|\sin t| \leq |t|$, $1 - \cos t \leq t^2/2$; elementary exponential/logarithmic bounds; rational bounds via denominator lower bounds on punctured neighborhoods.
6. **Dependence of δ on parameters.** (i) $\delta(\epsilon)$ on a set (uniform continuity); (ii) $\delta(\epsilon, x_0)$ (continuity at x_0 only); (iii) parameterized families $f(\cdot;\theta)$: continuity in x at fixed θ vs. joint continuity in (x, θ).
7. **Topological reformulations.** ϵ–δ $\Longleftrightarrow$ preimage of open sets is open; sequential characterization ($x_n \to x_0 \Rightarrow f(x_n) \to f(x_0)$). Recorded for completeness; the atlas emphasizes constructive δ.
8. **Role of the function space $F(\mathbb{R}, \mathbb{R})$.** Continuity-preserving operators: finite sums/products/scalar multiples/quotients (away from zeros), composition, max / min, absolute value; compact-set principles (e.g., uniform limits preserve continuity); and metrics/topologies on $F(\mathbb{R}, \mathbb{R})$ (pointwise vs. uniform) to situate when a single $\delta(\epsilon)$ exists. General theorems/schemata are stated here and instantiated by the examples.
9. **Piecewise construction and junction analysis.** (a) Continuity on each piece, (b) existence of one-sided limits at breakpoints, and (c) agreement with the function value.
10. **Discontinuity taxonomy (for contrast).** Removable, jump, and essential discontinuities, including oscillatory counterexamples, to delineate the scope of the methods.

1.3. Motivation and Scope

This *Atlas* addresses a recurring learning obstacle in early real analysis: students often grasp the idea of continuity but struggle to execute rigorous ϵ–δ arguments in practice. Our aim is to provide a structured collection of direct proofs in the function space $F(\mathbb{R}, \mathbb{R})$ that (i) minimize technical overhead, (ii) employ only basic algebra and a short list of standard equalities and inequalities, and (iii) foreground the ϵ–δ definition so that points of genuine difficulty are visible rather than hidden behind broad theorems. The hope is the result to be a curated, self-contained pedagogical-research reference that complements standard textbooks, supports independent study, offers classroom-ready exemplars for instructors, and—by way of a closing section on open problems—engages research mathematicians.

Audience and Objectives.

The Atlas is designed for four overlapping audiences: students in their first rigorous analysis course, instructors seeking transparent classroom demonstrations, independent learners, and research

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mathematicians. Across these groups, our objectives are: (i) to avoid unnecessarily lengthy derivations while preserving full rigor; (ii) to standardize bounding strategies via a small, reusable toolkit of equalities and inequalities; (iii) to highlight how the choice of method influences continuity claims in $F(\mathbb{R}, \mathbb{R})$; (iv) to build confidence through carefully scaffolded, direct ϵ–δ constructions; and (v) to articulate concise research questions that naturally emerge from the techniques and limitations exhibited by the proofs.

Benefits for Students.

As a supplement to a primary textbook, the Atlas offers a practical pathway from definition to proof:

1. **Direct method first.** Each argument begins with the ϵ–δ definition, revealing where the core difficulty arises and how to neutralize it.
2. **Minimal prerequisites.** Proofs rely on basic algebra and a concise catalogue of equalities and inequalities.
3. **Reusable schemata.** Standard proof schemes and patterns (e.g., factor-and-bound, local uniform bounds on cofactors, and "$\min\{\delta_0, \epsilon/C\}$" closures) are made explicit and portable across examples.
4. **Confidence building.** Worked examples progress in difficulty, helping students transition from imitation of templates to independent problem solving.
5. **Centralized reference.** Results that are typically scattered across chapters and sources are presented in one place for quick comparison and review.

Benefits for Instructors.

For course designers and lecturers, the Atlas functions as a ready-to-deploy teaching resource:

1. **Classroom-ready exemplars.** Short, fully rigorous proofs suitable for board work or slides, with clearly marked bounding steps.
2. **Varied demonstrations.** Multiple routes to a result (algebraic techniques, alternative equalities, alternative inequalities, or different local bounds) support adaptive instruction.
3. **Assessment alignment.** The proofs decompose naturally into graded checkpoints (e.g., problem framing, bounding choice, $\delta(\epsilon, x_0)$ construction, and quantifier closure).
4. **Curricular integration.** Side-by-side treatments under different methods and perspectives (e.g., pointwise vs. uniform) make abstract distinctions concrete.
5. **Economy of time.** Concise arguments minimize in-class derivation length without sacrificing logical completeness.

Benefits for Independent Learners.

For readers pursuing self-study outside a formal course, the Atlas is organized to support disciplined, incremental mastery:

1. **Self-paced progression.** Topics are sequenced from elementary to advanced, with clearly marked prerequisites and optional extensions.
2. **Diagnostic checkpoints.** Each proof highlights the key steps and decision points (e.g., choice of cofactor g, selection of bounds, calibration of C) to encourage metacognitive practice.
3. **Template portability.** Reusable schemata (e.g., factor-and-bound, local bounding, $\min\{\delta_0, \varepsilon/C\}$ closure) are summarized for adaptation to new functions.
4. **Minimal toolkit.** Only basic algebra and a compact list of equalities and inequalities are assumed, lowering the barrier to independent entry.
5. **Cross-Scheme insight.** Parallel treatments under different proof schemes and methods promote conceptual transfer beyond single-problem techniques.

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Benefits for Mathematicians.

Beyond pedagogy, the Atlas surfaces methodological and structural questions:

1. **Technique extraction.** The proofs isolate recurring schemes, steps and bounding moves that may generalize to broader and more abstract classes in $F(\mathbb{R},\mathbb{R})$ and related spaces.
2. **Metric sensitivity.** Side-by-side comparisons under distinct methods and perspectives highlight where continuity is stable and where it fails, suggesting avenues for characterization results.
3. **Problem generation.** Tight $\delta(\epsilon, x_0)$ bounds, extremal examples, and borderline cases naturally motivate conjectures recorded in the open-problems section.

Pedagogical Benefits.

The Atlas advances pedagogical practice by enforcing clarity at each stage of an ϵ–δ proof: the type of proof scheme, the steps and type of applied auxiliary equalities and inequalities. By standardizing these moves, the Atlas reduces cognitive load, exposes the transferable structure of continuity arguments, and lowers barriers to entry for learners who might otherwise be discouraged by fragmented or overly abstract presentations. As a curated collection, it fills an accessibility gap between terse textbook treatments and sprawling online materials.

Research Component and Open Problems.

To underscore that this Atlas is a *pedagogical–research* contribution, it concludes with a section on *Open Research Problems*. These problems crystallize from the techniques and limitations revealed by the proofs. The intent is to invite further investigation and subsequent publications, thereby connecting classroom clarity with active research directions.

1.4. Outline of the Paper

The paper is organized as follows. Section 2 reviews the necessary preliminaries from college algebra and introductory real analysis that are used in subsequent sections. Section 3 presents the main results: direct ϵ–δ proofs establishing the stability of continuity under finite algebraic operations and proofs across eight clusters of real-valued functions commonly treated in real analysis courses. Section 4 concludes with a discussion of the results, common pitfalls in ϵ–δ arguments, and directions for further advanced research.

2. Preliminaries

Readers who have studied the key topics of college Algebra and elementary real analysis are well-equipped with the following notations, definitions, and results in the areas of "algebraic comparisons" and “continuity” [8–11].

2.1. Foundations

2.1.1. Definitions

In this subsection we fix notation and recall the classical ϵ–δ formulation of continuity for real-valued functions. Throughout, we emphasize that the admissible δ may depend on both ϵ and the base point x_0 [8,9].

Definition 1 (Function Space $F(\mathbb{R},\mathbb{R})$)**.** *Let $\mathbb{R}$ denote the set of real numbers. Then, the set of all functions $F : \mathbb{R} \to \mathbb{R}$ is called the function space $F(\mathbb{R},\mathbb{R})$. This space, equipped with conventional addition $+$ and scalar multiplication $\cdot$ constitutes a vector space.*

Remark 1. *To ensure consistency on domains of the discussed functions, for any real-valued function f with domain $D_f \subsetneq \mathbb{R}$ we extend its definition to entire $\mathbb{R}$ by defining $f(x) = 0$ for all $x \in \mathbb{R} - D_f$, but our focus on its continuity points remains on its points in its original defined domain D_f.*

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Definition 2 (Point Continuity). *Let $f \in F(\mathbb{R}, \mathbb{R})$ and $x_0 \in D_f \subset \mathbb{R}$. Then, f is said to be continuous at $x = x_0$ denoted by $\lim_{x \to x_0} f(x) = f(x_0)$ if and only if for any given "output tolerance" $\epsilon > 0$ there exist an "input tolerance" $\delta = \delta(\epsilon, x_0) > 0$ such that for all $x \in \mathbb{R}$ if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon$. In mathematical logic language:*

$$\forall \epsilon > 0 \ \exists \delta = \delta(\epsilon, x_0) > 0 \ \forall x \in \mathbb{R}(|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon). \tag{1}$$

Remark 2. *In the definition 2, the notation $\delta = \delta(\epsilon, x_0)$ means that δ depends to the given (ϵ, x_0) rather than being a function of it. Trivially, any $\delta_* < \delta$ satisfies the condition "$|x - x_0| < \delta$" of continuity upon validity of the statement (1).*

Remark 3. *In the definition 2, without loss of generality we may assume $\epsilon > 0$ is sufficiently small enough. Trivially, when the statement (1) holds for shrunken $0 < \epsilon_* < \epsilon$, it holds for original epsilon too.*

Remark 4. *The contrary to the definition 2 is the concept of discontinuity. Here, f is said to be discontinuous at $x = x_0$ denoted by $\lim_{x \to x_0} f(x) \neq f(x_0)$ if and only if:*

$$\exists \epsilon > 0 \ \forall \delta = \delta(\epsilon, x_0) > 0 \ \exists x \in \mathbb{R}(|x - x_0| < \delta \ \& \ |f(x) - f(x_0)| \geq \epsilon). \tag{2}$$

Definition 3. *Let $f \in F(\mathbb{R}, \mathbb{R})$ with its original definition domain $D_f \subset \mathbb{R}$. Then, the set of all continuities of f is denoted by $C(f)$, the set of discontinuities of denoted by $D(f)$ and the set of roots of f are defined by, respectively:*

$$C(f) := \{x_0 \in D_f \subset \mathbb{R} | \lim_{x \to x_0} f(x) = f(x_0)\}, \tag{3}$$

$$D(f) := \{x_0 \in D_f \subset \mathbb{R} | \lim_{x \to x_0} f(x) \neq f(x_0)\}, \tag{4}$$

$$R(f) := \{x_0 \in D_f \subset \mathbb{R} | f(x_0) = 0\}. \tag{5}$$

Remark 5. *The set $C(f)$ in the Definition 3 is either empty, non-empty finite, denumerable, or uncountable.*

Definition 4 (Dirichlet-type functions). *Let $f : \mathbb{R} \to \mathbb{R}$. We classify f according to the size of $C(f)$ and $D(f)$ as follows:*

1. ***Type I Dirichlet function:*** $|C(f)| = 0$ *and* $|D(f)| = \mathfrak{c}$.
2. ***Type II Dirichlet function:*** $|C(f)| = n$*(for some* $n \in \mathbb{N}$*) and* $|D(f)| = \mathfrak{c}$.
3. ***Type III Dirichlet function:*** $|C(f)| = \aleph_0$ *and* $|D(f)| = \mathfrak{c}$.
4. ***Type IV Dirichlet function:*** $|C(f)| = \mathfrak{c}$ *and* $|D(f)| = \mathfrak{c}$.

2.1.2. Proof Workflow and Schemes

Remark 6 (Proof Workflow). *A step-by-step guide on how to approach an epsilon-delta proof, includes the typical workflow:*

1. *Given an $\epsilon > 0$, find a $\delta = \delta(\epsilon, x_0) > 0$.*
2. *Start with the inequality $|f(x) - f(x_0)| < \epsilon$ and work backward to find a relationship between $|x - x_0|$ and ϵ.*
3. *Show that choosing δ based on this relationship guarantees the original inequality holds.*

Remark 7 (Proof Scheme I). *The first useful process to prove statement (1) in Definition 2 is to begin with $|f(x) - f(x_0)|$ and bound it by a factor of $|x - x_0|$: find a function $g(x, x_0)$ such that*

$$|f(x) - f(x_0)| \leq |x - x_0| \, |g(x, x_0)|.$$

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Choose $\delta_0 > 0$ and a constant $C > 0$ so that $|g(x, x_0)| < C$ whenever $|x - x_0| < \delta_0$. Then, for any $\epsilon > 0$, if $|x - x_0| < \min\{\delta_0, \epsilon/C\}$ we have

$$|f(x) - f(x_0)| \leq |x - x_0|\,|g(x, x_0)| < \left(\frac{\epsilon}{C}\right)C = \epsilon, \tag{6}$$

which verifies the ϵ–δ condition. Hence we may take:

$$\delta = \delta(\epsilon, x_0) = \min\{\delta_0, \tfrac{\epsilon}{C}\}. \tag{7}$$

Remark 8 (Proof Scheme II)**.** *The second useful process to prove statement (1) in the definition 2 is to start with the inequality including the expression $|f(x) - f(x_0)|$ and find its equivalent inequality including expression $|x - x_0|$ that is for some positive bi-variate functions g, h we have:*

$$|f(x) - f(x_0)| < \epsilon \quad \Longleftrightarrow \quad -g(\epsilon, x_0) < x - x_0 < h(\epsilon, x_0) \tag{8}$$

Now, using Symmetric-anti Symmetric inequality (29) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{g(\epsilon, x_0), h(\epsilon, x_0)\}. \tag{9}$$

Remark 9 (Proof Scheme III)**.** *The third useful process to prove statement (1) in the definition 2 is to present the function f as composite $f = f_2 \circ f_1$ where the $\epsilon - \delta$ information on the individual function f_2 and f_1 in already known. Here, for given $\epsilon > 0$ since f_2 is continuous at $y = f_1(x_0)$ there is $\eta = \eta(\epsilon, f_1(x_0)) > 0$ such that the statement (1) holds. Furthermore, for the found $\eta > 0$ since the function f_1 is continuous at $x = x_0$ there is a $\delta = \delta(\eta, x_0) > 0$ such that the statement (1) holds too. Substituting of the value of $\eta > 0$, it follows that for the case of composition function f it is sufficient to consider:*

$$\delta = \delta(\eta(\epsilon, f_1(x_0)), x_0). \tag{10}$$

2.2. Auxiliary Equalities and Inequalities

The following list of identities and inequalities play a prominent role in after-mentioned Proof Schema I and Proof Schema II in Remarks 7-8 [10,11].

Notation. Given a sequence of reals $(x_k)_{k=0}^{+\infty}$. We define $\prod_{k=u}^{v} x_k = 1 (u > v)$, and $\sum_{k=u}^{v} x_k = 0^+, (u > v)$.

2.2.1. Auxiliary Equalities

We record several identities used throughout the paper.

Lemma 1 (Algebraic and trigonometric identities)**.** *Let $\mathbb{R}$ denote the real numbers. Then:*

1. ***General Product Decomposition (GPD).*** *For any integer $n \geq 2$ and elements $x_1, \dots, x_n, y_1, \dots, y_n \in \mathbb{R}$,*

$$\prod_{k=1}^{n} y_k - \prod_{k=1}^{n} x_k = \sum_{k=1}^{n}\left(\left(\prod_{i=1}^{k-1} y_i\right)(y_k - x_k)\left(\prod_{i=k+1}^{n} x_i\right)\right). \tag{11}$$

2. ***Difference of Powers Factorization (DPF).*** *For any $x, y \in \mathbb{R}$ and integer $n \geq 1$,*

$$x^n - y^n = (x - y)\sum_{k=0}^{n-1} x^{n-1-k} y^k. \tag{12}$$

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3. ***Max/min Recursive.*** *For any* $x_1, \cdots, x_{n-1}, x_n \in \mathbb{R}$,

$$\max\{x_1, \cdots, x_{n-1}, x_n\} = \max\{\max\{x_1, \cdots, x_{n-1}\}, x_n\}, \tag{13}$$
$$\min\{x_1, \cdots, x_{n-1}, x_n\} = \min\{\min\{x_1, \cdots, x_{n-1}\}, x_n\}. \tag{14}$$

4. ***Max/min via absolute value.*** *For any* $x, y \in \mathbb{R}$,

$$\max\{x, y\} = \frac{x + y + |x - y|}{2}, \tag{15}$$
$$\min\{x, y\} = \frac{x + y - |x - y|}{2}. \tag{16}$$

5. ***Trigonometric subtraction (sum–to–product).*** *For any* $x, y \in \mathbb{R}$,

$$\sin(x) - \sin(y) = +2\cos(\frac{x+y}{2})\sin(\frac{x-y}{2}), \tag{17}$$
$$\cos(x) - \cos(y) = -2\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}), \tag{18}$$
$$\tan(x) - \tan(y) = +(\cos(x)\cos(y))^{-1}\sin(x - y), \tag{19}$$
$$\cot(x) - \cot(y) = -(\sin(x)\sin(y))^{-1}\sin(x - y), \tag{20}$$
$$\sec(x) - \sec(y) = +2(\cos(x)\cos(y))^{-1}\sin(\frac{x+y}{2})\sin(\frac{x-y}{2}), \tag{21}$$
$$\csc(x) - \csc(y) = -2(\sin(x)\sin(y))^{-1}\cos(\frac{x+y}{2})\sin(\frac{x-y}{2}). \tag{22}$$

6. ***Hyperbolic subtraction properties.*** *For any* $x, y \in \mathbb{R}$,

$$\sinh(x - y) = \sinh(x)\cosh(y) - \cosh(x)\sinh(y), \tag{23}$$
$$|\sinh(x - y)| = \sinh(|x - y|). \tag{24}$$

2.2.2. Auxiliary Inequalities

We record several inequalities used throughout the paper.

Lemma 2 (Inequalities used in the sequel). *Let* $n \in \mathbb{N}$, *let* $(x_k)_{k=1}^{n} \subset \mathbb{R}$, *and let* $x, y \in \mathbb{R}$. *Then:*

1. ***Generalized triangle inequality.***

$$\Big|\sum_{k=1}^{n} x_k\Big| \leq \sum_{k=1}^{n} |x_k|. \tag{25}$$

2. ***Two–Sided Bernoulli Power Bounds(TSBPB).*** *For any* $x, y \geq 0$ *and* $r \in \mathbb{R}$,

$$0 \leq r \leq 1: \quad r\,x^{r-1}(x - y) \leq x^r - y^r \leq r\,y^{r-1}(x - y), \tag{26}$$
$$r \leq 0 \text{ or } r \geq 1: \quad r\,x^{r-1}(x - y) \geq x^r - y^r \geq r\,y^{r-1}(x - y). \tag{27}$$

3. ***Reverse triangle inequality.*** *For any* $x, y \in \mathbb{R}$,

$$\big||x| - |y|\big| \leq |x - y|. \tag{28}$$

4. ***Symmetric-anti Symmetric inequality.*** *For any* $x, y \in \mathbb{R}, \delta_1, \delta_2 > 0$, *and* $|x - y| < \min\{\delta_1, \delta_2\}$,

$$-\delta_1 < x - y < \delta_2. \tag{29}$$

5. ***Chord–arc bound for sine.*** *For any* $x, y \in \mathbb{R}, d \in [0, 1]$

$$|\sin(x - y)| \leq |x - y|^d. \tag{30}$$

6. ***Distance reverse triangle inequality*** **.** *For any $x, y \in \mathbb{R}$,*

$$|1.7em(x,\mathbb{Z}) - 1.7em(y,\mathbb{Z})| \;\leq\; |x-y|. \tag{31}$$

3. Main Results

3.1. Stability of Continuity under Algebraic Operations

In real analysis, algebraic operations on continuous functions are typically established by first giving a direct ϵ–δ proof for the binary case $n = 2$, and then extending the *statement* to general n by mere mathematical induction, while the underlying proof method is left implicit [7,12–15]. In contrast, in this section we work directly with ϵ–δ estimates for a finite family of functions, proving the continuity of finite sums, finite products, and finite compositions without invoking induction on n. This perspective makes explicit the quantitative bounds (shared δ's, uniform envelopes, and cascaded tolerances) that drive the arguments. Finally, since naively passing from "finite" to "infinite" operations is not valid, we provide counterexamples showing why the finiteness assumptions are essential: infinite sums, products, iterated compositions or maximum (minimum) of continuous functions may fail to define a real-valued function at a point, and even when defined, need not be continuous.

Theorem 1. *Let $f_k : \mathbb{R} \to \mathbb{R}$ $(1 \leq k \leq n < +\infty)$ be real-valued functions, and let $x_0 \in \mathbb{R}$. If each f_k is continuous at x_0, then the finite sum $S_n(x) = \sum_{k=1}^n f_k(x)$ is continuous at x_0; that is, $\lim\limits_{x\to x_0} \sum_{k=1}^n f_k(x) = \sum_{k=1}^n f_k(x_0)$.*

Proof. Given $\epsilon > 0$. Then, by assumption and Definition 2 there are $\delta_k = \delta_k(\frac{\epsilon}{n}, x_0) > 0$ $(1 \leq k \leq n)$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_k$, then $|f_k(x) - f_k(x_0)| < \frac{\epsilon}{n}$. Now, define:

$$\delta = \delta(\epsilon, x_0) = \min_{1\leq k\leq n} \{\delta_k(\frac{\epsilon}{n}, x_0)\}. \tag{32}$$

Then, we have:

$$\forall k \in \mathbb{N}_n \forall x \in \mathbb{R}(|x - x_0| < \delta \to |f_k(x) - f_k(x_0)| < \frac{\epsilon}{n}). \tag{33}$$

Next, using the General triangle inequality (25), for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta$, it follows that:

$$|S_n(x) - \sum_{k=1}^n f_k(x_0)| = |\sum_{k=1}^n (f_k(x) - f_k(x_0))| \leq \sum_{k=1}^n |(f_k(x) - f_k(x_0))| < n(\frac{\epsilon}{n}) = \epsilon. \tag{34}$$

This completes the proof. □

Counterexample 3.1. *In Theorem 1, the condition of finite n is necessary. To see the necessity of this restriction, it is sufficient to consider $f_k(x) = \dfrac{(1+|x|)}{k}$, $(1 \leq k < +\infty)$. As the harmonic series is divergent, it follows that $S_{+\infty}$ given by $S_{+\infty}(x) = +\infty$ (for all $x \in \mathbb{R}$) is not a real-valued function and hence discussing its continuity is irrelevant.*

Theorem 2. *Let $f_k : \mathbb{R} \to \mathbb{R}$ $(1 \leq k \leq n)$ be real-valued functions, and let $x_0 \in \mathbb{R}$. If each f_k is continuous at x_0, then the finite product $P_n(x) = \prod_{k=1}^n f_k(x)$ is continuous at x_0; that is, $\lim\limits_{x\to x_0} \prod_{k=1}^n f_k(x) = \prod_{k=1}^n f_k(x_0)$.*

Proof. The proof of the theorem is accomplished in three steps as follows:
Step 1. (Uniform Bound Existence) Let $\epsilon_1 = 1$. Then, by assumption and Definition 2 there are $\delta_{1,k} =$

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$\delta_{1,k}(1,x_0) > 0$ $(1 \leq k \leq n)$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_{1,k}$, then $|f_k(x) - f_k(x_0)| < 1$. But, the latest statement yields $|f_k(x)| < \max\{|f_k(x_0) - 1|, |f_k(x_0) + 1|\} \leq |f_k(x_0)| + 1$. Now, define:

$$\delta_{1,*} = \delta_{1,*}(1,x_0) = \min_{1\leq k\leq n} \{\delta_{1,k}(1,x_0)\}, \tag{35}$$

and $C_1 = \max\limits_{1\leq k\leq n} \{|f_k(x_0)| + 1\}$. Then, we have:

$$\forall k \in \mathbb{N}_n \forall x \in \mathbb{R}(|x - x_0| < \delta_{1,*} \to |f_k(x)| < C_1). \tag{36}$$

Step 2. (Uniform Approximation Existence) Let $\epsilon_2 = \dfrac{\epsilon}{nC_1^{n-1}}$ $(\epsilon > 0)$. Again, by assumption and Definition 2 there are $\delta_{2,k} = \delta_{2,k}(\dfrac{\epsilon}{nC_1^{n-1}}, x_0) > 0$ $(1 \leq k \leq n)$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_{2,k}$, then $|f_k(x) - f_k(x_0)| < \dfrac{\epsilon}{nC_1^{n-1}}$. Now, define:

$$\delta_{2,*} = \delta_{2,*}(\epsilon, x_0) = \min_{1\leq k\leq n} \{\delta_{2,k}(\frac{\epsilon}{nC_1^{n-1}}, x_0)\}. \tag{37}$$

Then, we have:

$$\forall k \in \mathbb{N}_n \forall x \in \mathbb{R}(|x - x_0| < \delta_{2,*} \to |f_k(x) - f_k(x_0)| < \frac{\epsilon}{nC_1^{n-1}}). \tag{38}$$

Step 3. GPD Application Let $\epsilon > 0$. Take $\delta_* = \min\{\delta_{1,*}, \delta_{2,*}\}$ as defined in Step 1 and Step 2 and given by:

$$\delta = \delta(\epsilon, x_0) = \min_{1\leq k\leq n} \{\delta_{1,k}(1,x_0), \delta_{2,k}(\frac{\epsilon}{nC_1^{n-1}}, x_0)\} : \ C_1 = \max_{1\leq k\leq n} \{|f_k(x_0)| + 1\}. \tag{39}$$

Finally, by an application of GPD (11), for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_*$, it follows that:

$$\begin{aligned}
|P_n(x) - \prod_{k=1}^{n} f_k(x_0)| &= |\prod_{k=1}^{n} f_k(x) - \prod_{k=1}^{n} f_k(x_0)| \\
&= |\sum_{k=1}^{n} \left(\left(\prod_{i=1}^{k-1} f_i(x) \right) (f_k(x) - f_k(x_0)) \left(\prod_{i=k+1}^{n} f_i(x_0) \right) \right)| \\
&\leq \sum_{k=1}^{n} \left(\left(\prod_{i=1}^{k-1} |f_i(x)| \right) (|f_k(x) - f_k(x_0)|) \left(\prod_{i=k+1}^{n} |f_i(x_0)| \right) \right) \\
&\leq \sum_{k=1}^{n} |(f_k(x) - f_k(x_0))| C_1^{n-1} \\
&< n(\frac{\epsilon}{nC_1^{n-1}}) C_1^{n-1} = \epsilon.
\end{aligned} \tag{40}$$

This completes the proof. □

Counterexample 3.2. *In Theorem 2, the condition of finite n is necessary. To see the necessity of this restriction, it is sufficient to consider $f_k(x) = (k + |x|)$, $(1 \leq k < +\infty)$. As the factorial function diverges to infinity, it follows that $P_{+\infty}$ given by $P_{+\infty}(x) = +\infty$ (for all $x \in \mathbb{R}$) is not a real-valued function and hence discussing its continuity is irrelevant.*

Exercise 3.3. *A modification of above proofs can be applied for the epsilon-delta continuity proof of the following special cases:*

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(a) *the Arithmetic Mean:* $AM_n(x) = \dfrac{\sum_{k=1}^{n} f_k(x)}{n}$ $(n \in \mathbb{N})$,

(b) *the Geometric Mean:* $GM_n(x) = (\prod_{k=1}^{n} f_k(x))^{\frac{1}{n}}$ $(f_k(x) \geq 0, n \in \mathbb{N})$,

(c) *the Harmonic Mean:* $HM_n(x) = \dfrac{1}{\frac{1}{n}\sum_{k=1}^{n} \frac{1}{f_k(x)}}$ $(f_k(x) \neq 0, n \in \mathbb{N})$.

Theorem 3. *Let $f_k : \mathbb{R} \to \mathbb{R}$ $(1 \leq k \leq n)$ be real-valued functions, and let $x_0 \in \mathbb{R}$. If each f_k is continuous at $f_{k-1}(x_0)$ with $f_0(x_0) = x_0$ $(1 \leq k \leq n)$, then the finite composition $C_n(x) = (\bigcirc_{k=1}^{n} f_k)(x)$ is continuous at x_0; that is, $\lim_{x \to x_0} (\bigcirc_{k=1}^{n} f_k)(x) = (\bigcirc_{k=1}^{n} f_k)(x_0)$.*

Proof. We apply the cascade epsilon-delta method as follows. Since f_k $(1 \leq k \leq n)$ is continuous at $(f_{k-1} \circ \cdots \circ f_1)(x_0)$, for any given $\eta_k > 0$ there exists $\eta_{k-1} = \eta_{k-1}(\eta_k, (f_{k-1} \circ \cdots \circ f_1)(x_0)) > 0$ such that:

$$\forall y \in \mathbb{R}(|y - (f_{k-1} \circ \cdots \circ f_1)(x_0)| < \eta_{k-1} \quad \to \quad |f_k(y) - f_k((f_{k-1} \circ \cdots \circ f_1)(x_0))| < \eta_k). \tag{41}$$

Now, let $\epsilon = \eta_n > 0$ be given. Proceeding backward in this way, we obtain tolerances $\epsilon = \eta_n, \eta_{n-1}, \ldots, \eta_{k-1}, \eta_{k-2}, \ldots, \eta_1$ and finally a $\delta > 0$ such that:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta \quad \to \quad |f_1(x) - f_1(x_0)| < \eta_1). \tag{42}$$

Hence, the by considering cascade of implications and $\delta > 0$ given by:

$$\delta = \delta(\epsilon, x_0) = \delta(\eta_1(\eta_2(\eta_3(\cdots(\eta_{n-1}(\epsilon, (f_{n-1} \circ \cdots \circ f_2 \circ f_1)(x_0)))\cdots), f_2 \circ f_1(x_0)), f_1(x_0)), x_0) \tag{43}$$

it follows that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta$ we obtain $|C_n(x) - C_n(x_0)| < \epsilon$. This completes the proof. □

Counterexample 3.4. *In Theorem 3, the condition of finite n is necessary. To see the necessity of this restriction, it is sufficient to consider $f_k(x) = |x|^{\frac{1}{k}}$, $(1 \leq k < +\infty)$. As the factorial function diverges to infinity, it follows that $C_{+\infty}$ given by $C_{+\infty}(x) = 1_{\{|x| \geq 1\}}(x)$ (for all $x \in \mathbb{R}$) is not a continuous function at $x = 1$.*

Exercise 3.5. *A similar proof to the above shows that for the continuous function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = (x^2 - 1).1_{[-1,1]}(x) + 1$, $C_n(x) = f^{\circ n}(x)$ is continuous at $x = \pm 1$ for any $n \in \mathbb{N}$, but not for $n = +\infty$.*

Theorem 4. *Let $f_k : \mathbb{R} \to \mathbb{R}$ $(1 \leq k \leq n < +\infty)$ be real-valued functions, and let $x_0 \in \mathbb{R}$. If each f_k is continuous at x_0, then the finite maximum $M_n(x) = \max_{1 \leq k \leq n} \{f_k(x)\}$ is continuous at x_0; that is, $\lim_{x \to x_0} \max_{1 \leq k \leq n} \{f_k(x)\} = \max_{1 \leq k \leq n} \{f_k(x_0)\}$.*

Proof. We prove the assertion in two steps as follows:

Step 1. (Upper Bound Existence) Fix $2 \le k \le n$. Then, for all $x \in \mathbb{R}$ using equation (13) , (15) and inequality (28) it follows that:

$$\begin{aligned}
|M_k(x) - M_k(x_0)| &= |\max\{M_{k-1}(x), f_k(x)\} - \max\{M_{k-1}(x_0), f_k(x_0)\}| \\
&= |\frac{M_{k-1}(x) + f_k(x) + |M_{k-1}(x) - f_k(x)|}{2} - \frac{M_{k-1}(x_0) + f_k(x_0) + |M_{k-1}(x_0) - f_k(x_0)|}{2}| \\
&= |\frac{(M_{k-1}(x) - M_{k-1}(x_0)) + (f_k(x) - f_k(x_0))}{2} + \frac{|M_{k-1}(x) - f_k(x)| - |M_{k-1}(x_0) - f_k(x_0)|}{2}| \\
&\le |\frac{(M_{k-1}(x) - M_{k-1}(x_0)) + (f_k(x) - f_k(x_0))}{2}| + |\frac{|M_{k-1}(x) - f_k(x)| - |M_{k-1}(x_0) - f_k(x_0)|}{2}| \\
&\le |\frac{(M_{k-1}(x) - M_{k-1}(x_0)) + (f_k(x) - f_k(x_0))}{2}| + |\frac{(M_{k-1}(x) - f_k(x)) - (M_{k-1}(x_0) - f_k(x_0))}{2}| \\
&\le |\frac{(M_{k-1}(x) - M_{k-1}(x_0)) + (f_k(x) - f_k(x_0))}{2}| + |\frac{(M_{k-1}(x) - M_{k-1}(x_0)) - (f_k(x) - f_k(x_0))}{2}| \\
&\le 2(\frac{|M_{k-1}(x) - M_{k-1}(x_0)| + |f_k(x) - f_k(x_0)|}{2}) \\
&= |M_{k-1}(x) - M_{k-1}(x_0)| + |f_k(x) - f_k(x_0)|, \ (2 \le k \le n) \qquad (44)
\end{aligned}$$

or equivalently:

$$|M_k(x) - M_k(x_0)| - |M_{k-1}(x) - M_{k-1}(x_0)| \le |f_k(x) - f_k(x_0)|. \ (2 \le k \le n) \tag{45}$$

Next, taking sums on both sides of inequality (45) over $(2 \le k \le n)$ it follows that:

$$|M_n(x) - M_n(x_0)| \le \sum_{k=1}^{n} |f_k(x) - f_k(x_0)|. \tag{46}$$

Step 2. (The delta Assessment) Given $\epsilon > 0$. Then, repeating the similar argument as in the proof of Theorem 1, it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min_{1 \le k \le n} \{\delta_k(\frac{\epsilon}{n}, x_0)\}. \tag{47}$$

This completes the proof. □

Counterexample 3.6. *In Theorem 4, the condition of finite n is necessary. To see the necessity of this restriction, it is sufficient to consider $f_k(x) = k + |x|$, $(1 \le k < +\infty)$. As the sequence of natural numbers is divergent, it follows that $M_{+\infty}$ given by $M_{+\infty}(x) = +\infty$ (for all $x \in \mathbb{R}$) is not a real-valued function and hence discussing its continuity is irrelevant.*

Exercise 3.7. *The twin results for the case of the Minimum function $m_n(x) = \min_{1 \le k \le n} \{f_k(x)\}$ $(n \in \mathbb{N})$:*

(a) *A modification of above proof for the maximum function M_n can be applied for the epsilon-delta continuity proof of minimum function m_n.*
(b) *A modification of above counterexample for the maximum function M_n can be applied for the associated counterexample of minimum function m_n.*

Remark 10. *In contrast to addition, multiplication and composition, division does not admit a well-defined generalized n-ary form in standard algebra. Specifically, there exists no established algebraic operation or special symbol representing a "generalized division" of $n > 2$ functions. This limitation arises from the fact that division is not an associative operation; that is, for functions (or real numbers) f_1, f_2, f_3, one generally has:*

$$(f_1/f_2)/f_3 \neq f_1/(f_2/f_3). \tag{48}$$

Consequently, the notion of division is inherently restricted to the binary case $n = 2$, where its meaning remains unambiguous.

3.2. Atlas of Epsilon-Delta Continuity Proofs

In this section we present rigorous ϵ–δ continuity proofs for eight clusters of real-valued functions: (i) algebraic [16], (ii) exponential and logarithmic [17–21], (iii) trigonometric [22], (iv) inverse trigonometric [22], (v) hyperbolic [22], (vi) inverse-hyperbolic [22], (vii) elementary piecewise [23,24], and (viii) pathological functions [25–33]. Each cluster comprises several special cases, totaling 54 functions. For each case, given $\epsilon > 0$ and a point of continuity x_0, we provide a sample choice $\delta = \delta(\epsilon, x_0)$ that satisfies the condition in Definition 2. In every proof we apply one of three deduction templates—Proof Scheme I, Proof Scheme II, or Proof Scheme III. Where helpful, we include targeted exercises that invite readers to apply the same methods to closely related instances and thereby deepen their understanding.

3.2.1. Algebraics

Theorem 5. ***(Polynomials)*** *Let* $f_{1,1}(x) = \sum_{k=0}^{n} a_k x^k, \quad a_k \in \mathbb{R}$. *Then,* $C(f_{1,1}) = \mathbb{R}$.

Proof. Fix $x_0 \in \mathbb{R}$ and given $\epsilon > 0$. We accomplish the proof in few steps:

Step 1. (Individual Cases) We consider three cases as follows:

Case 1: The case for $n = 0$ is trivial by taking $\delta = \delta(\epsilon, x_0) = 1$.

Case 2: The case for $n = 1$ is trivial by taking $\delta = \delta(\epsilon, x_0) = \frac{\epsilon}{|a_1|}$.

Case 3: The case for $n \geq 2$ involves the general scheme in Remark 7. First, using DPF equality (12) it follows that:

$$f_{1,1}(x) - \sum_{k=0}^{n} a_k x_0^k = \sum_{k=1}^{n} a_k (x^k - x_0^k) = \sum_{k=1}^{n} a_k ((x - x_0) \sum_{j=0}^{k-1} x^{k-1-j} x_0^j) = (x - x_0)(\sum_{k=1}^{n} \sum_{j=0}^{k-1} a_k x^{k-1-j} x_0^j). \quad (49)$$

Set $g_{1,1}(x, x_0) = \sum_{k=1}^{n} \sum_{j=0}^{k-1} a_k x^{k-1-j} x_0^j$ and take $\delta_0 = 1$ so that $|x - x_0| < 1$. Then, as the latest bound on x implies $|x| \leq |x_0| + 1$ it follows that:

$$|g(x, x_0)| \leq \sum_{k=1}^{n} \sum_{j=0}^{k-1} |a_k||x|^{k-1-j}|x_0|^j \leq \sum_{k=1}^{n} \sum_{j=0}^{k-1} |a_k|(|x_0| + 1)^{k-1} = \sum_{k=1}^{n} k|a_k|(|x_0| + 1)^{k-1} = C. \quad (50)$$

Now, it is sufficient to consider $\delta = \delta(\epsilon, x_0) = \min\{1, \frac{\epsilon}{(\sum_{k=1}^{n} k|a_k|(|x_0| + 1)^{k-1})}\}$.

Step 2. (Merging cases) Finally, to combine all three cases, it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{1, \frac{\epsilon}{(\sum_{k=1}^{n} k|a_k|(|x_0| + 1)^{k-1})}\}. \quad (51)$$

This completes the proof. □

Exercise 3.8. *The above proof can be applied for the following special polynomial functions:*

(a) *the basic form polynomials (constant, linear, quadratic, cubic),*
(b) *special families of orthogonal polynomials (Legendre, Chebyshev, Hermite, Laguerre, Bernoulli, Euler),*
(c) *the Cyclotomic polynomials,*
(d) *the Minimal polynomials,*
(e) *the Smoothstep function* ($f(x) = 3x^2 - 2x^3$)*,*
(f) *the base Logistic map* ($f(x) = rx(1 - x), r > 0$)*.*

Theorem 6. ***(Powers)*** *Let* $f_{1,2}(x) = x^r, \quad r \in \mathbb{Q}$. *Then,* $C(f_{1,2}) = \mathbb{R}^+$ *if* $r = \frac{\pm p}{q}, (p, q) = 1, q$: *even;* $\mathbb{R}$ *if* $r = \frac{+p}{q}, (p, q) = 1, q$: *odd;* $\mathbb{R} - \{0\}$ *if* $r = \frac{-p}{q}, (p, q) = 1, q$: *odd.*

Proof. We accomplish the proof in few steps:

Step 1. (Establishing Reference Case) Reference Case: Fix $r \in \mathbb{R}$, $x_0 \in \mathbb{R}$ and given $\epsilon > 0$. First, when $(r > 0, x_0 = 0)$, it is sufficient to take $\delta = \delta(\epsilon, x_0) = \epsilon^{\frac{1}{r}}$. Second, we assume $x_0 \neq 0$. We again apply general scheme in Remark 7. First, by two applications of TSBPB inequality (26), (27) it follows that whenever x^r, x_0^r are defined:

$$|f_{1,2}(x) - f_{1,2}(x_0)| \leq \max\{|r||x|^{r-1}|x - x_0|, |r||x_0|^{r-1}|x - x_0|\} = |r|.\max\{|x|^{r-1}, |x_0|^{r-1}\}|x - x_0| \ \ (r \in \mathbb{R}). \quad (52)$$

Set $g_{1,2}(x, x_0) = |r|.\max\{|x|^{r-1}, |x_0|^{r-1}\}$ and take $\delta_0 = \frac{|x_0|}{2} > 0$ so that $|x - x_0| < \frac{|x_0|}{2}$. Then, using the reverse triangle inequality (28) $||x| - |x_0|| \leq |x - x_0|$ it follows that $||x| - |x_0|| \leq \frac{|x_0|}{2}$ or equivalently $\frac{|x_0|}{2} < |x| < 3\frac{|x_0|}{2}$. Next, two applications of the latest inequality for $r < 1$, and $r \geq 1$, respectively yield:

$$|x|^{r-1} < (1_{[1,\infty)}(r)(\frac{3}{2})^{r-1} + 1_{(-\infty,1)}(r)(\frac{1}{2})^{r-1})|x_0|^{r-1}. \quad (53)$$

Accordingly:

$$\begin{aligned} |g_{1,2}(x, x_0)| &= |r|.\max\{|x|^{r-1}, |x_0|^{r-1}\} \\ &\leq |r|.\max\{(1_{[1,\infty)}(r)(\frac{3}{2})^{r-1} + 1_{(-\infty,1)}(r)(\frac{1}{2})^{r-1}), 1\}|x_0|^{r-1} \\ &= |r|(1_{[1,\infty)}(r)(\frac{3}{2})^{r-1} + 1_{(-\infty,1)}(r)(\frac{1}{2})^{r-1})|x_0|^{r-1} \\ &= |r|(1_{[1,\infty)}(r)3^{r-1} + 1_{(-\infty,1)}(r))|\frac{x_0}{2}|^{r-1}. \end{aligned} \quad (54)$$

Now, it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{|\frac{x_0}{2}|, \frac{\epsilon}{|r|(1_{[1,\infty)}(r)3^{r-1} + 1_{(-\infty,1)}(r))|\frac{x_0}{2}|^{r-1}}\}. \quad (55)$$

Step 2. (Case by Case Proofs) Second, we consider two cases as follows:

Case 1: Fix $r \in \mathbb{Q}$ $(r = \frac{\pm p}{q}, (p, q) = 1, q : even)$, $x_0 > 0$: We simply repeat the proof in Reference Case for domain area of $\mathbb{R}^+$.

Case 2: Fix $r > 0$ $(r = \frac{p}{q}, (p, q) = 1, q : odd)$, $x_0 \in \mathbb{R}$. Here, We simply repeat the proof in Reference Case for domain area $\mathbb{R}$.

Case 3: Fix $r < 0$ $(r = \frac{-p}{q}, (p, q) = 1, q : odd)$, $x_0 \neq 0$. Here, we simply repeat the proof in Reference Case for domain area $\mathbb{R} - \{0\}$.

This completes the proof. □

Exercise 3.9. *The above proof can be applied for the following special power cases:*

(a) *the integer powers (e.g., $f(x) = x^{-5}$)*
(b) *the rational powers (e.g., $f(x) = x^{2/3}$)*

Remark 11 (Transcendental Powers)**.** *The Reference Case in the proof of Theorem 6 is valid for the special Case $r \in \mathbb{R}^+ - \mathbb{Q}$ as well. Hence,this proof establishes the continuity of the (non-algebraic) transcendental powers too. Some prominent cases include the following:*

(a) $f(x) = x^{\sqrt{2}} : \sqrt{2} = 1.41421\cdots$,
(b) $f(x) = x^{\phi} : \phi = 1.61803\cdots$,
(c) $f(x) = x^{e} : e = 2.71828\cdots$,
(d) $f(x) = x^{\pi} : \pi = 3.14159\cdots$.

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Theorem 7. ***(General Rationals)*** *Let* $f_{1,3}(x) = \dfrac{p(x)}{q(x)} = \dfrac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{m} b_k x^k}$, $n, m \in \mathbb{N}, a_k, b_k \in \mathbb{R}$. *Then,* $C(f_{1,3}) = \mathbb{R} - R(q)$.

Proof. Fix $x_0 \in \mathbb{R}$ with $|\sum_{k=0}^{m} b_k x_0^k| > 0$. We accomplish the proof in four steps:

Step 1. (Upper Bound Representation) Let $p(x) = \sum_{k=0}^{n} a_k x^k$, and $q(x) = \sum_{k=0}^{m} b_k x^k$. Then:

$$\begin{aligned} |f_{1,3}(x) - f_{1,3}(x_0)| &= |\frac{p(x)}{q(x)} - \frac{p(x_0)}{q(x_0)}| \\ &= |\frac{q(x_0)(p(x) - p(x_0)) - p(x_0)(q(x) - q(x_0))}{q(x)q(x_0)}| \\ &\leq |p(x) - p(x_0)| \times (\frac{|q(x_0)|}{|q(x)q(x_0)|}) + |q(x) - q(x_0)| \times (\frac{|p(x_0)|}{|q(x)q(x_0)|}). \end{aligned} \tag{56}$$

Step 2. (Upper Bound for Factors) Let $\epsilon_b = \dfrac{|q(x_0)|}{2} > 0$. Since, q is continuous at $x = x_0$ there exists $\delta_b > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_b$ we have: $|q(x) - q(x_0)| < \dfrac{|q(x_0)|}{2}$. Hence, using the reverse triangle inequality (28) $||q(x)| - |q(x_0)|| \leq |q(x) - q(x_0)|$ we have $||q(x)| - |q(x_0)|| < \dfrac{|q(x_0)|}{2}$. This latest inequality yields $|q(x)| > \dfrac{|q(x_0)|}{2}$ or equivalently $\dfrac{1}{|q(x)|} \leq \dfrac{2}{|q(x_0)|}$. Accordingly, the right hand of inequality (56) has upper bound as in the following:

$$|p(x) - p(x_0)| \times (\frac{2|q(x_0)|}{|q(x_0)|^2}) + |q(x) - q(x_0)| \times (\frac{2|p(x_0)|}{|q(x_0)|^2}). \tag{57}$$

Step 3. (Neighborhood Approximations) Given $\epsilon > 0$. Define $\epsilon_p = (\dfrac{2|q(x_0)|}{|q(x_0)|^2})^{-1} \times \dfrac{\epsilon}{2} > 0$ and $\epsilon_q = (\dfrac{2|p(x_0)|}{|q(x_0)|^2})^{-1} \times \dfrac{\epsilon}{2} > 0$. By continuity of p and q at $x = x_0$ there are $\delta_p > 0$, and $\delta_q > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_p$ we have $|p(x) - p(x_0)| < \epsilon_p$, and, whenever $|x - x_0| < \delta_q$ we have $|q(x) - q(x_0)| < \epsilon_q$. Equivalently:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_p \rightarrow |p(x) - p(x_0)| \times (\frac{2|q(x_0)|}{|q(x_0)|^2}) < \frac{\epsilon}{2}), \tag{58}$$

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_q \rightarrow |q(x) - q(x_0)| \times (\frac{2|p(x_0)|}{|q(x_0)|^2}) < \frac{\epsilon}{2}). \tag{59}$$

Step 4. (The delta Assessment). Let $\delta = \min\{\delta_b, \delta_p, \delta_q\} > 0$. Then, combining inequalities (56)-(59) we have:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta \rightarrow |f_{1,3}(x) - f_{1,3}(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon). \tag{60}$$

Finally, the formulae of $\delta = \delta(\epsilon, x_0)$ via three times usage of equation (51) for $\epsilon_b = \frac{|q(x_0)|}{2}$, $\epsilon_p = \frac{|q(x_0)|\epsilon}{4}$ and $\epsilon_q = \frac{|q(x_0)|^2\epsilon}{4|p(x_0)|}$, respectively, is given by:

$$\begin{aligned} \delta = \delta(\epsilon, x_0) &= \min\{\delta_b(\epsilon_b, x_0), \delta_p(\epsilon_p, x_0), \delta_q(\epsilon_q, x_0)\} : && (61) \\ \delta_b(\epsilon_b, x_0) &= \min\{1, \frac{|\sum_{k=0}^{m} b_k x_0^k|}{2(\sum_{k=1}^{m} k|b_k|(|x_0| + 1)^{k-1})}\}, && (62) \\ \delta_p(\epsilon_p, x_0) &= \min\{1, \frac{|\sum_{k=0}^{m} b_k x_0^k|\epsilon}{4(\sum_{k=1}^{n} k|a_k|(|x_0| + 1)^{k-1})}\}, && (63) \\ \delta_q(\epsilon_q, x_0) &= \min\{1, \frac{(|\sum_{k=0}^{m} b_k x_0^k|^2 / |\sum_{k=0}^{n} a_k x_0^k|)\epsilon}{4(\sum_{k=1}^{m} k|b_k|(|x_0| + 1)^{k-1})}\}. && (64) \end{aligned}$$

This is simplified to:

$$\delta = \delta(\epsilon, x_0) = \min\Big\{1,\ \frac{\beta}{2B},\ \frac{\beta\epsilon}{4A},\ \frac{\beta^2\epsilon}{4\alpha B}\Big\}: \\ \beta = \Big|\sum_{k=0}^{m} b_k x_0^k\Big|,\ B = \sum_{k=1}^{m} k|b_k|(|x_0|+1)^{k-1},\ \alpha = \Big|\sum_{k=0}^{n} a_k x_0^k\Big|,\ A = \sum_{k=1}^{n} k|a_k|(|x_0|+1)^{k-1}. \tag{65}$$

This completes the proof. □

Exercise 3.10. *The above proof can be applied for the following special rational functions:*

(a) *the Möbius (linear fractional) functions* ($f(x) = \frac{ax+b}{cx+d}$),
(b) *the Quadratic-over-linear rationals,*
(c) *the Cubic-over-quadratic rationals,*
(d) *the Runge function* ($f(x) = \frac{1}{1+x^2}$),
(e) *the Hill function* ($f(x) = \frac{x^n}{K^n + x^n}$).

3.2.2. Exponential & Logarithmic Family

Proposition 1. ***(Logarithm family)*** *Let* $f_{2,1}(x) = log_a(x),\quad 1 \neq a > 0$. *Then,* $C(f_{2,1}) = \mathbb{R}^+$.

Proof. We accomplish the proof in few steps:
Step 1. (Establishing individual cases) We consider two cases as follows:
Case 1 ($a > 1$): Given $\epsilon > 0$ and fix $x_0 > 0$. Then, for all $x \in \mathbb{R}$, by a series of algebraic operations we have:

$$|f_{2,1}(x) - f_{2,1}(x_0)| = |\log_a(x) - \log_a(x_0)| < \epsilon \iff -x_0(1 - a^{-\epsilon}) < x - x_0 < x_0(a^{\epsilon} - 1). \tag{66}$$

Now, using Symmetric-anti Symmetric inequality (29) it is sufficient to consider $\delta = \delta(\epsilon, x_0) = \min\{x_0(1 - a^{-\epsilon}), x_0(a^{\epsilon} - 1)\} = x_0(1 - a^{-\epsilon})$.
Case 2($0 < a < 1$): Here, with similar argument as in above it sufficient to consider $\delta = \delta(\epsilon, x_0) = x_0(1 - a^{\epsilon})$.
Step 2. (Merging cases) Finally, both cases can be combined by considering:

$$\delta = \delta(\epsilon, x_0) = x_0(1 - a^{(\mathrm{sgn}(1-a))\epsilon}). \tag{67}$$

This completes the proof. □

Proposition 2. ***(Exponential family)*** *Let* $f_{2,2}(x) = a^x,\quad a > 0$. *Then,* $C(f_{2,2}) = \mathbb{R}$.

Proof. We accomplish the proof in few steps:
Step 1.(Establishing individual cases) We consider two cases as follows (Case $a = 1$ is trivial):
Case 1 ($a > 1$): Given $\epsilon > 0$ and fix $x_0 \in \mathbb{R}$. Then, for all $x \in \mathbb{R}$, by a series of algebraic operations we have:

$$|f_{2,2}(x) - f_{2,2}(x_0)| = |a^x - a^{x_0}| < \epsilon \iff -\log_a\Big(\frac{1}{1 - a^{-x_0}\epsilon}\Big) < x - x_0 < \log_a(1 + a^{-x_0}\epsilon). \tag{68}$$

Now, using Symmetric-anti Symmetric inequality (29) it is sufficient to consider $\delta = \delta(\epsilon, x_0) = \min\{\log_a(\frac{1}{1 - a^{-x_0}\epsilon}), \log_a(1 + a^{-x_0}\epsilon)\} = \log_a(1 + a^{-x_0}\epsilon)$.

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Case 2($0 < a < 1$): Here, with a similar argument to the Case 1 we have, $\delta = \delta(\epsilon, x_0) = -\log_a(1 + a^{-x_0}\epsilon)$.

Step 2. (Merging cases) Finally, both cases can be combined by considering:

$$\delta = \delta(\epsilon, x_0) = \operatorname{sgn}(a-1)\big(\log_a(1 + a^{-x_0}\epsilon)\big). \tag{69}$$

This completes the proof.□

Proposition 3. ***(Inverse Lambert W function)*** *Let $f_{2,3}(x) = x\exp(x)$. Then, $C(f_{2,3}) = \mathbb{R}$.*

Proof. Fix $x_0 \in \mathbb{R}$. Similar to the proof of Theorem 5, we accomplish the proof in few steps:
Step 1. (Upper Bound Representation) Using GPD equality (11), we have:

$$\begin{aligned} |f_{2,3}(x) - f_{2,3}(x_0)| &= |x\exp(x) - x_0\exp(x_0)| \\ &= |(x - x_0)\exp(x_0) + x(\exp(x) - \exp(x_0))| \\ &\leq |x - x_0|\exp(x_0) + |x||\exp(x) - \exp(x_0)|. \end{aligned} \tag{70}$$

Step 2. (Upper Bound for Factor) Let $\delta_1 = 1$. Then, for all $x \in \mathbb{R}$ with $|x - x_0| < 1$, we have $|x| < |x_0| + 1$. Accordingly, the right hand of inequality (70) has upper bound as in the following:

$$|x - x_0| \times (\exp(x_0)) + |\exp(x) - \exp(x_0)| \times (|x_0| + 1). \tag{71}$$

Step 3. (Neighborhood Approximations) Given $\epsilon > 0$. Define $\epsilon_2 = \frac{\exp(-x_0)\epsilon}{2}$ and $\epsilon_3 = \frac{(|x_0|+1)^{-1}\epsilon}{2}$. Then, by continuity of identity function I and the exponential function $\exp(.)$ at $x = x_0$, there are $\delta_2 > 0$, and $\delta_3 > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_2$ we have $|x - x_0| < \epsilon_2$, and whenever $|x - x_0| < \delta_3$ we have $|\exp(x) - \exp(x_0)| < \epsilon_3$. Equivalently:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_2 \to |x - x_0| \times (\exp(x_0)) < \frac{\epsilon}{2}), \tag{72}$$

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_3 \to |\exp(x) - \exp(x_0)| \times (|x_0| + 1) < \frac{\epsilon}{2}). \tag{73}$$

Step 4. (The delta Assessment). Let $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$. Then, combining inequalities (70)-(73) we have:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta \to |f_{2,3}(x) - f_{2,3}(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2}) = \epsilon. \tag{74}$$

Finally, using equation (51) for $\epsilon_2 = \frac{\exp(-x_0)\epsilon}{2}$ and equation (69) for $a = e, \epsilon_3 = \frac{(|x_0|+1)^{-1}\epsilon}{2}$, respectively, we have $\delta_2 = \min\{1, \frac{\exp(-x_0)\epsilon}{2}\}$ and $\delta_3 = \ln(1 + \frac{\exp(-x_0)\epsilon}{2(|x_0|+1)})$, respectively. Hence:

$$\delta = \delta(\epsilon, x_0) = \min\{1, \frac{\exp(-x_0)\epsilon}{2}, \ln(1 + \frac{\exp(-x_0)\epsilon}{2(|x_0|+1)})\}. \tag{75}$$

This completes the proof.□

Proposition 4. ***(The Negative-entropy potential)*** *Let $f_{2,4}(x) = x\ln(x), \quad x > 0$. Then, $C(f_{2,4}) = \mathbb{R}^+$.*

Proof. The proof is essentially the repetition of the proof of previous proposition with $\exp(.)$ replaced by $\ln(.)$ and minor changes. First, fix $x_0 > 0$ then the inequality (70) is replaced by:

$$|f_{2,4}(x) - f_{2,4}(x_0)| \leq |x - x_0||\ln(x_0)| + |x||\ln(x) - \ln(x_0)|. \tag{76}$$

Second, take $\delta_1 = \frac{x_0}{2} > 0$, to obtain an upper bound for factor for all $x \in \mathbb{R}$ with $|x - x_0| < \delta_1$:

$$|x - x_0|| \ln(x_0)| + | \ln(x) - \ln(x_0)|(\frac{3x_0}{2}). \tag{77}$$

Third, use equation (51) for $\epsilon_2 = \frac{\epsilon}{2|\ln(x_0)|}$, and, equation (67) for $a = e, \epsilon_3 = \frac{\epsilon}{3x_0}$ to obtain $\delta_2 = \frac{\epsilon}{2(|\ln(x_0)|)}$ and, $\delta_3 = x_0(1 - \exp(\frac{-\epsilon}{3x_0}))$, respectively. Accordingly, by $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$:

$$\delta = \delta(\epsilon, x_0) = \min\{\frac{x_0}{2}, \frac{\epsilon}{2(|\ln(x_0)|)}, x_0(1 - \exp(\frac{-\epsilon}{3x_0}))\}. \tag{78}$$

This completes the proof. □

Proposition 5. ***(The Tetration function)*** *Let* $f_{2,5}(x) = x^x, \quad x > 0$. *Then,* $C(f_{2,5}) = \mathbb{R}^+$.

Proof. We accomplish the proof in few steps:

Step 1. (Composite representation) We apply the method of the proof of Theorem 3 given the identity $f_{2,5}(x) = x^x = \exp(x \ln(x)) = f_{2,2}(f_{2,4}(x))$ for all $x > 0$.

Step 2. (Component cascade estimations) Fix $x_0 > 0$ and given $\epsilon > 0$. First, as the function $f_{2,2}(y) = \exp(y)$ is continuous at $y = y_0 = x_0 \ln(x_0)$ we have:

$$\exists \eta = \eta(\epsilon, y_0) > 0 \, \forall y \in \mathbb{R}(|y - x_0 \ln(x_0)| < \eta \to | \exp(y) - \exp(x_0 \ln(x_0))| < \epsilon), \tag{79}$$

where by the equation (69) for $a = e$:

$$\eta = \eta(\epsilon, x_0 \ln(x_0)) = \ln(1 + e^{-x_0 \ln(x_0)}\epsilon) = \ln(1 + x_0^{-x_0}\epsilon). \tag{80}$$

Second, as the function $f_{2,4}(x) = x \ln(x)$ is continuous at $x = x_0$ for the found $\eta > 0$ in (79) we have:

$$\exists \delta(\eta, x_0) > 0 \, \forall x \in \mathbb{R}(|x - x_0| < \delta \to |x \ln(x) - x_0 \ln(x_0)| < \eta), \tag{81}$$

where by the equation (78):

$$\delta = \delta(\eta, x_0) = \min\{\frac{x_0}{2}, \frac{\eta}{2(|\ln(x_0)|)}, x_0(1 - \exp(\frac{-\eta}{3x_0}))\}. \tag{82}$$

Step 3. (Final assessment) Finally, substituting the value of η from equation (80) in to equation (82) it follows that:

$$\delta = \delta(\epsilon, x_0) = \min\{\frac{x_0}{2}, \frac{\ln(1 + x_0^{-x_0}\epsilon)}{2(|\ln(x_0)|)}, x_0(1 - (1 + x_0^{-x_0}\epsilon)^{\frac{-1}{3x_0}})\}. \tag{83}$$

This completes the proof. □

Proposition 6. ***(The Normal density)*** *Let* $f_{2,6}(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2\sigma^2}(x - \mu)^2) : \mu \in \mathbb{R}, \sigma > 0$. *Then,* $C(f_{2,6}) = \mathbb{R}$.

Proof. The proof is essentially similar to the proof of previous proposition. Here, we have the identity $f_{2,6}(x) = f_{2,2}(f_{1,1}(x))$ where $f_{2,2}(x) = \exp(x)$, and $f_{1,1}(x) = \frac{-1}{2\sigma^2}x^2 + \frac{\mu}{\sigma^2}x - \frac{\mu^2}{2\sigma^2} - \ln(\sqrt{2\pi}\sigma)$ for all $x \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Then, using equation (69) for $a = e$, and equation (51) for

$n = 2(a_2 = \frac{-1}{2\sigma^2}, a_1 = \frac{\mu}{\sigma^2})$, respectively, the corresponding $\eta = \eta(\epsilon, f_{1,1}(x_0)) > 0$, and $\delta = \delta(\eta, x_0) > 0$, respectively, are given by:

$$\eta = \eta(\epsilon, f_{1,1}(x_0)) = \ln(1 + e^{-f_{1,1}(x_0)}\epsilon), \tag{84}$$

$$\delta = \delta(\eta, x_0) = \min\{1, \frac{\eta}{|\frac{\mu}{\sigma^2}| + 2|\frac{1}{2\sigma^2}|(|x_0| + 1)}\}. \tag{85}$$

Finally, substituting the value of η from equation (84) in to equation (85) it follows that:

$$\delta = \delta(\epsilon, x_0) = \min\{1, \frac{\sigma^2(\ln(1 + (\sqrt{2\pi}\sigma \exp(\frac{1}{2\sigma^2}(x_0 - \mu)^2))\epsilon))}{|\mu| + |x_0| + 1}\}. \tag{86}$$

This completes the proof.□

Proposition 7. ***(The Logistic function)*** *Let* $f_{2,7}(x) = \frac{L_{max}}{1+Q.\exp(-k.x)}$: $L_{max}, Q, k > 0$. *Then,* $C(f_{2,7}) = \mathbb{R}$.

Proof. The proof is essentially similar to the proof of previous proposition too. Here, we have the identity $f_{2,7}(x) = f_{1,3}(f_{2,2}(x))$ where $f_{1,3}(x) = \frac{L_{max}}{1+Qx}$, and $f_{2,2}(x) = \exp(-kx)$ for all $x \in \mathbb{R}$. Fix $x_0 \in \mathbb{R}$ and $\epsilon > 0$. Then, using equation (65) for $(\beta = |1 + Q\exp(-kx_0)|, B = Q, \alpha = L_{max}, A = 0^+)$, and, equation (69) for $a = e^{-k}$, respectively, the corresponding $\eta = \eta(\epsilon, f_{2,2}(x_0)) > 0$, and $\delta = \delta(\eta, x_0) > 0$, respectively, are given by:

$$\eta = \eta(\epsilon, f_{2,2}(x_0)) = \min\{1, \frac{1 + Q.\exp(-k.x_0)}{2Q}, \frac{(1 + Q.\exp(-k.x_0))^2\epsilon}{4L_{max}Q}\}, \tag{87}$$

$$\delta = \delta(\eta, x_0) = \frac{1}{k}\ln(1 + \exp(k.x_0)\eta). \tag{88}$$

Finally, substituting the value of η from equation (87) in to equation (88) it follows that:

$$\delta = \delta(\epsilon, x_0) = \frac{1}{k}\ln\left(1 + \exp(k.x_0) \times \min\{1, \frac{1 + Q.\exp(-k.x_0)}{2Q}, \frac{(1 + Q.\exp(-k.x_0))^2\epsilon}{4L_{max}Q}\}\right). \tag{89}$$

This completes the proof.□

Exercise 3.11. *The above proofs can be applied for the following functions:*

(a) *the bump function* $(f(x) = \exp(\frac{-1}{1-x^2})1_{(-1,1)}(x))$,

(b) *the standard normal density* $(f(x) = \frac{\exp(-\frac{x^2}{2})}{\sqrt{2\pi}})$,

(c) *the Softplus function* $(f(x) = \ln(1 + e^x))$,

(d) *the standard logistic function* $(f(x) = \frac{1}{1+e^{-x}})$,

(e) *the logit map* $(f(x) = \log(\frac{x}{1-x}))$.

Remark 12. *The last three functions in above list are members of the elementary transforms and sigmoids family of functions.*

3.2.3. Trigonometrics

Proposition 8. ***(The Trigonometric functions)*** *Let* $f_3(x) = Trig(x)$. *Then,* $C(f_3) = \mathbb{R} \setminus B(f_3)$ *for* $f_3 \in \{\sin, \cos, \tan, \cot, \sec, \csc\}$, *where* $B(f_3) = \emptyset$, *if* $f_3 \in \{\sin, \cos\}$; $\frac{\pi}{2} + \pi\mathbb{Z}$, *if* $f_3 \in \{\tan, \sec\}$; $\pi\mathbb{Z}$, *if* $f_3 \in \{\cot, \csc\}$.

Proof. Given $\epsilon > 0$. We prove the continuity as follows:

1. Let $f_{3,1}(x) = \sin(x)$, and fix $x_0 \in \mathbb{R}$. By equation (17) and inequality (30) it follows that:

$$|f_{3,1}(x) - f_{3,1}(x_0)| = |+2||\cos(\frac{x+x_0}{2})||\sin(\frac{x-x_0}{2})| \leq 2(1)|\frac{x-x_0}{2}| = |x-x_0|. \tag{90}$$

Accordingly, it is sufficient to take:

$$\delta = \delta(\epsilon, x_0) = \epsilon. \tag{91}$$

2. Let $f_{3,2}(x) = \cos(x)$, and fix $x_0 \in \mathbb{R}$. By equation (18) and inequality (30) it follows that:

$$|f_{3,2}(x) - f_{3,2}(x_0)| = |-2||\sin(\frac{x+x_0}{2})||\sin(\frac{x-x_0}{2})| \leq 2(1)|\frac{x-x_0}{2}| = |x-x_0|. \tag{92}$$

Accordingly, it is sufficient to take:

$$\delta = \delta(\epsilon, x_0) = \epsilon. \tag{93}$$

3. Let $f_{3,3}(x) = \tan(x)$, and fix $x_0 \notin \frac{\pi}{2} + \pi\mathbb{Z}$. First, using equality (19) and inequality (30) it follows that:

$$\begin{aligned} |f_{3,3}(x) - f_{3,3}(x_0)| &= |+(\cos(x)\cos(x_0))^{-1}||\sin(x-x_0)| \\ &\leq |+(\cos(x)\cos(x_0))^{-1}||x-x_0|. \end{aligned} \tag{94}$$

Second, set $\epsilon_1 = \frac{|\cos(x_0)|}{2} > 0$. Since $\cos(.)$ is continuous at $x = x_0$, by equation (93) there is $\delta_1 = \frac{|\cos(x_0)|}{2}$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta_1$ we have $|\cos(x) - \cos(x_0)| < \frac{|\cos(x_0)|}{2}$. As before, an application of reverse triangle inequality (28) $||\cos(x)| - |\cos(x_0)|| \leq |\cos(x) - \cos(x_0)|$ yields $||\cos(x)| - |\cos(x_0)|| \leq \frac{|\cos(x_0)|}{2}$. This latest inequality implies $|\cos(x)| \geq \frac{|\cos(x_0)|}{2}$ or equivalently:

$$|+(\cos(x)\cos(x_0))^{-1}| \leq 2|\cos(x_0)|^{-2}. \tag{95}$$

Third, combining inequalities (95) and (94), and, considering $\delta_2 = \frac{|\cos(x_0)|^2\epsilon}{2}$ for all $x \in \mathbb{R}$ with $|x - x_0| < \delta = \min\{\delta_1, \delta_2\}$ we have:

$$|f_{3,3}(x) - f_{3,3}(x_0)| \leq 2|\cos(x_0)|^{-2}|x-x_0| < \epsilon. \tag{96}$$

Finally, $\delta = \min\{\delta_1, \delta_2\} > 0$ is given by:

$$\delta = \delta(\epsilon, x_0) = \min\{\frac{|\cos(x_0)|}{2}, \frac{|\cos(x_0)|^2\epsilon}{2}\}. \tag{97}$$

4. Let $f_{3,4}(x) = \sec(x)$, and fix $x_0 \notin \frac{\pi}{2} + \pi\mathbb{Z}$. First, using equality (21) and inequality (30) it follows that:

$$\begin{aligned} |f_{3,4}(x) - f_{3,4}(x_0)| &= |+2(\cos(x)\cos(x_0))^{-1}||\sin(\frac{x+x_0}{2})||\sin(\frac{x-x_0}{2})| \\ &\leq |+(\cos(x)\cos(x_0))^{-1}||x-x_0|. \end{aligned} \tag{98}$$

Given that inequality (98) has the exact upper bound as in the inequality (94), the left over proof is -by word by word-similar to the previous one. Hence:

$$\delta = \delta(\epsilon, x_0) = \min\{\frac{|\cos(x_0)|}{2}, \frac{|\cos(x_0)|^2\epsilon}{2}\}. \quad (99)$$

5. Let $f_{3,5}(x) = \cot(x)$, and fix $x_0 \notin \pi\mathbb{Z}$. First, using equality (20) and inequality (30) it follows that:

$$\begin{aligned} |f_{3,5}(x) - f_{3,5}(x_0)| &= |-(\sin(x)\sin(x_0))^{-1}||\sin(x - x_0)| \\ &\leq |-(\sin(x)\sin(x_0))^{-1}||x - x_0|. \end{aligned} \quad (100)$$

Given that inequality (100) has the similar upper bound as in the inequality (94) with cos's replaced by sin's, the left over proof is similar to the previous one. Hence:

$$\delta = \delta(\epsilon, x_0) = \min\{\frac{|\sin(x_0)|}{2}, \frac{|\sin(x_0)|^2\epsilon}{2}\}. \quad (101)$$

6. Let $f_{3,6}(x) = \csc(x)$, and fix $x_0 \notin \pi\mathbb{Z}$. First, using equality (22) and inequality (30) it follows that:

$$\begin{aligned} |f_{3,6}(x) - f_{3,6}(x_0)| &= |-2(\sin(x)\sin(x_0))^{-1}||\cos(\frac{x + x_0}{2})||\sin(\frac{x - x_0}{2})| \\ &\leq |-(\sin(x)\sin(x_0))^{-1}||x - x_0|. \end{aligned} \quad (102)$$

Given that inequality (102) has the exact upper bound as in the inequality (100), the left over proof is -by word by word-similar to the previous one. Hence:

$$\delta = \delta(\epsilon, x_0) = \min\{\frac{|\sin(x_0)|}{2}, \frac{|\sin(x_0)|^2\epsilon}{2}\}. \quad (103)$$

This completes the proof. □

Exercise 3.12. *The above proofs can be applied for the following function:*

(a) *the Haversine function* ($f(x) = \sin^2(\frac{x}{2})$).

3.2.4. Inverse-Trigonometrics

Proposition 9. ***(The Inverse-Trigonometric functions)*** *Let* $f_4(x) = InvTrig(x)$. *Then,* $C(f_4) = [-1,1]$, *if* $f_4 \in \{\arcsin, \arccos\}; \mathbb{R}$, *if* $f_4 \in \{\arctan, \text{arccot}\}; \mathbb{R} - (-1,1)$, *if* $f_4 \in \{\text{arcsec}, \text{arccsc}\}$.

Proof. Without loss of generality, we assume the given $\epsilon > 0$ is sufficiently small such that all quantities of the form $Trig(InvTrig(x_0) \pm \epsilon)$ are well defined [i.e., $0 < \epsilon < 1.7em(y_0, \partial I_{InvTrig})$, $y_0 = InvTrig(x_0), I_{InvTrig}$:= range branch of InvTrig, $y_0 \pm \epsilon \subset I_{InvTrig}$]. We prove the continuity using Remark 8 as follows:

1. Let $f_{4,1}(x) = \arcsin(x)$, and fix $x_0 \in [-1,1]$. Then, for all $x \in [-1,1]$, by a series of algebraic operations we have:

$$\begin{aligned} |f_{4,1}(x) - f_{4,1}(x_0)| = |\arcsin(x) - \arcsin(x_0)| < \epsilon \iff \\ -(-\sin(\arcsin(x_0) - \epsilon) + x_0) < x - x_0 < \sin(\arcsin(x_0) + \epsilon) - x_0. \end{aligned} \quad (104)$$

Now, using Symmetric-anti Symmetric inequality (29) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{-\sin(\arcsin(x_0) - \epsilon) + x_0, \sin(\arcsin(x_0) + \epsilon) - x_0\}. \quad (105)$$

2. Let $f_{4,2}(x) = \arccos(x)$, and fix $x_0 \in [-1,1]$. Then, for all $x \in [-1,1]$, by a series of algebraic operations we have:

$$|f_{4,2}(x) - f_{4,2}(x_0)| = |\arccos(x) - \arccos(x_0)| < \epsilon \iff$$
$$-(-\cos(\arccos(x_0) + \epsilon) + x_0) < x - x_0 < \cos(\arccos(x_0) - \epsilon) - x_0. \quad (106)$$

Now, using Symmetric-anti Symmetric inequality (29) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{-\cos(\arccos(x_0) + \epsilon) + x_0, \cos(\arccos(x_0) - \epsilon) - x_0\}. \quad (107)$$

3. Let $f_{4,3}(x) = \arctan(x)$, and fix $x_0 \in \mathbb{R}$. Then, by similar proof to the case of $f_{4,1}(x) = \arcsin(x)$ with sin replaced by tan, and arcsin replaced by arctan, respectively, we have:

$$\delta = \delta(\epsilon, x_0) = \min\{-\tan(\arctan(x_0) - \epsilon) + x_0, \tan(\arctan(x_0) + \epsilon) - x_0\}. \quad (108)$$

4. Let $f_{4,4}(x) = \operatorname{arccot}(x)$, and fix $x_0 \in \mathbb{R}$. Then, by similar proof to the case of $f_{4,2}(x) = \arccos(x)$ with cos replaced by cot, and arccos replaced by arccot, respectively, we have:

$$\delta = \delta(\epsilon, x_0) = \min\{-\cot(\operatorname{arccot}(x_0) + \epsilon) + x_0, \cot(\operatorname{arccot}(x_0) - \epsilon) - x_0\}. \quad (109)$$

5. Let $f_{4,5}(x) = \operatorname{arcsec}(x)$, and fix $x_0 \in \mathbb{R} - (-1,1)$. Then, by similar proof to the case of $f_{4,1}(x) = \arcsin(x)$ with sin replaced by sec, and arcsin replaced by arcsec, respectively, we have:

$$\delta = \delta(\epsilon, x_0) = \min\{-\sec(\operatorname{arcsec}(x_0) - \epsilon) + x_0, \sec(\operatorname{arcsec}(x_0) + \epsilon) - x_0\}. \quad (110)$$

6. Let $f_{4,6}(x) = \operatorname{arccsc}(x)$, and fix $x_0 \in \mathbb{R} - (-1,1)$. Then, by similar proof to the case of $f_{4,2}(x) = \arccos(x)$ with cos replaced by csc, and arccos replaced by arccsc, respectively, we have:

$$\delta = \delta(\epsilon, x_0) = \min\{-\csc(\operatorname{arccsc}(x_0) + \epsilon) + x_0, \csc(\operatorname{arccsc}(x_0) - \epsilon) - x_0\}. \quad (111)$$

This completes the proof.□

Exercise 3.13. *An alternative $\epsilon - \delta$ method to prove continuity of the above inverse trigonometric functions is consideration of the following equations and applying the proof scheme presented in Remark (7):*

(a) $\arcsin(x) - \arcsin(x_0) = \arcsin\big(x\sqrt{1-x_0^2} - x_0\sqrt{1-x^2}\big)$.

(b) $\arccos(x) - \arccos(x_0) = \arccos\big(x\,x_0 + \sqrt{1-x^2}\sqrt{1-x_0^2}\big)$,

(c) $\arctan(x) - \arctan(x_0) = \arctan\big(\frac{x-x_0}{1+x\,x_0}\big)$ *(principal value; add $\pm\pi$ if the fraction crosses a branch cut),*

(d) $\operatorname{arccot}(x) - \operatorname{arccot}(x_0) = \operatorname{arccot}\big(\frac{x\,x_0+1}{x_0-x}\big)$ *(with $\operatorname{arccot} \in (0,\pi)$),*

(e) $\operatorname{arcsec}(x) - \operatorname{arcsec}(x_0) = \operatorname{arcsec}\left(\dfrac{1}{\dfrac{1}{x\,x_0} + \sqrt{1-\dfrac{1}{x^2}}\sqrt{1-\dfrac{1}{x_0^2}}}\right)$,

(f) $\operatorname{arccsc}(x) - \operatorname{arccsc}(x_0) = \operatorname{arccsc}\left(\dfrac{1}{\dfrac{1}{x}\sqrt{1-\dfrac{1}{x_0^2}} - \dfrac{1}{x_0}\sqrt{1-\dfrac{1}{x^2}}}\right)$.

3.2.5. Hyperbolics

Proposition 10. ***(The Hyperbolic functions)*** *Let $f_5(x) = Hyperb(x)$. Then, $C(f_5) = \mathbb{R}$, if $f_5 \in \{\sinh, \cosh, \tanh, \operatorname{sech}\}$; $\mathbb{R} - \{0\}$, if $f_5 \in \{\coth, \operatorname{csch}\}$.*

Proof. Given $\epsilon > 0$. We prove the continuity as follows:

1. Let $f_{5,1}(x) = \sinh(x)$, and fix $x_0 \in \mathbb{R}$. We accomplish the proof in few steps:

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Step 1. (Upper Bound Representation). By definition:

$$\begin{aligned} |f_{5,1}(x) - f_{5,1}(x_0)| &= |\frac{\exp(x) - \exp(-x)}{2} - \frac{\exp(x_0) - \exp(-x_0)}{2}| \\ &= |\frac{\exp(x) - \exp(x_0)}{2} - (\frac{\exp(-x) - \exp(-x_0)}{2})| \\ &\leq \frac{1}{2}|\exp(x) - \exp(x_0)| + \frac{1}{2}|\exp(-x) - \exp(-x_0)|. \end{aligned} \tag{112}$$

Step 2. (Neighborhood Approximations). Given $\epsilon > 0$. Define $\epsilon_1 = \epsilon > 0$ and $\epsilon_2 = \epsilon > 0$. By continuity of $\exp(x)$ at the point $x = x_0$ and, $\exp(-x)$ at the point $x = x_0$ there are $\delta_1 > 0$ and $\delta_2 > 0$, respectively, such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_1$ we have $|\exp(x) - \exp(x_0)| < \epsilon_1$, and, whenever $|x - x_0| < \delta_2$ we have $|\exp(-x) - \exp(-x_0)| < \epsilon_2$, respectively. Equivalently:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_1 \to \frac{1}{2}|\exp(x) - \exp(x_0)| < \frac{\epsilon}{2}), \tag{113}$$

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_2 \to \frac{1}{2}|\exp(-x) - \exp(-x_0)| < \frac{\epsilon}{2}). \tag{114}$$

Step 3. (The delta Assessment). Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, combining inequalities (112)-(114) we have:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta \to |f_{5,1}(x) - f_{5,1}(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon). \tag{115}$$

Finally, the formula of $\delta = \delta(\epsilon, x_0) > 0$ via two times application of equation (69) for $\epsilon_1 = \epsilon, a = e$ and $\epsilon_2 = \epsilon, a = e^{-1}$, respectively, is given by:

$$\delta = \delta(\epsilon, x_0) = \min\{\ln(1 + e^{-x_0}\epsilon), \ln(1 + e^{x_0}\epsilon)\}. \tag{116}$$

2. Let $f_{5,2}(x) = \cosh(x)$, and fix $x_0 \in \mathbb{R}$. Then, given that $f_{5,2}(x) = \frac{\exp(x) + \exp(-x)}{2}$, a similar proof for above presented proof for the case of $f_{5,1}(x)$ applies. Hence:

$$\delta = \delta(\epsilon, x_0) = \min\{\ln(1 + e^{-x_0}\epsilon), \ln(1 + e^{x_0}\epsilon)\}. \tag{117}$$

3. Let $f_{5,3}(x) = \tanh(x)$, and fix $x_0 \in \mathbb{R}$. By definition, and equalities (23), (24) and lower bound of 1 for the function $\cosh(x)$ it follows that:

$$\begin{aligned} |f_{5,3}(x) - f_{5,3}(x_0)| &= |\frac{\sinh(x)}{\cosh(x)} - \frac{\sinh(x_0)}{\cosh(x_0)}|, \\ &= \frac{|\sinh(x)\cosh(x_0) - \sinh(x_0)\cosh(x)|}{\cosh(x)\cosh(x_0)}, \\ &\leq |\sinh(x - x_0)|, \\ &= \sinh(|x - x_0|). \end{aligned} \tag{118}$$

Now, it is sufficient to take:

$$\delta = \delta(\epsilon, x_0) = \operatorname{arsinh}(\epsilon). \tag{119}$$

Then, by (118) we have:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta \to |f_{5,3}(x) - f_{5,3}(x_0)| \leq \sinh(|x - x_0|) < \sinh(\operatorname{arsinh}(\epsilon)) = \epsilon). \tag{120}$$

4. Let $f_{5,4}(x) = \text{sech}(x)$, and fix $x_0 \in \mathbb{R}$. By definition, and lower bound of 1 for the function $\cosh(x)$ it follows that:

$$\begin{aligned} |f_{5,4}(x) - f_{5,4}(x_0)| &= |\frac{1}{\cosh(x)} - \frac{1}{\cosh(x_0)}|, \\ &= \frac{|\cosh(x) - \cosh(x_0)|}{\cosh(x)\cosh(x_0)}, \\ &\leq |\cosh(x) - \cosh(x_0)|. \end{aligned} \tag{121}$$

So any δ that makes $|\cosh(x) - \cosh(x_0)| < \epsilon$ also makes $|\text{sech}(x) - \text{sech}(x_0)| < \epsilon$. In this way the continuity of sech at $x = x_0$ is "dominated" by (indeed, follows from) the continuity of cosh at $x = x_0$. Hence, it is sufficient to take:

$$\delta = \delta(\epsilon, x_0) = \min\{\ln(1 + e^{-x_0}\epsilon), \ln(1 + e^{x_0}\epsilon)\}. \tag{122}$$

5. Let $f_{5,5}(x) = \coth(x)$, and fix $x_0 \neq 0$. We accomplish the proof in few steps:
Step 1.(Upper Bound Representation). By definition and equalities (23), (24):

$$\begin{aligned} |f_{5,5}(x) - f_{5,5}(x_0)| &= |\frac{\cosh(x)}{\sinh(x)} - \frac{\cosh(x_0)}{\sinh(x_0)}|, \\ &= \frac{|\cosh(x)\sinh(x_0) - \cosh(x_0)\sinh(x)|}{|\sinh(x)\sinh(x_0)|}, \\ &= \frac{|\sinh(x - x_0)|}{|\sinh(x)\sinh(x_0)|}, \\ &= \frac{\sinh(|x - x_0|)}{|\sinh(x)\sinh(x_0)|}. \end{aligned} \tag{123}$$

Step 2.(Neighborhood Approximation). Set $\epsilon_1 = \frac{|\sinh(x_0)|}{2} > 0$. Since $\sinh(.)$ is continuous at $x = x_0$, by equation (116) there is $\delta_1 = \min\{\ln(1 + e^{-x_0}(\frac{|\sinh(x_0)|}{2})), \ln(1 + e^{x_0}(\frac{|\sinh(x_0)|}{2}))\} > 0$ such that for all $x \in \mathbb{R}$ with $|x - x_0| < \delta_1$ we have $|\sinh(x) - \sinh(x_0)| < \frac{|\sinh(x_0)|}{2}$. As before, an application of reveres triangle inequality (28) $||\sinh(x)| - |\sinh(x_0)|| \leq |\sinh(x) - \sinh(x_0)|$ yields $||\sinh(x)| - |\sinh(x_0)|| \leq \frac{|\sinh(x_0)|}{2}$. This latest inequality implies $|\sinh(x)| \geq \frac{|\sinh(x_0)|}{2}$ or equivalently:

$$\frac{1}{|\sinh(x)\sinh(x_0)|} \leq 2|\sinh(x_0)|^{-2}. \tag{124}$$

Step 3. (The delta Assessment). Combining the equality (123) and inequality (124), and considering $\delta_2 = \text{arsinh}(\frac{|\sinh(x_0)|^2\epsilon}{2})$ for all $x \in \mathbb{R}$ with $|x - x_0| < \delta = \min\{\delta_1, \delta_2\}$ we have:

$$\begin{aligned} |f_{5,5}(x) - f_{5,5}(x_0)| &\leq 2|\sinh(x_0)|^{-2}\sinh(|x - x_0|) \\ &< 2|\sinh(x_0)|^{-2}\sinh(\text{arsinh}(\frac{|\sinh(x_0)|^2\epsilon}{2})) = \epsilon. \end{aligned} \tag{125}$$

Finally, $\delta = \min\{\delta_1, \delta_2\} > 0$ is given by:

$$\delta = \delta(\epsilon, x_0) = \min\{\ln(1 + e^{-x_0}(\frac{|\sinh(x_0)|}{2})), \ln(1 + e^{x_0}(\frac{|\sinh(x_0)|}{2})), \text{arsinh}(\frac{|\sinh(x_0)|^2\epsilon}{2})\}. \tag{126}$$

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6. Let $f_{5,6}(x) = \text{csch}(x)$, and fix $x_0 \neq 0$. The proof is essentially similar to he previous proof with some minor modifications as follows. First, the equation (123) is replaced by:

$$|f_{5,6}(x) - f_{5,6}(x_0)| = \frac{|\sinh(x) - \sinh(x_0)|}{|\sinh(x)\sinh(x_0)|}. \tag{127}$$

Second, for the same $\epsilon_1 = \frac{|\sinh(x_0)|}{2} > 0$, and $\delta_1 = \min\{\ln(1 + e^{-x_0}(\frac{|\sinh(x_0)|}{2})), \ln(1 + e^{x_0}(\frac{|\sinh(x_0)|}{2}))\} > 0$ inequality (124) holds. Third, considering $\delta_2 = \min\{\ln(1 + e^{-x_0}(\frac{|\sinh(x_0)|^2\epsilon}{2})), \ln(1 + e^{x_0}(\frac{|\sinh(x_0)|^2\epsilon}{2}))\}$ for all $x \in \mathbb{R}$ with $|x - x_0| < \delta = \min\{\delta_1, \delta_2\}$ we have:

$$|f_{5,6}(x) - f_{5,6}(x_0)| \quad \leq \quad 2|\sinh(x_0)|^{-2}|\sinh(x) - \sinh(x_0)| < \epsilon. \tag{128}$$

Finally, $\delta = \min\{\delta_1, \delta_2\} > 0$ is given by:

$$\delta = \delta(\epsilon, x_0) = \min_{b=1,2}\{\ln(1 + e^{-x_0}(\frac{|\sinh(x_0)|^b\epsilon^{b-1}}{2})), \ln(1 + e^{x_0}(\frac{|\sinh(x_0)|^b\epsilon^{b-1}}{2}))\}. \tag{129}$$

This completes the proof.□

3.2.6. Inverses-Hyperbolics

Proposition 11. ***(The Inverse-Hyperbolic functions)*** *Let* $f_6(x) = InvHyperb(x)$. *Then,* $C(f_6) = \mathbb{R}$, *if* $f_6 = \text{arsinh}; [1, \infty)$, *if* $f_6 = \text{arcosh}; (-1, +1)$, *if* $f_6 = \text{artanh}; \mathbb{R} - [-1, 1]$, *if* $f_6 = \text{arcoth}; (0, 1]$, *if* $f_6 = \text{arsech}; \mathbb{R} - \{0\}$, *if* $f_6 = \text{arcsch}$.

Proof. For each case below, fix $\epsilon > 0$ small enough so that the perturbed arguments $InvHyperb(x_0) \pm \epsilon$ remain in the domain of the corresponding direct map (e.g., for arcosh, arsech require $InvHyperb(x_0) - \epsilon \geq 0$; for arcoth, arcsch require $InvHyperb(x_0) \neq 0$ and $\epsilon < |InvHyperb(x_0)|$ to stay on the same branch). The proof schema is using Remark 8 similar to that of Inverse-Trigonometric functions. The list of functions and their corresponding δ's are as follows:

1. $f_{6,1}(x) = \text{arsinh}(x), \quad x_0 \in \mathbb{R}$:

$$\delta = \delta(\epsilon, x_0) = \min\{\sinh(\text{arsinh}(x_0) + \epsilon) - x_0,\ x_0 - \sinh(\text{arsinh}(x_0) - \epsilon)\}. \tag{130}$$

2. $f_{6,2}(x) = \text{arcosh}(x), \quad x_0 \in [1, \infty),\ \text{arcosh}(x_0) \geq 0$:

$$\delta = \delta(\epsilon, x_0) = \min\{\cosh(\text{arcosh}(x_0) + \epsilon) - x_0,\ x_0 - \cosh(\text{arcosh}(x_0) - \epsilon)\}. \tag{131}$$

3. $f_{6,3}(x) = \text{artanh}(x), \quad x_0 \in (-1, 1)$:

$$\delta = \delta(\epsilon, x_0) = \min\{\tanh(\text{artanh}(x_0) + \epsilon) - x_0,\ x_0 - \tanh(\text{artanh}(x_0) - \epsilon)\}. \tag{132}$$

4. $f_{6,4}(x) = \text{arcoth}(x), \quad x_0 \in (1, \infty)$ or $x_0 \in (-\infty, -1)$:

$$\delta = \delta(\epsilon, x_0) = \min\{\coth(\text{arcoth}(x_0) - \epsilon) - x_0,\ x_0 - \coth(\text{arcoth}(x_0) + \epsilon)\}. \tag{133}$$

5. $f_{6,5}(x) = \text{arsech}(x), \quad x_0 \in (0, 1],\ \text{arsech}(x_0) \geq 0$:

$$\delta = \delta(\epsilon, x_0) = \min\{\text{sech}(\text{arsech}(x_0) - \epsilon) - x_0,\ x_0 - \text{sech}(\text{arsech}(x_0) + \epsilon)\}. \tag{134}$$

6. $f_{6,6}(x) = \text{arcsch}(x), \quad x_0 \in \mathbb{R} \setminus \{0\}$:

$$\delta = \delta(\epsilon, x_0) = \min\{\text{csch}(\text{arcsch}(x_0) - \epsilon) - x_0,\ x_0 - \text{csch}(\text{arcsch}(x_0) + \epsilon)\}. \tag{135}$$

This completes the proof. □

Exercise 3.14. *An alternative $\epsilon - \delta$ method to prove continuity of the above inverse hyperbolic functions is consideration of the following equations and applying the proof scheme presented in Remark (7):*

(a) $\operatorname{arsinh}(x) - \operatorname{arsinh}(x_0) = \operatorname{arsinh}\Big(x\sqrt{1+x_0^2} - x_0\sqrt{1+x^2}\Big)$,

(b) $\operatorname{arccosh}(x) - \operatorname{arccosh}(x_0) = \operatorname{arccosh}\Big(xx_0 - \sqrt{(x^2-1)(x_0^2-1)}\Big)$,

(c) $\operatorname{artanh}(x) - \operatorname{artanh}(x_0) = \operatorname{artanh}\Big(\frac{x-x_0}{1-xx_0}\Big)$,

(d) $\operatorname{arcoth}(x) - \operatorname{arcoth}(x_0) = \operatorname{arcoth}\Big(\frac{1-xx_0}{x-x_0}\Big)$,

(e) $\operatorname{arsech}(x) - \operatorname{arsech}(x_0) = \operatorname{arccosh}\Big(\frac{1}{xx_0} - \sqrt{\Big(\frac{1}{x^2}-1\Big)\Big(\frac{1}{x_0^2}-1\Big)}\Big)$,

(f) $\operatorname{arcsch}(x) - \operatorname{arcsch}(x_0) = \operatorname{arsinh}\Big(\frac{1}{x}\sqrt{1+\frac{1}{x_0^2}} - \frac{1}{x_0}\sqrt{1+\frac{1}{x^2}}\Big)$.

3.2.7. Elementary Piecewise functions

Proposition 12. ***(The Elementary Piecewise functions)*** *Let $f_7(x) = ElPi(x)$. Then, $C(f_7) = \mathbb{R}$, if $f_7 \in \{absolute\ value, lower-clipping, upper-clipping, clamp-clip\}; \mathbb{R}-\{0\}$, if $f_7 = signum; \mathbb{R}-\mathbb{Z}$, if $f_7 \in \{floor, ceiling, fractional\}$.*

Proof. Given $\epsilon > 0$. We prove the continuity as follows:

1. Let $f_{7,1}(x) = |x|$, and fix $x_0 \in \mathbb{R}$. Using Reverse triangle inequality (28) we have:

$$|f_{7,1}(x) - f_{7,1}(x_0)| \le |x - x_0|. \tag{136}$$

Accordingly, by (136) it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = \epsilon. \tag{137}$$

2. Let $f_{7,2}(x) = \max\{x, c\}$, fix $x_0 \in \mathbb{R}$. Then, using equality (15) and inequality (28) we have:

$$\begin{aligned} |f_{7,2}(x) - f_{7,2}(x_0)| &= |\frac{x-x_0}{2} + \frac{|x-c| - |x_0-c|}{2}| \\ &\le \frac{|x-x_0|}{2} + \frac{||x-c| - |x_0-c||}{2} \\ &\le \frac{|x-x_0|}{2} + \frac{|(x-c) - (x_0-c)|}{2} \\ &= |x-x_0|. \end{aligned} \tag{138}$$

Accordingly, by (138) it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = \epsilon. \tag{139}$$

3. Let $f_{7,3}(x) = \min\{x, c\}$, fix $x_0 \in \mathbb{R}$. Then, using equality (16) and inequality (28) and similar to the process above, we have:

$$|f_{7,3}(x) - f_{7,3}(x_0)| \le |x - x_0|. \tag{140}$$

Accordingly, by (140) it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = \epsilon. \tag{141}$$

4. Let $f_{7,4}(x) = \max\{a, \min\{x, b\}\}$, fix $x_0 \in \mathbb{R}$. Then, by $f_{7,4}(x) = f_{7,2}(f_{7,3}(x))$ and inequalities (138), (140) it follows that:

$$|f_{7,4}(x) - f_{7,4}(x_0)| = |f_{7,2}(f_{7,3}(x)) - f_{7,2}(f_{7,3}(x_0))| \leq |f_{7,3}(x) - f_{7,3}(x_0)| \leq |x - x_0|. \tag{142}$$

Accordingly, by (142) it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = \epsilon. \tag{143}$$

5. Let $f_{7,5}(x) = \text{sgn}(x)$, fix $x_0 \neq 0$. Set $\delta = |x_0|$. Then:

$$|x - x_0| < |x_0| \iff x_0 - |x_0| < x < x_0 + |x_0|. \tag{144}$$

Next, from inequality (144), it follows that if $x_0 > 0$ then $x > 0$, and, if $x_0 < 0$ then $x < 0$. Hence, in both cases $\text{sgn}(x) = \text{sgn}(x_0)$ or equivalently: $|\text{sgn}(x) - \text{sgn}(x_0)| = 0 < \epsilon$. Consequently, it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = |x_0|. \tag{145}$$

6. Let $f_{7,6}(x) = \lfloor x \rfloor$, fix $x_0 \in \mathbb{R} - \mathbb{Z}$. Set $\delta = \min\{x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\}$. Then:

$$\begin{aligned} &|x - x_0| < \min\{x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\} \iff \\ \lfloor x_0 \rfloor \leq \max\{\lfloor x_0 \rfloor, 2x_0 - \lfloor x_0 \rfloor - 1\} < x < &\min\{2x_0 - \lfloor x_0 \rfloor, \lfloor x_0 \rfloor + 1\} \leq \lfloor x_0 \rfloor + 1. \end{aligned} \tag{146}$$

Next, from inequality (146), it follows that $\lfloor x \rfloor = \lfloor x_0 \rfloor$ or equivalently $|\lfloor x \rfloor - \lfloor x_0 \rfloor| = 0 < \epsilon$. Consequently, it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = \min\{x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\}. \tag{147}$$

7. Let $f_{7,7}(x) = \lceil x \rceil$, fix $x_0 \in \mathbb{R} - \mathbb{Z}$. Set $\delta = \min\{\lceil x_0 \rceil - x_0, 1 - (\lceil x_0 \rceil - x_0)\}$. Then:

$$\begin{aligned} &|x - x_0| < \min\{\lceil x_0 \rceil - x_0, 1 - (\lceil x_0 \rceil - x_0)\} \iff \\ \lceil x_0 \rceil - 1 \leq \max\{2x_0 - \lceil x_0 \rceil, \lceil x_0 \rceil - 1\} < x < &\min\{\lceil x_0 \rceil, 2x_0 - \lceil x_0 \rceil + 1\} \leq \lceil x_0 \rceil. \end{aligned} \tag{148}$$

Next, from inequality (148), it follows that $\lceil x \rceil = \lceil x_0 \rceil$ or equivalently $|\lceil x \rceil - \lceil x_0 \rceil| = 0 < \epsilon$. Consequently, it is sufficient to set:

$$\delta = \delta(\epsilon, x_0) = \min\{\lceil x_0 \rceil - x_0, 1 - (\lceil x_0 \rceil - x_0)\}. \tag{149}$$

8. Let $f_{7,8}(x) = \{x\}$, fix $x_0 \in \mathbb{R} - \mathbb{Z}$. Using triangle inequality we have:

$$\begin{aligned} |f_{7,8}(x) - f_{7,8}(x_0)| &= |(x - \lfloor x \rfloor) - (x_0 - \lfloor x_0 \rfloor)| \\ &= |(x - x_0) - (\lfloor x \rfloor - \lfloor x_0 \rfloor)| \\ &\leq |x - x_0| + |\lfloor x \rfloor - \lfloor x_0 \rfloor|. \end{aligned} \tag{150}$$

Next, for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_1 = \epsilon$, we have $|x - x_0| < \epsilon$, and whenever $|x - x_0| < \delta_2 = \min\{x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\}$, we have $|\lfloor x \rfloor - \lfloor x_0 \rfloor| = 0$, respectively. Hence, it is sufficient to take $\delta = \min\{\delta_1, \delta_2\}$ to ensure the right hand side of inequality (150) is less than ϵ. Finally:

$$\delta = \delta(\epsilon, x_0) = \min\{\epsilon, x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\}. \tag{151}$$

This completes the proof. □

Exercise 3.15. *The modification of above proofs can be applied for the following functions:*

(a) *the Rectified Linear Unit function* $(\mathrm{ReLU}(x) = \max\{0, x\})$,

(b) *the Huber loss function,*$\delta > 0(l_\delta(x) = (\frac{x^2}{2})1_{\{|x|\leq\delta\}}(x) + \delta(|x| - \frac{1}{2}\delta)1_{\{|x|>\delta\}}(x)$,

(c) *the Softsign function* $(\mathrm{softsign}(x) = \frac{x}{1+|x|})$,

(d) *the Clamped / HardTanh function* $(h(x) = \min\{1, \max\{-1, x\}\})$,

(e) *the Hinge loss function* $(\ell(x) = \max\{0, 1 - x\})$,

(f) *the Ramp function:* $((x)_+ = \max\{0, x\})$,

(g) *the Heaviside function:* $(H(x) = 1_{\{x\geq 0\}}(x))$,

(h) *the Indicator function:* $(Ind(x) = 1_{\{x>c\}}(x))$,

(i) *the Sawtooth norm:* $(\|x\| = \min_{k\in\mathbb{Z}} |x - k|)$,

(j) *the Tent map:* $(T_r(x) = r(1 - 2|x - \frac{1}{2}|) \qquad (0 \leq x \leq 1,\ 0 < r \leq 2))$,

(k) *the Sawtooth wave function:* $(\ \mathrm{saw}(x) = 2(\{x\} - \frac{1}{2}) \qquad (\{x\} = x - \lfloor x \rfloor))$,

(l) *the Triangle wave function:* $(\tau(x) = 1 - 2\,\|x\| \qquad (\|x\| = \min_{k\in\mathbb{Z}} |x - k|))$,

(m) *the Square wave function:* $(q(x) = \mathrm{sgn}(\sin x))$,

(n) *the Rect(boxcar) kernel:* $(\mathrm{rect}(x) = 1_{\{|x|\leq\frac{1}{2}\}}(x))$,

(o) *the Triangle Kernel function:* $(\Lambda(x) = \max\{0, 1 - |x|\})$.

Remark 13. *The last six functions in above list are members of waves.*

3.2.8. Pathologic Functions

Proposition 13. ***(The Volterra-type power–sine family functions)*** *Let* $f_8(x) = x^a \sin(x^b), a, b \in \mathbb{R}, f_8(0) = 0$. *Then,* $C(f_8) = \mathbb{R},\ if\ (a, b) \in \{(2, -1), (+1, -1)\}; \mathbb{R} - \{0\},\ if\ (a, b) = (0, -1)$.

Proof. Given $\epsilon > 0$. We prove the continuity as follows:

1. Let $f_{8,1}(x) = x^a \sin(x^b)$ $a > 0 > b$, and fix $x_0 \in \mathbb{R}$. First, when $x_0^a \sin(x_0^b) = 0$, it is sufficient to consider $\delta = \delta(\epsilon, x_0) = \min\{|x_0 + \epsilon^{\frac{1}{a}}|, |x_0 - \epsilon^{\frac{1}{a}}|\}$. Second, assume $|x_0^a \sin(x_0^b)| > 0$. We accomplish the proof in few steps.
Step 1.(Upper Bound Representation). Using inequality (30) it follows that:

$$\begin{aligned} |f_{8,1}(x) - f_{8,1}(x_0)| &= |x^a \sin(x^b) - x_0^a \sin(x_0^b)| \\ &= |(x^a - x_0^a)\sin(x_0^b) + x_0^a(\sin(x^b) - \sin(x_0^b))| \\ &\leq |(x^a - x_0^a)||\sin(x_0^b)| + |x_0^a||(\sin(x^b) - \sin(x_0^b))| \\ &\leq |(x^a - x_0^a)||\sin(x_0^b)| + |x_0^a||x^b - x_0^b|. \end{aligned} \tag{152}$$

Step 2. (Neighborhood Approximation). Set $\epsilon_1 = \frac{|\sin(x_0^b)|^{-1}\epsilon}{2}$ and $\epsilon_2 = \frac{|x_0|^{-a}\epsilon}{2}$.By continuity of the functions x^a and x^b at point $x = x_0$, respectively, there are $\delta_1 > 0$ and $\delta_2 > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_1$ we have $|(x^a - x_0^a)| < \epsilon_1$, and whenever $|x - x_0| < \delta_2$ we have $|(x^b - x_0^b)| < \epsilon_2$, respectively. Equivalently:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_1 \to |(x^a - x_0^a)||\sin(x_0^b)| < \frac{\epsilon}{2}), \tag{153}$$

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_2 \to |x_0^a||(x^b - x_0^b)| < \frac{\epsilon}{2}). \tag{154}$$

Step 3. (The delta Assessment). Let $\delta = \min\{\delta_1, \delta_2\} > 0$. Then, combining inequalities (152)-(154) we have:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta \to |f_{8,1}(x) - f_{8,1}(x_0)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \tag{155}$$

Finally, the formulae of $\delta = \delta(\epsilon, x_0) > 0$ via two times application of equation (55) for $r_1 = a, \epsilon_1 = \frac{|\sin(x_0^b)|^{-1}\epsilon}{2}$ and $r_2 = b, \epsilon_2 = \frac{|x_0|^{-a}\epsilon}{2}$, respectively is given by:

$$\delta = \delta(\epsilon, x_0) \quad = \quad \min\{|\frac{x_0}{2}|, \frac{2^{a-2}|x_0|^{-(a-1)}|\sin(x_0^b)|^{-1}\epsilon}{|a|(1_{[1,\infty)}(a)3^{a-1} + 1_{(-\infty,1)}(a))}, \frac{2^{b-2}|x_0|^{-a-b+1}\epsilon}{|b|}\}. \tag{156}$$

2. Let $f_{8,2}(x) = \sin(x^b)$ $0 > b$, and fix $x_0 \neq 0$. Then, using inequality (30) it follows that:

$$|f_{8,2}(x) - f_{8,2}(x_0)| = |\sin(x^b) - \sin(x_0^b)| \leq |x^b - x_0^b|. \tag{157}$$

Hence, by an application of equation (55) for $r_1 = b, \epsilon_1 = \epsilon$ it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{|\frac{x_0}{2}|, \frac{\epsilon}{|b||\frac{x_0}{2}|^{b-1}}\}. \tag{158}$$

This completes the proof. □

Proposition 14. *(The Cantor function) Let* $f_{8,4}(x := \sum_{n=1}^{+\infty} \frac{a_n(x)}{3^n}) = \sum_{n=1}^{N(x)-1} \frac{a_n(x)}{2^{n+1}} + \frac{\mathbf{1}_{\{N(x)<\infty\}}}{2^{N(x)}}$: $N(x) := \min\{n \in \mathbb{N} : a_n(x) = 1\}$, $N(x) = \infty$ *if no such n. Then,* $C(f_{8,4}) = [0,1]$.

Proof. Given $\epsilon > 0$. Fix $x_0 \in [0,1]$. We accomplish the proof in few steps as follows:

Step 1.(Key prefix–tail bounds (from the formula)) Set $x := \sum_{n=1}^{+\infty} \frac{a_n(x)}{3^n}$ and $x_0 := \sum_{n=1}^{+\infty} \frac{a_n(x_0)}{3^n}$. Then:

- **Exact prefix match:** If $a_1(x) = a_1(x_0), \ldots, a_n(x) = a_n(x_0)$, then:

$$|f_{8,4}(x) - f_{8,4}(x_0)| \leq \sum_{k>n} \frac{1}{2^k} = 2^{-n}. \tag{159}$$

- **Adjacent prefix (first difference at position n):** If the first index where a_n differ is n (so the first $n-1$ digits agree), then:

$$|f_{8,4}(x) - f_{8,4}(x_0)| \leq 2^{-(n-1)}. \tag{160}$$

(Worst case: one side contributes a 2^{-n} "first-1 spike" plus a tail $< 2^{-n}$.)

Step 2.(Turning closeness in x into a prefix agreement) Partition $[0,1]$ into level-n triadic intervals ("cylinders") determined by the first n ternary digits. If $|x - x_0| < 3^{-n}$, then x and x_0 lie either

- in the **same** level-n cylinder (first n digits agree), or
- in **adjacent** level-n cylinders (first $n-1$ digits agree and they differ at digit n).

Therefore, for all x, x_0 with $|x - x_0| < 3^{-n}$:

$$|f_{8,4}(x) - f_{8,4}(x_0)| \leq 2^{-(n-1)}. \tag{161}$$

Step 3.(The delta Assessment) Choose: $n = \lceil \log_2(\frac{2}{\epsilon}) \rceil$ and set $\delta = 3^{-n}$. Then, whenever $|x - x_0| < \delta$, we are in one of the two cases above, so $|f_{8,4}(x) - f_{8,4}(x_0)| \leq 2^{-(n-1)} = 2^{-(\lceil \log_2(\frac{2}{\epsilon}) \rceil - 1)} \leq \epsilon$. Hence, it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = 3^{-\lceil \log_2(\frac{2}{\epsilon}) \rceil}. \tag{162}$$

This completes the proof. □

Proposition 15. *(The Thomae function) Let* $f_{8,5}(\frac{m}{n}\mathbf{1}_{\mathbb{Q}}(x := \frac{m}{n}) + x\,\mathbf{1}_{\mathbb{Q}^c}(x)) \;=\; \frac{1}{n}\mathbf{1}_{\mathbb{Q}}(x)$: $(m,n) = 1$, *Then,* $C(f_{8,5}) = \mathbb{Q}^c$.

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Proof. Given $\epsilon > 0$ and without loss of generality we assume $0 < \epsilon < 1$. We accomplish the proof in two steps as follows:

Step 1. (Filter the "big spikes" = small denominators) Fix $x_0 \in \mathbb{Q}^c$. Consider the highest values of $f_{8,5}$ candidate set of $A(\epsilon) = \{x | x = \frac{p}{q} \in \mathbb{Q}, (p,q) = 1, 1 \leq q \leq \lceil \frac{1}{\epsilon} \rceil\}$. Here, for any $x \in A(\epsilon)$, we have $f_{8,5}(x) > \frac{1}{\lceil \frac{1}{\epsilon} \rceil} \geq \frac{\epsilon}{2}$. So, if we consider a neighborhood of x_0 such that the set $A(\epsilon)$ is removed, we will prove the statement.

Step 2 (Punch a small hole around x_0 that misses those spikes) Define:

$$\delta = \delta(\epsilon, x_0) = \frac{1}{2} \min\{|x - x_0| | x = \frac{p}{q} \in \mathbb{Q}, (p,q) = 1, 1 \leq q \leq \lceil \frac{1}{\epsilon} \rceil\}. \tag{163}$$

Then, for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta$ we have the following cases:

Case 1: Fix $x \in \mathbb{Q}^c$. Then, $|f_{8,5}(x) - f_{8,5}(x_0)| = |0 - 0| = 0 < \epsilon$.

Case 2: Fix $x \in \mathbb{Q}$. Then, $x = \frac{a_x}{b_x}$: $(a_x, b_x) = 1$, and $b_x > \lceil \frac{1}{\epsilon} \rceil$ (Otherwise, if $b_x \leq \lceil \frac{1}{\epsilon} \rceil$, then $\delta > |x - x_0| \geq 2\delta > \delta$, yielding $\delta > \delta$ a contradiction !). Accordingly, $|f_{8,5}(x) - f_{8,5}(x_0)| = |\frac{1}{b_x} - 0| < \frac{1}{\lceil \frac{1}{\epsilon} \rceil} < \epsilon$.

This completes the proof. □

Proposition 16. *(The Riemann function) Let* $f_{8,6}(x) = \sum_{n=1}^{+\infty} \dfrac{\sin(n^2 x)}{n^2}$. *Then,* $C(f_{8,6}) = \mathbb{R}$.

Proof. Given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$. We accomplish the proof in few steps as follows:

Step 1.(Upper Bound Representation) First, using inequality (90), for any $x \in \mathbb{R}$ we have $|\sin(x) - \sin(x_0)| \leq \min\{|x - x_0|, 2\}$. Hence:

$$\begin{aligned}
|f_{8,6}(x) - f_{8,6}(x_0)| &= |\sum_{n=1}^{+\infty} \frac{\sin(n^2 x) - \sin(n^2 x_0)}{n^2}| \\
&\leq \sum_{n=1}^{+\infty} \frac{|\sin(n^2 x) - \sin(n^2 x_0)|}{n^2} \\
&\leq \sum_{n=1}^{+\infty} \frac{\min\{n^2|x - x_0|, 2\}}{n^2} \\
&= \sum_{n=1}^{N} |x - x_0| + \sum_{n=N+1}^{+\infty} \frac{2}{n^2} \; : \; N = \lfloor \sqrt{\frac{2}{|x - x_0|}} \rfloor.
\end{aligned} \tag{164}$$

Note that $N \in \mathbb{N}$, is granted by setting $\delta_0 = 2$, such that $|x - x_0| < 2$.

Step 2. (Upper Bounds for Components) Second, we find an upper bound in terms $|x - x_0|$ for each component in the right hand side of inequality (164). Here, the first component is bounded as in:

$$\sum_{n=1}^{N} |x - x_0| = N|x - x_0| = \lfloor \sqrt{\frac{2}{|x - x_0|}} \rfloor |x - x_0| \leq 2\sqrt{|x - x_0|}. \tag{165}$$

Next, let $k_0 \in \mathbb{N}$ satisfy $2^{k_0} \leq N + 1 \leq 2^{k_0+1} - 1 < 2^{k_0+1}$ and for all $k \geq k_0$ consider the bins of 2^k positive integers $2^k \leq n \leq 2^{k+1} - 1$. Accordingly:

$$\begin{aligned}
\sum_{n=N+1}^{+\infty} \frac{2}{n^2} \leq \sum_{n=2^{k_0}}^{+\infty} \frac{2}{n^2} \leq \sum_{k=k_0}^{+\infty} 2^k \frac{2}{2^{2k}} &= \sum_{k=k_0}^{+\infty} \frac{2}{2^k} = \frac{2}{2^{k_0 - 1}} = \frac{8}{2^{k_0+1}} \\
&< \frac{8}{N+1} < \frac{8}{N} = \frac{8}{\lfloor \sqrt{\frac{2}{|x - x_0|}} \rfloor} \\
&\leq 8(2\sqrt{|x - x_0|}) = 16\sqrt{|x - x_0|}.
\end{aligned} \tag{166}$$

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Step 3.(The delta Assessment) Third, by inequalities (164)-(166) we have:

$$|f_{8,6}(x) - f_{8,6}(x_0)| \leq 2\sqrt{|x-x_0|} + 16\sqrt{|x-x_0|} = 18\sqrt{|x-x_0|}. \quad (167)$$

Finally, using upper bound in inequality (167) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{2, (\frac{\epsilon}{18})^2\}. \quad (168)$$

This completes the proof. □

Proposition 17. ***(The Takagi function)*** *Let* $f_{8,7}(x) = \sum_{n=0}^{+\infty} \frac{1.7em(2^n x,\mathbb{Z})}{2^n}$*. Then,* $C(f_{8,7}) = \mathbb{R}$*.*

Proof. Given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$. We accomplish the proof in few steps as follows:

Step 1.(Upper Bound Representation) First, using inequality (31) and bounds $|1.7em(x,\mathbb{Z})| \leq \frac{1}{2}$, for any $x \in \mathbb{R}$ we have:

$$\begin{aligned}
|f_{8,7}(x) - f_{8,7}(x_0)| &= |\sum_{n=0}^{+\infty} \frac{1.7em(2^n x,\mathbb{Z})}{2^n} - \sum_{n=0}^{+\infty} \frac{1.7em(2^n x_0,\mathbb{Z})}{2^n}| \\
&= |(\sum_{n=0}^{N} \frac{1.7em(2^n x,\mathbb{Z})}{2^n} + \sum_{n=N+1}^{+\infty} \frac{1.7em(2^n x,\mathbb{Z})}{2^n}) - (\sum_{n=0}^{N} \frac{1.7em(2^n x_0,\mathbb{Z})}{2^n} + \sum_{n=N+1}^{+\infty} \frac{1.7em(2^n x_0,\mathbb{Z})}{2^n})| \\
&\leq |\sum_{n=0}^{N} \frac{1.7em(2^n x,\mathbb{Z})}{2^n} - \sum_{n=0}^{N} \frac{1.7em(2^n x_0,\mathbb{Z})}{2^n}| + \sum_{n=N+1}^{+\infty} \frac{1.7em(2^n x,\mathbb{Z})}{2^n} + \sum_{n=N+1}^{+\infty} \frac{1.7em(2^n x_0,\mathbb{Z})}{2^n} \\
&\leq \sum_{n=0}^{N} \frac{|1.7em(2^n x,\mathbb{Z}) - 1.7em(2^n x_0,\mathbb{Z})|}{2^n} + \sum_{n=N+1}^{+\infty} \frac{(1.7em(2^n x,\mathbb{Z}) + 1.7em(2^n x_0,\mathbb{Z}))}{2^n} \\
&\leq \sum_{n=0}^{N} \frac{2^n|x-x_0|}{2^n} + \sum_{n=N+1}^{+\infty} \frac{2(\frac{1}{2})}{2^n} \\
&= (N+1)|x-x_0| + 2^{-N}. \quad (169)
\end{aligned}$$

Step 2. (Upper Bounds for Components) Second, using the right hand side components in inequality (169), for $\lceil \log_2(\frac{2}{\epsilon}) \rceil \leq N$ we have $2^{-N} < \frac{\epsilon}{2}$. Furthermore, for $\delta = \frac{\epsilon}{2(N+1)}$ whenever $|x-x_0| < \delta$ we have $(N+1)|x-x_0| < \frac{\epsilon}{2}$.

Step 3.(The delta Assessment) Finally, from previous step, substituting N in the δ, the $\delta = \delta(\epsilon, x_0)$ is given by:

$$\delta = \delta(\epsilon, x_0) = \frac{\epsilon}{2(\lceil \log_2(\frac{2}{\epsilon}) \rceil + 1)}. \quad (170)$$

This completes the proof. □

Proposition 18. ***(The Weierstrass function)*** *Let* $f_{8,8}(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x) : 0 < a < 1, b \in 2\mathbb{N}+1, ab > 1 + \frac{3\pi}{2}$*. Then,* $C(f_{8,8}) = \mathbb{R}$*.*

Proof. Given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$. We accomplish the proof in few steps as follows:

Step 1.(Upper Bound Representation) First, using equality (18) and inequality (30), respectively, for any $x \in \mathbb{R}$ we have $|\cos(x) - \cos(x_0)| \leq 2^{1-d}|x - x_0|^d, \ \ 0 \leq d \leq 1$. Hence:

$$
\begin{aligned}
|f_{8,8}(x) - f_{8,8}(x_0)| &= |\sum_{n=0}^{+\infty}(a^n \cos(b^n \pi x) - a^n \cos(b^n \pi x_0))| \\
&\leq \sum_{n=0}^{+\infty} a^n |\cos(b^n \pi x) - \cos(b^n \pi x_0)| \\
&\leq \sum_{n=0}^{+\infty} a^n (2^{1-d}\pi^d b^{dn}|x - x_0|^d) \\
&= 2^{1-d}\pi^d |x - x_0|^d \sum_{n=0}^{+\infty}(ab^d)^n : \ |ab^d| < 1, \\
&= \frac{2(\pi/2)^d}{1 - ab^d}|x - x_0|^d : \ 0 < d < -\log_b(a) < 1 - \log_b(1 + \frac{3\pi}{2}) < 1. \quad (171)
\end{aligned}
$$

Step 2.(Neighborhood Approximation). Set $\epsilon_1 = (\frac{1-ab^d}{2(\pi/2)^d})^{\frac{1}{d}}\epsilon^{\frac{1}{d}}$. Then, for $\delta_1 = \epsilon_1$ for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_1$ we have $|x - x_0| < \epsilon_1$, or equivalently:

$$\forall x \in \mathbb{R}(|x - x_0| < \delta_1 \to \frac{2(\pi/2)^d}{1 - ab^d}|x - x_0|^d < \epsilon). \tag{172}$$

Step 3.(The delta Assessment) Finally, by inequalities (171) and (172) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = (\frac{1 - ab^d}{2(\pi/2)^d})^{\frac{1}{d}}\epsilon^{\frac{1}{d}} : \ \ 0 < d < -\log_b(a). \tag{173}$$

This completes the proof. □

Proposition 19. *(The Type I Dirichlet function) Let $f_{8,9}(x) = 1_{\mathbb{Q}}(x)$. Then, $C(f_{8,9}) = \emptyset$.*

Proof. Given $\epsilon > 0$. Without loss of generality we may assume $0 < \epsilon < 1$. Fix $x_0 \in \mathbb{R}$. We accomplish the proof in few steps as follows:

Step 1. (Restriction of Discussion Domain). Given that $f_{8,9}$ has period $T = 1$ and for $x_0 = \lfloor x_0 \rfloor + \{x_0\}$, $\lfloor x_0 \rfloor \in \mathbb{Z}$, we have $f_{8,9}(x_0) = f_{8,9}(\{x_0\})$, it is sufficient to assume $x_0 \in [0, 1)$.

Step 2. (Proof by Contradiction) Assume there is $1 > \delta = \delta(\epsilon, x_0) > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$. We consider two separate cases as follows:

Case 1: Assume $x_0 \notin \mathbb{Q}$. Then, consider the unique base 10 representation $x_0 = \sum_{n=1}^{+\infty} \frac{a_{n,x_0}}{10^n}$ where $a_{n,x_0} \in \mathbb{N}_{10} - 1$ for all $n \in \mathbb{N}$. Take the rational number $x = \sum_{n=1}^{\lceil \log_{10}(\frac{10}{\delta}) \rceil} \frac{a_{n,x_0}}{10^n}$. Then:

$$|x - x_0| = \sum_{n=\lceil \log_{10}(\frac{10}{\delta}) \rceil + 1}^{+\infty} \frac{a_{n,x_0}}{10^n} \leq \sum_{n=\lceil \log_{10}(\frac{10}{\delta}) \rceil + 1}^{+\infty} \frac{9}{10^n} = \frac{1}{10^{\lceil \log_{10}(\frac{10}{\delta}) \rceil}} < \delta. \tag{174}$$

But, $|f(x) - f(x_0)| = |1 - 0| = 1 < \epsilon < 1$, a contradiction.

Case 2: Assume $x_0 \in \mathbb{Q}$. Take the irrational number $x = x_0 + \frac{\pi}{10^{\lceil \log_{10}(\frac{10\pi}{\delta}) \rceil}}$. Then:

$$|x - x_0| = \frac{\pi}{10^{\lceil \log_{10}(\frac{10\pi}{\delta}) \rceil}} < \delta. \tag{175}$$

But, $|f(x) - f(x_0)| = |0 - 1| = 1 < \epsilon < 1$, again a contradiction.

This completes the proof. □

Proposition 20. *(The Type II Dirichlet function) Let* $f_{8,10}(x) = \left(\prod_{k=1}^{n}(x-k)\right)1_{\mathbb{Q}}(x)$*. Then,* $C(f_{8,10}) = \mathbb{N}_n$.

Proof. Given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$. We consider two distinct cases as follows:
Case 1: Assume $x_0 = k_0$ for some $k_0 \in \mathbb{N}_n$. Then, using equation (51) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \min\{1, \frac{\epsilon}{\prod_{1\le k\ne k_0\le n}(|x_0|+k+1)}\}. \tag{176}$$

Case 2: Assume $x_0 \neq k_0$ for all $k_0 \in \mathbb{N}_n$. Define $g_2(x) = \prod_{k=1}^{n}(x-k)$ and let $\epsilon_g = \dfrac{|g_2(x_0)|}{2} > 0$. Since, g_2 is continuous at $x = x_0$ there exists $\delta_{g_2} > 0$ such that for all $x \in \mathbb{R}$ whenever $|x - x_0| < \delta_{g2}$ we have: $|g_2(x) - g_2(x_0)| < \dfrac{|g_2(x_0)|}{2}$. Hence, using the reverse triangle inequality (28) $||g_2(x)| - |g_2(x_0)|| \le |g_2(x) - g_2(x_0)|$ we have $||g_2(x)| - |g_2(x_0)|| < \dfrac{|g_2(x_0)|}{2}$. This latest inequality yields $|g_2(x)| > \dfrac{|g_2(x_0)|}{2} = C(x_0)$ and, furthermore:

$$|f_{8,10}(x)| = \left(\prod_{k=1}^{n}|(x-k)|\right)1_{\mathbb{Q}}(x) \ge C(x_0)1_{\mathbb{Q}}(x), \text{ for all } x \in (x_0 - \delta_{g_2}, x_0 + \delta_{g_2}). \tag{177}$$

Finally, it is sufficient to repeat the slightly modified version of the proof in Proposition 19 for the function $g(x) = C(x_0)1_{\mathbb{Q}}(x)$ on the neighborhood $(x_0 - \delta_{g_2}, x_0 + \delta_{g_2})$ to ensure discontinuity at the point $x = x_0$.
This completes the proof. □

Proposition 21. *(The Type III Dirichlet function) Let* $f_{8,11}(x) = \sin(\pi x)1_{\mathbb{Q}}(x)$*. Then,* $C(f_{8,11}) = \mathbb{Z}$.

Proof. Given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$. We consider two distinct cases as follows:
Case 1: Assume $x_0 = k_0$ for some $k_0 \in \mathbb{Z}$. Then, using equations (90) and (91) it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = \frac{\epsilon}{\pi}. \tag{178}$$

Case 2: Assume $x_0 \neq k_0$ for all $k_0 \in \mathbb{Z}$. Then, it is sufficient to consider $g_3(x) = \sin(\pi x)$ and repeat the similar argument to the one in the proof of Proposition 20 to ensure discontinuity at the point $x = x_0$.
This completes the proof. □

Proposition 22. *(The Type IV Dirichlet function) Let* $f_{8,12}$ *be defined on the closed unit interval by :*

$$f_{8,12}(x) = \left(\sum_{n=1}^{+\infty}\sum_{k=1}^{2^{n-1}}\left[\sqrt{3}\left(\frac{3^{-n}}{2} - \left|x - \frac{a_{n,k}+b_{n,k}}{2}\right|\right)\mathbf{1}_{(a_{n,k},b_{n,k})}(x)\right]\right)1_{\mathbb{Q}}(x),$$

where $a_{n,k}$ *and* $b_{n,k}$ *are defined by a unique sequence of digits* $a_{n,k,i} = 0, 2(1 \le i \le n-1), a_{n,k,n} = 1$ *by* $a_{n,k} = \sum_{i=1}^{n}\frac{a_{n,k,i}}{3^i}$ *and* $b_{n,k} = a_{n,k} + \frac{1}{3^n}$*, respectively for all* $k \in \{1, \ldots, 2^{n-1}\}, n \in \mathbb{N}$*. Then,* $C(f_{8,12}) = C$*, where* C *is the classical ternary Cantor-set.*

Proof. Given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$. We consider two distinct cases as follows:
Case 1: Assume $x_0 \in C$ with base-3 representation $x_0 = \sum_{n=1}^{+\infty}\frac{a_n(x_0)}{3^n}$: $a_n(x_0) = 0, 2$ $(n \in \mathbb{N})$. Then, by definition, $\mathbf{1}_{(a_{n,k},b_{n,k})}(x_0) = 0$ for all $k \in \mathbb{N}_{2^{n-1}}, n \in \mathbb{N}$. Hence, $f_{8,12}(x_0) = 0$. We modify the proof of Proposition 14 as follows:
Step 1.(Key prefix–tail bounds (from the formula)) Set $x := \sum_{n=1}^{+\infty}\frac{a_n(x)}{3^n}$ and $x_0 := \sum_{n=1}^{+\infty}\frac{a_n(x_0)}{3^n}$. Then:

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- **Exact prefix match:** If $a_1(x) = a_1(x_0), \ldots, a_n(x) = a_n(x_0)$, then:

$$|f_{8,12}(x) - f_{8,12}(x_0)| = f_{8,12}(x) \leq \max_{m \geq n+1} \left(\sqrt{3}\, \frac{3^{-m}}{2} \right) = \frac{\sqrt{3}}{2} 3^{-(n+1)}. \tag{179}$$

- **Adjacent prefix (first difference at position n):** If the first index where a_n differ is n (so the first $n-1$ digits agree), then:

$$|f_{8,12}(x) - f_{8,12}(x_0)| = f_{8,12}(x) \leq \frac{\sqrt{3}}{2} 3^{-n}.. \tag{180}$$

(*Key point: intervals do* not *pile up; for each x only the first deletion level contributes, which is why the bounds are exactly $\frac{\sqrt{3}}{2} 3^{-(n+1)}$ and $\frac{\sqrt{3}}{2} 3^{-n}$.)*

Step 2.(Turning closeness in x into a prefix agreement) Partition $[0,1]$ into level-n triadic intervals ("cylinders") determined by the first n ternary digits. If $|x - x_0| < 3^{-n}$, then x and x_0 lie either

- in the **same** level-n cylinder (first n digits agree), or
- in **adjacent** level-n cylinders (first $n-1$ digits agree and they differ at digit n).

Therefore, for all x, x_0 with $|x - x_0| < 3^{-n}$:

$$|f_{8,12}(x) - f_{8,12}(x_0)| \leq \frac{\sqrt{3}}{2} 3^{-n}. \tag{181}$$

Step 3.(The delta Assessment) Choose: $n = \left\lceil \log_3\!\left(\frac{\sqrt{3}}{2\epsilon}\right) \right\rceil$ and set $\delta = 3^{-n}$. Then, whenever $|x - x_0| < \delta$, we are in one of the two cases above, so $|f_{8,12}(x) - f_{8,12}(x_0)| \leq \frac{\sqrt{3}}{2} 3^{-n} = \frac{\sqrt{3}}{2} 3^{-\left\lceil \log_3\left(\frac{\sqrt{3}}{2\epsilon}\right) \right\rceil} \leq \epsilon$. Hence, it is sufficient to consider:

$$\delta = \delta(\epsilon, x_0) = 3^{-\left\lceil \log_3\left(\frac{\sqrt{3}}{2\epsilon}\right) \right\rceil}. \tag{182}$$

Case 2: Assume $x_0 \notin C$. Then, it is sufficient to consider $g_4(x) = \sqrt{3}\left(\frac{3^{-n}}{2} - \left|x - \frac{a_{n,k}+b_{n,k}}{2}\right|\right)\mathbf{1}_{(a_{n,k},b_{n,k})}(x)$ for some $a_{n,k}, b_{n,k}$ and repeat the similar argument to the one in the proof of Proposition 20 to ensure discontinuity at the point $x = x_0$.

This completes the proof. □

Exercise 3.16. *Given $f : \mathbb{R} \to \mathbb{R}$. Assume for some continuous function $g : \mathbb{R} \to \mathbb{R}$ we have $f(x) = g(x)1_{\mathbb{Q}}(x)$ for all $x \in \mathbb{R}$. Using presented ideas in the proofs of above Dirichlet type functions it follows that:*

(a) f *is type I Dirichlet function whenever* $|R(g)| = 0$,
(b) f *is type II Dirichlet function whenever* $|R(g)| = n$ *(for some* $n \in \mathbb{N}$*)*,
(c) f *is type III Dirichlet function whenever* $|R(g)| = \aleph_0$,
(d) f *is type IV Dirichlet function whenever* $|R(g)| = |R(g)^c| = \mathfrak{c}$.

4. Discussion

4.1. Summary and Contributions

This work offers a survey of direct ϵ–δ continuity proofs across eight clusters of real-valued functions, encompassing 54 representative functions. Table (1) presents summary of all related results. It also develops direct ϵ–δ proofs for finite algebraic operations—sum, product, composition and maximum (minimum)—on a prescribed collection of n functions $f : \mathbb{R} \to \mathbb{R}$. A notable feature of the exposition is its reliance on only minimal auxiliary results (elementary equalities and inequalities) and basic facts about infinite series, deliberately avoiding advanced theorems and their corollaries.

Table 1. Summary of prominent real valued functions and associated δ for their continuity at point $x = x_0$.

#	Cluster –Function ($f(x)$)	Sample $\delta = \delta(\epsilon, x_0)$
1	Algebraics	
1-1	—Polynomials	
	$f_{1,1}(x) = \sum_{k=0}^{n} a_k x^k$	$\min\{1, \frac{\epsilon}{(\sum_{k=1}^{n} k\lvert a_k\rvert(\lvert x_0\rvert + 1)^{k-1})}\}$
1-2	—Powers	
	$f_{1,2}(x) = x^r$	$\min\{\lvert\frac{x_0}{2}\rvert, \frac{\epsilon}{\lvert r\rvert(1_{[1,\infty)}(r)3^{r-1}+1_{(-\infty,1)}(r))\lvert\frac{x_0}{2}\rvert^{r-1}}\}$
1-3	—General Rationals	
	$f_{1,3}(x) = \frac{\sum_{k=0}^{n} a_k x^k}{\sum_{k=0}^{m} b_k x^k}$	$\min\left\{1, \frac{\beta}{2B}, \frac{\beta\epsilon}{4A}, \frac{\beta^2\epsilon}{4\alpha B}\right\} : \begin{cases} \beta = \left\lvert\sum_{k=0}^{m} b_k x_0^k\right\rvert, B = \sum_{k=1}^{m} k\,\lvert b_k\rvert\,(\lvert x_0\rvert + 1)^{k-1}, \\ \alpha = \left\lvert\sum_{k=0}^{n} a_k x_0^k\right\rvert, A = \sum_{k=1}^{n} k\,\lvert a_k\rvert\,(\lvert x_0\rvert + 1)^{k-1}. \end{cases}$
2	Exponential and Logarithmic	
2-1	—Logarithm	
	$f_{2,1}(x) = \log_a(x)$	$x_0(1 - a^{(\mathrm{sgn}(1-a))\epsilon})$
2-2	—Exponential	
	$f_{2,2}(x) = a^x$	$\mathrm{sgn}(a-1)\big(\log_a(1 + a^{-x_0}\epsilon)\big)$
2-3	—Inverse Lambert	
	$f_{2,3}(x) = x\exp(x)$	$\min\{1, \frac{\exp(-x_0)\epsilon}{2}, \ln(1 + \frac{\exp(-x_0)\epsilon}{2(\lvert x_0\rvert + 1)})\}$
2-4	—Neg. Entropy Potential	
	$f_{2,4}(x) = x\ln(x)$	$\min\{\frac{x_0}{2}, \frac{\epsilon}{2(\lvert\ln(x_0)\rvert)}, x_0(1 - \exp(\frac{-\epsilon}{3x_0}))\}$
2-5	—Tetration	
	$f_{2,5}(x) = x^x$	$\min\{\frac{x_0}{2}, \frac{\ln(1 + x_0^{-x_0}\epsilon)}{2(\lvert\ln(x_0)\rvert)}, x_0(1 - (1 + x_0^{-x_0}\epsilon)^{\frac{-1}{3x_0}})\}$
2-6	—Normal density	
	$f_{2,6}(x) = \frac{\exp(\frac{-(x-\mu)^2}{2\sigma^2})}{\sqrt{2\pi}\sigma}$	$\min\{1, \frac{\sigma^2(\ln(1+\left(\sqrt{2\pi}\sigma\exp(\frac{1}{2\sigma^2}(x_0-\mu)^2)\right)\epsilon))}{\lvert\mu\rvert+\lvert x_0\rvert+1}\}$
2-7	—Logistic	
	$f_{2,7}(x) = \frac{L_{max}}{1 + Q.\exp(-k.x)}$	$\frac{1}{k}\ln\left(1 + \exp(k.x_0) \times \min\{1, \frac{1+Q.\exp(-k.x_0)}{2Q}, \frac{(1+Q.\exp(-k.x_0))^2\epsilon}{4L_{max}Q}\}\right)$
3	Trigonometric	
3-1	—Trig 1	
	$f_{3,1}(x) = \sin(x)$	ϵ
3-2	—Trig 2	
	$f_{3,2}(x) = \cos(x)$	ϵ
3-3	—Trig 3	
	$f_{3,3}(x) = \tan(x)$	$\min\{\frac{\lvert\cos(x_0)\rvert}{2}, \frac{\lvert\cos(x_0)\rvert^2\epsilon}{2}\}$
3-4	—Trig 4	
	$f_{3,4}(x) = \cot(x)$	$\min\{\frac{\lvert\sin(x_0)\rvert}{2}, \frac{\lvert\sin(x_0)\rvert^2\epsilon}{2}\}$
3-5	—Trig 5	

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#	Cluster –Function ($f(x)$)	Sample $\delta = \delta(\epsilon, x_0)$
	$f_{3,5}(x) = \sec(x)$	$\min\{\frac{\lvert\cos(x_0)\rvert}{2}, \frac{\lvert\cos(x_0)\rvert^2\epsilon}{2}\}$
3-6	—Trig 6	
	$f_{3,6}(x) = \csc(x)$	$\min\{\frac{\lvert\sin(x_0)\rvert}{2}, \frac{\lvert\sin(x_0)\rvert^2\epsilon}{2}\}$
4	Inverse-Trigonometric	
4-1	—Inv-Trig 1	
	$f_{4,1}(x) = \arcsin(x)$	$\min\{-\sin(\arcsin(x_0) - \epsilon) + x_0, \sin(\arcsin(x_0) + \epsilon) - x_0\}$
4-2	—Inv-Trig 2	
	$f_{4,2}(x) = \arccos(x)$	$\min\{-\cos(\arccos(x_0) + \epsilon) + x_0, \cos(\arccos(x_0) - \epsilon) - x_0\}$
4-3	—Inv-Trig 3	
	$f_{4,3}(x) = \arctan(x)$	$\min\{-\tan(\arctan(x_0) - \epsilon) + x_0, \tan(\arctan(x_0) + \epsilon) - x_0\}$
4-4	—Inv-Trig 4	
	$f_{4,4}(x) = \mathrm{arccot}(x)$	$\min\{-\cot(\mathrm{arccot}(x_0) + \epsilon) + x_0, \cot(\mathrm{arccot}(x_0) - \epsilon) - x_0\}$
4-5	—Inv-Trig 5	
	$f_{4,5}(x) = \mathrm{arcsec}(x)$	$\min\{-\sec(\mathrm{arcsec}(x_0) - \epsilon) + x_0, \sec(\mathrm{arcsec}(x_0) + \epsilon) - x_0\}$
4-6	—Inv-Trig 6	
	$f_{4,6}(x) = \mathrm{arccsc}(x)$	$\min\{-\csc(\mathrm{arccsc}(x_0) + \epsilon) + x_0, \csc(\mathrm{arccsc}(x_0) - \epsilon) - x_0\}$
5	Hyperbolic	
5-1	—Hyperb 1	
	$f_{5,1}(x) = \sinh(x)$	$\min\{\ln(1 + e^{-x_0}\epsilon), \ln(1 + e^{x_0}\epsilon)\}$
5-2	—Hyperb 2	
	$f_{5,2}(x) = \cosh(x)$	$\min\{\ln(1 + e^{-x_0}\epsilon), \ln(1 + e^{x_0}\epsilon)\}$
5-3	—Hyperb 3	
	$f_{5,3}(x) = \tanh(x)$	$\mathrm{arsinh}(\epsilon)$
5-4	—Hyperb 4	
	$f_{5,4}(x) = \coth(x)$	$\min\left\{\ln(1 + \mathrm{e}^{-x_0}\frac{\lvert\sinh(x_0)\rvert}{2}),\ \ln(1 + \mathrm{e}^{x_0}\frac{\lvert\sinh(x_0)\rvert}{2}),\ \mathrm{arsinh}(\frac{\lvert\sinh(x_0)\rvert^2\epsilon}{2})\right\}$
5-5	—Hyperb 5	
	$f_{5,5}(x) = \mathrm{sech}(x)$	$\min\{\ln(1 + e^{-x_0}\epsilon), \ln(1 + e^{x_0}\epsilon)\}$
5-6	—Hyperb 6	
	$f_{5,6}(x) = \mathrm{csch}(x)$	$\min\limits_{b=1,2}\{\ln(1 + e^{-x_0}(\frac{\lvert\sinh(x_0)\rvert^b\epsilon^{b-1}}{2})), \ln(1 + e^{x_0}(\frac{\lvert\sinh(x_0)\rvert^b\epsilon^{b-1}}{2}))\}$
6	Inverse-Hyperbolic	
6-1	—Inv-Hyperb 1	
	$f_{6,1}(x) = \mathrm{arsinh}(x)$	$\min\{\sinh(\mathrm{arsinh}(x_0) + \epsilon) - x_0,\ x_0 - \sinh(\mathrm{arsinh}(x_0) - \epsilon)\}$
6-2	—Inv-Hyperb 2	
	$f_{6,2}(x) = \mathrm{arcosh}(x)$	$\min\{\cosh(\mathrm{arcosh}(x_0) + \epsilon) - x_0,\ x_0 - \cosh(\mathrm{arcosh}(x_0) - \epsilon)\}$
6-3	—Inv-Hyperb 3	
	$f_{6,3}(x) = \mathrm{artanh}(x)$	$\min\{\tanh(\mathrm{artanh}(x_0) + \epsilon) - x_0,\ x_0 - \tanh(\mathrm{artanh}(x_0) - \epsilon)\}$
6-4	—Inv-Hyperb 4	
	$f_{6,4}(x) = \mathrm{arcoth}(x)$	$\min\{\coth(\mathrm{arcoth}(x_0) - \epsilon) - x_0,\ x_0 - \coth(\mathrm{arcoth}(x_0) + \epsilon)\}$
6-5	—Inv-Hyperb 5	
	$f_{6,5}(x) = \mathrm{arsech}(x)$	$\min\{\mathrm{sech}(\mathrm{arsech}(x_0) - \epsilon) - x_0,\ x_0 - \mathrm{sech}(\mathrm{arsech}(x_0) + \epsilon)\}$
6-6	—Inv-Hyperb 6	

Continued on next page

#	Cluster –Function ($f(x)$)	Sample $\delta = \delta(\epsilon, x_0)$
	$f_{6,6}(x) = \operatorname{arcsch}(x)$	$\min\{\operatorname{csch}(\operatorname{arcsch}(x_0) - \epsilon) - x_0,\ x_0 - \operatorname{csch}(\operatorname{arcsch}(x_0) + \epsilon)\}$
7	Elementary Piecewise	
7-1	—Absolute value	
	$f_{7,1}(x) = \lvert x \rvert$	ϵ
7-2	—Lower clipping	
	$f_{7,2}(x) = \max\{x, c\}$	ϵ
7-3	—Upper clipping	
	$f_{7,3}(x) = \min\{x, c\}$	ϵ
7-4	—Clamp clip	
	$f_{7,4}(x) = \max\{a, \min\{x, b\}\}$	ϵ
7-5	—Signum	
	$f_{7,5}(x) = \operatorname{sgn}(x)$	$\lvert x_0 \rvert$
7-6	—Floor	
	$f_{7,6}(x) = \lfloor x \rfloor$	$\min\{x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\}$
7-7	—Ceiling	
	$f_{7,7}(x) = \lceil x \rceil$	$\min\{\lceil x_0 \rceil - x_0, 1 - (\lceil x_0 \rceil - x_0)\}$
7-8	—Fractional	
	$f_{7,8}(x) = \{x\}$	$\min\{\epsilon, x_0 - \lfloor x_0 \rfloor, 1 - (x_0 - \lfloor x_0 \rfloor)\}$
8	Pathological	
8-1	—Standard Volterra's	
	$f_{8,1}(x) = x^2 \sin(1/x)$	$\min\{\lvert \frac{x_0}{2} \rvert, \frac{\epsilon}{6\lvert x_0 \sin(1/x_0) \rvert}, \frac{\epsilon}{8}\}$
8-2	—Damped topologist's sine	
	$f_{8,2}(x) = x \sin(1/x)$	$\min\{\lvert \frac{x_0}{2} \rvert, \frac{\epsilon}{2\lvert \sin(1/x_0) \rvert}, \frac{\epsilon \lvert x_0 \rvert}{8}\}$
8-3	—Topologist's sine	
	$f_{8,3}(x) = \sin(1/x)$	$\min\{\lvert \frac{x_0}{2} \rvert, \lvert \frac{x_0}{2} \rvert^2 \epsilon\}$
8-4	—Cantor	
	$f_{8,4}(x) = \sum_{n=1}^{N(x)-1} \frac{a_n(x)}{2^{n+1}} + \frac{\mathbf{1}_{\{N(x)<\infty\}}}{2^{N(x)}}$	$3^{-\lceil \log_2(\frac{2}{\epsilon}) \rceil}$
8-5	—Thomae	
	$f_{8,5}\Big(\frac{m}{n} + x\,\mathbf{1}_{\mathbb{Q}^c}(x)\Big) = \frac{1}{n}\mathbf{1}_{\mathbb{Q}}(x),$	$\frac{1}{2}\min\{\lvert x - x_0 \rvert \mid x = \frac{p}{q} \in \mathbb{Q}, (p,q) = 1, 1 \le q \le \lceil \frac{1}{\epsilon} \rceil\}$
8-6	—Riemann	
	$f_{8,6}(x) = \sum_{n=1}^{+\infty} \frac{\sin(n^2 x)}{n^2}$	$\min\{2, (\frac{\epsilon}{18})^2\}$
8-7	—Takagi	
	$f_{8,7}(x) = \sum_{n=0}^{+\infty} \frac{1.7em(2^n x, \mathbb{Z})}{2^n}$	$\frac{\epsilon}{2(\lceil \log_2(\frac{2}{\epsilon}) \rceil + 1)}$
8-8	—Weierstrass	
	$f_{8,8}(x) = \sum_{n=0}^{+\infty} a^n \cos(b^n \pi x)$	$(\frac{1-ab^d}{2(\pi/2)^d})^{\frac{1}{d}} \epsilon^{\frac{1}{d}}\ :\ \ 0 < d < -\log_b(a)$
8-9	—Type I Dirichlet	
	$f_{8,9}(x) = 1_{\mathbb{Q}}(x)$	DNE
8-10	—Type II Dirichlet	
	$f_{8,10}(x) = \big(\prod_{k=1}^{n}(x-k)\big) 1_{\mathbb{Q}}(x)$	$\min\{1, \frac{\epsilon}{\prod_{1 \le k \ne k_0 \le n}(\lvert x_0 \rvert + k + 1)}\},\ \ x_0 \in \mathbb{N}_n$

Continued on next page

#	Cluster –Function ($f(x)$)	Sample $\delta = \delta(\epsilon, x_0)$
8-11	—Type III Dirichlet $f_{8,11}(x) = \sin(\pi x) 1_{\mathbb{Q}}(x)$	$\dfrac{\epsilon}{\pi}$, $x_0 \in \mathbb{Z}$
8-12	—Type IV Dirichlet $f_{8,12}(x) = \left(\sum_{n=1}^{+\infty} \sum_{k=1}^{2^{n-1}} \left[\sqrt{3} \left(\frac{3^{-n}}{2} - \left| x - \frac{a_{n,k} + b_{n,k}}{2} \right| \right) \mathbf{1}_{(a_{n,k}, b_{n,k})}(x) \right] \right) 1_{\mathbb{Q}}(x)$	$3^{-\left\lceil \log_3 \left(\frac{\sqrt{3}}{2\epsilon} \right) \right\rceil}$, $x_0 \in C$: Cantor set

Taken together, these contributions reinforce the central role of ϵ–δ arguments in (i) teaching the foundations of real analysis and (ii) motivating further mathematical research in this area.

4.2. Common Pitfalls

Common pitfalls arise frequently in ϵ–δ proofs. We highlight several below.

1. **Treating scratch work as a finished proof.** Scratch work often contains tentative guesses—chains of equalities and inequalities relating ϵ and δ—some of which are unsuitable or too advanced for the final argument. For example, for $f(x) = \arctan(x)$ one might invoke the Mean Value Theorem (MVT) to deduce $|f(x) - f(x_0)| \leq |x - x_0|$. At the level considered here, however, the MVT is beyond the intended toolkit (which is limited to elementary algebraic identities, inequalities, and basic series facts). Our Appendix provides classroom aids designed to reinforce the structure of a complete and correct written proof.
2. **Treating δ as a unique function of ϵ.** By definition, δ depends on the prescribed $\epsilon > 0$, but it need not be unique: any smaller $\delta^* \in (0, \delta]$ also suffices. Moreover, continuity at a point x_0 typically yields $\delta = \delta(\epsilon, x_0)$, reflecting dependence on both the tolerance and the base point.
3. **Ignoring the function's domain.** The definition requires x and x_0 to lie in the domain D_f. This is often overlooked. A practical remedy—adopted in this paper—is to extend f to $\mathbb{R}$ when convenient, while explicitly formulating pointwise continuity with respect to the points x_0 in the original domain D_f.
4. **Algebraic slips.** Errors in algebraic manipulation (equalities or inequalities) can invalidate the inferred relationship between δ and ϵ. A final pass checking each logical implication in the chain of estimates is essential.
5. **Assuming a unique proof.** The definition of continuity neither requires nor favors a single method. Our presentation provides one admissible construction among many. Alternative approaches can yield different, equally valid δ's. For instance, for $f(x) = \tanh(x)$ one may start from the auxiliary identity:

$$\tanh(x) - \tanh(x_0) \;=\; \big(1 - \tanh(x)\,\tanh(x_0)\big)\,\tanh(x - x_0), \tag{183}$$

which leads to a distinct but legitimate ϵ–δ development.

4.3. Future Work and Open Research Problems

Up to this point, we have reviewed and consolidated ϵ–δ continuity proofs for 54 prominent real-valued functions in mathematical analysis. A natural question for the curious reader is whether there remains scope for inquiry beyond the exercises presented here. To the best of the author's knowledge, the answer is yes: there is room to move beyond the exercise tier and into genuine research. In particular, two principal families of open problems invite further investigation by interested mathematicians, as outlined below.

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4.3.1. Point Continuity

First, let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$ and define:

$$\Delta_f(\epsilon, x_0) = \{\delta > 0 | \forall x \in \mathbb{R}(|x - x_0| < \delta \to |f(x) - f(x_0)| < \epsilon)\}. \tag{184}$$

It is trivial that $\Delta_f(\epsilon, x_0) \neq \emptyset$ if and only if $x_0 \in C(f)$. In this paper, we gave one sample example of elements of $\Delta_f(\epsilon, x_0)$ for 45 prominent functions in real-analysis presented in Table (1). Now, we consider the following new quantity:

Definition 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and given $\epsilon > 0$. Fix $x_0 \in \mathbb{R}$ and define:*

$$\delta_f(\epsilon, x_0) = \sup(\Delta_f(\epsilon, x_0)). \tag{185}$$

We present two examples as follows. First, for $f_1(x) = x$, given $\epsilon > 0$, and $x_0 \in \mathbb{R}$ we have $\delta_{f_1}(\epsilon, x_0) = \epsilon$. Second, for $f_2(x) = 1$, given $\epsilon > 0$, and $x_0 \in \mathbb{R}$ we have $\delta_{f_2}(\epsilon, x_0) = +\infty$. These examples motivates the following set of Open Problems:

Research Problem 4.1. *Compute $\delta_f(\epsilon, x_0)$ for all 54 functions presented in Table (1).*

Research Problem 4.2. *Establish finiteness conditions on $\delta_f(\epsilon, x_0)$ as follows:*

1. *What is the necessary condition for f and x_0 such that $\delta_f(\epsilon, x_0) < +\infty$?*
2. *What is the sufficient condition for f and x_0 such that $\delta_f(\epsilon, x_0) < +\infty$?*
3. *Is there a necessary and sufficient condition for f and x_0 such that $\delta_f(\epsilon, x_0) < +\infty$?*

4.3.2. Uniform Continuity

Second, consider an stronger version of the concept of "Point Continuity" presented in Definition (2) called "Uniform Continuity" as in follows [34,35]:

Definition 6 (Uniform Continuity). *Let $f \in F(\mathbb{R}, \mathbb{R})$. Then, f is said to be uniform continuous in its domain if and only if for any given "output tolerance" $\epsilon > 0$ there exists an "input tolerance" $\delta^U = \delta^U(\epsilon) > 0$ such that for all $x \in \mathbb{R}$ and for all $y \in \mathbb{R}$ if $|x - y| < \delta^U$ then $|f(x) - f(y)| < \epsilon$. In mathematical logic language:*

$$\forall \epsilon > 0 \, \exists \delta^U = \delta^U(\epsilon) > 0 \, \forall x \in \mathbb{R} \, \forall y \in \mathbb{R} \, (|x - y| < \delta^U \to |f(x) - f(y)| < \epsilon). \tag{186}$$

We present two examples as follows. First, for $f_1(x) = \sin(x)$, given $\epsilon > 0$, we have $\delta^U(\epsilon) = \epsilon$. Second, for $f_2(x) = 1$, given $\epsilon > 0$, we have $\delta^U(\epsilon) = 1$. These examples motivate the following Open Problem:

Research Problem 4.3. *Compute $\delta^U(\epsilon)$ for all 54 functions presented in Table (1).*

Next, similar to the case of "Point Continuity", for given $f \in F(\mathbb{R}, \mathbb{R})$ and $\epsilon > 0$ we define:

$$\Delta_f^U(\epsilon) = \{\delta^U > 0 | \forall x \in \mathbb{R} \, \forall y \in \mathbb{R}(|x - y| < \delta^U \to |f(x) - f(y)| < \epsilon)\}. \tag{187}$$

and, consequently, the following "twin" of Definition (5) for the case of "Uniform Continuity":

Definition 7. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function and given $\epsilon > 0$. Set:*

$$\delta_f^U(\epsilon) = \sup(\Delta_f^U(\epsilon)). \tag{188}$$

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Here, for the same presented examples $f_(x) = \sin(x)$ and $f_2(x) = 1$ we have $\delta^{U}_{f_1}(\epsilon) = \epsilon$, and $\delta^{U}_{f_2}(\epsilon) = +\infty$. Similar to the "Core Continuity" these examples motivates the following set of Open Problems:

Research Problem 4.4. *Compute $\delta^{U}_{f}(\epsilon)$ for all 54 functions presented in Table (1).*

Research Problem 4.5. *Establish finiteness conditions on $\delta^{U}_{f}(\epsilon, x_0)$ as follows:*

1. *What is the necessary condition for f such that $\delta^{U}_{f}(\epsilon, x_0) < +\infty$?*
2. *What is the sufficient condition for f such that $\delta^{U}_{f}(\epsilon) < +\infty$?*
3. *Is there a necessary and sufficient condition for f such that $\delta^{U}_{f}(\epsilon) < +\infty$?*

5. Conclusion

In conclusion, this atlas provides a comprehensive, systematic survey of direct ϵ–δ continuity proofs for 54 prominent real-valued functions. By meticulously analyzing these functions and categorizing them into eight distinct clusters, the work highlights recurring proof structures and methodologies. For each function, the paper presents a step-by-step proof alongside an explicit formula for δ in terms of ϵ and the point of continuity x_0. This systematic approach serves as a robust pedagogical resource designed to demystify the proof-writing process and foster a deeper intuition for the core principles of continuity. By transforming a collection of challenging proofs into an accessible and navigable reference, the atlas offers clarity where confusion often resides and opens new avenues for further research.

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Abbreviations

The following abbreviations are used in this manuscript:

DNE: Does Not Exist; DPF: Difference of Powers Factorization; GPD: General Product Decomposition; TSBPB: Two-Sided Bernoulli Power Bounds

Appendix A. Classroom Aids: $\epsilon - \delta$ Continuity Toolkit

This Appendix provides ready-to-use "Classroom Aids" that scaffold ϵ–δ continuity proofs from first principles [36–40]. The toolkit includes a fully worked worksheet (illustrated for x^{π}), a concise analytic rubric that operationalizes proof quality (bounding strategy, explicit $\delta(\epsilon, x_0)$ construction, and quantifier discipline), and three concept questions for rapid formative assessment. Each aid emphasizes a transferable pattern—introduce an auxiliary radius to freeze cofactors, derive a bound, and choose $\delta = \delta(\epsilon, x_0) = \min\{\text{radius}, \ \epsilon\text{-based constraint}\}$—with any domain restrictions made explicit. Instructors can adapt the same template verbatim to other prominent functions in real analysis.

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Appendix A.1. Worksheet (Fully Worked Example: $f(x) = x^{\pi} : \pi = 3.14159\cdots$)

Domain and goal.

Let $r = \pi > 1$. Work on the natural real domain

$$D_{\pi} = \mathbb{R}_0^+ = [0, \infty).$$

Fix $x_0 \in D_{\pi}$. Prove, by the ϵ–δ definition, that $f(x) = x^{\pi}$ is continuous at any $x_0 > 0$. Here, we want to prove that for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\forall x \in D_{\pi}(\ |x - x_0| < \delta \ \rightarrow \ |x^{\pi} - x_0^{\pi}| < \epsilon).$$

Step 1 (Algebraic bound via TSBPB).

By the Two–Sided Bernoulli Power Bounds for $r \geq 1$ and $x, x_0 \geq 0$,

$$\left|x^{\pi} - x_0^{\pi}\right| \ \leq \ \pi \ \max\{x^{\pi-1},\ x_0^{\pi-1}\}\ |x - x_0|.$$

Impose the auxiliary radius

$$|x - x_0| < \frac{|x_0|}{2} \quad \rightarrow \quad \frac{x_0}{2} \leq x \leq 3\frac{x_0}{2}.$$

Hence

$$\max\{x^{\pi-1},\ x_0^{\pi-1}\} \ \leq \ \max\{(3\frac{x_0}{2})^{\pi-1},\ x_0^{\pi-1}\} =: M_{\pi}(x_0).$$

Therefore, under $|x - x_0| < \frac{|x_0|}{2}$ we have the linear control

$$|x^{\pi} - x_0^{\pi}| \ \leq \ \pi\, M_{\pi}(x_0)\, |x - x_0|.$$

Step 2 ($\epsilon - \delta$ planning).

It suffices to force

$$\pi\, M_{\pi}(x_0)\, |x - x_0| < \epsilon \quad \Longleftrightarrow \quad |x - x_0| < \frac{\epsilon}{\pi\, M_{\pi}(x_0)}.$$

Step 3 (Candidate δ).

Take

$$\boxed{\delta = \delta(\epsilon, x_0) = \min\Big\{\ \frac{|x_0|}{2},\ \frac{\epsilon}{\pi\, M_{\pi}(x_0)}\ \Big\}, \qquad M_{\pi}(x_0) = \max\{(\frac{3x_0}{2})^{\pi-1},\ x_0^{\pi-1}\}.}$$

Step 4 (Verification).

If $x \in D_{\pi}$ and $0 < |x - x_0| < \delta$, then in particular $|x - x_0| < \frac{|x_0|}{2}$, so Step 1 yields

$$|x^{\pi} - x_0^{\pi}| \ \leq \ \pi\, M_{\pi}(x_0)\, |x - x_0| \ < \ \pi\, M_{\pi}(x_0) \cdot \frac{\epsilon}{\pi\, M_{\pi}(x_0)} \ = \ \epsilon.$$

As $\epsilon > 0$ was arbitrary, $x \mapsto x^{\pi}$ is continuous at x_0 on D_{π}.

Quantifier audit. $\forall \epsilon > 0\ \exists \delta > 0\ \forall x \in D_{\pi}(\ |x - x_0| < \delta \rightarrow |x^{\pi} - x_0^{\pi}| < \epsilon)$. ✓

Instructor note (transfer pattern). The technique generalizes: factor the difference, impose a local radius (often 1) to bound the "cofactor," then choose $\delta = \min\{bound, \text{some function of } \epsilon\}$. Students should explicitly (i) introduce the auxiliary bound (here, $\frac{|x_0|}{2}$), (ii) box the candidate $\delta(\epsilon, x_0)$, and (iii) complete the quantifier audit. Here, the factorization is done via the the TSBPB bound $|x^{\pi} - x_0^{\pi}| \leq \pi \max\{x^{\pi-1}, x_0^{\pi-1}\}|x - x_0|$. Keep the same *auxiliary radius* $|x - x_0| < \frac{|x_0|}{2}$ to freeze the cofactor, then choose $\delta = \min\{\frac{|x_0|}{2}, \frac{\epsilon}{(\pi M_{\pi}(x_0))}\}$. The domain restriction to $[0, \infty)$ is essential for real-valued x^{π}.

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Appendix A.2. Analytic Rubric (Short Form; 10 Points)

Criterion	Exceeds / Meets / Not Yet (pts)
Problem framing (2pts)	Correct domain $D_\pi = [0, \infty)$ identified; precise $\epsilon - \delta$ target for x^π at x_0. *Exceeds (2pts)*: domain stated and goal restated; *Meets (1pts)*: generally correct; *NY (0pts)*: missing/misstated.
Bounding strategy (3pts)	Invokes TSBPB (case $r \geq 1$) to obtain $\|x^\pi - x_0^\pi\| \leq \pi \max\{x^{\pi-1}, x_0^{\pi-1}\}\|x - x_0\|$ and justifies a *local* bound $\|x - x_0\| < \frac{\|x_0\|}{2} \to x \leq \frac{3\|x_0\|}{2}$. *Exceeds (3pts)*: fully justified; *Meets (2pts)*: idea present with minor gaps; *NY (0–1pts)*: incorrect or unjustified.
$\delta(\epsilon, x_0)$ construction (3pts)	Gives explicit $\delta = \min\{\frac{\|x_0\|}{2}, \frac{\epsilon}{(\pi M_\pi(x_0))}\}$ with $M_\pi(x_0) = \max\{(\frac{3x_0}{2})^{\pi-1}, x_0^{\pi-1}\}$. *Exceeds (3pts)*: correct, boxed, and motivated; *Meets (2pts)*: correct but thinly justified; *NY (0–1pts)*: incorrect/implicit.
Quantifiers & closure (2pts)	Final implication verified and quantifiers audited. *Exceeds (2pts)*: explicit audit; *Meets (1pts)*: verification present; *NY (0pts)*: omitted.

Feedback keys for common issues. (i) **Domain slip**: ensure $x, x_0 \in [0, \infty)$; (ii) **Unjustified cofactor bound**: explicitly state $|x - x_0| < \frac{|x_0|}{2}$ and derive $M_\pi(x_0)$; (iii) **Quantifier order**: keep "$\forall\epsilon\,\exists\delta\,\forall x$ in D_π".

Appendix A.3. Concept Questions (Peer Instruction/Clickers)

Each question is designed for a 1–2 minute vote, 2–3 minute peer discussion, and a brief plenary debrief.

Q1 (Bounding the cofactor for x^π).

Let $x_0 > 0$. Which line *correctly* justifies a bound on the cofactor in the TSBPB estimate?

(a) Assume $|x - x_0| < \frac{|x_0|}{2}$, then $x \leq \frac{3|x_0|}{2}$, hence $\max\{x^{\pi-1}, x_0^{\pi-1}\} \leq \max\{(\frac{3|x_0|}{2})^{\pi-1}, x_0^{\pi-1}\}$.

(b) Assume $|x - x_0| < \epsilon$, then $\max\{x^{\pi-1}, x_0^{\pi-1}\} \leq \epsilon^{\pi-1}$.

(c) $\max\{x^{\pi-1}, x_0^{\pi-1}\} \leq x^{\pi-1} + x_0^{\pi-1} \leq 2x^{\pi-1}$; no auxiliary bound is needed.

(d) Replace TSBPB by $|x^\pi - x_0^\pi| \leq \pi|x - x_0|$ for all $x \geq 0$.

Q2 (Selecting δ).

With the setup above, which $\delta(\epsilon, x_0) : x_0 > 1$ *works*?

(a) $\delta(\epsilon, x_0) = \dfrac{\epsilon}{\pi\, M_\pi(x_0)}$

(b) $\delta(\epsilon, x_0) = \min\Big\{\dfrac{|x_0|}{2}, \dfrac{\epsilon}{\pi\, M_\pi(x_0)}\Big\}$

(c) $\delta(\epsilon, x_0) = \epsilon$

(d) $\delta(\epsilon, x_0) = \min\Big\{1,\ \epsilon \cdot \pi\, M_\pi(x_0)\Big\}$

Q3 (Quantifier sensitivity).

Which rephrasing preserves the correct quantifier order for continuity at $x_0 > 0$ on D_π?

(a) $\exists\delta > 0\ \forall\epsilon > 0(\ |x - x_0| < \delta \to |x^\pi - x_0^\pi| < \epsilon$ for all $x \in D_\pi)$.

(b) $\forall\epsilon > 0\ \exists\delta > 0\ \forall x \in D_\pi(\ |x - x_0| < \delta \to |x^\pi - x_0^\pi| < \epsilon)$.

(c) $\forall x \in D_\pi\ \exists\delta > 0\ \forall\epsilon > 0(\ |x - x_0| < \delta \to |x^\pi - x_0^\pi| < \epsilon)$.

(d) $\exists\epsilon > 0\ \exists\delta > 0$ such that the implication holds for all $x \in D_\pi$.

Instructor key. Q1: (a). Q2: (b). Q3: (b).

Adaptation notes. The same template applies to any fixed $r \geq 1$ on $[0, \infty)$ by replacing π with r and $(\pi - 1)$ with $(r - 1)$.

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