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Posted Date: 30 October 2025

doi: 10.20944/preprints202510.2323.v1

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Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 30 October 2025 doi:10.20944/preprints202510.2323.v1

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Symmetry-Preserving Fixed Points in FLRW Cosmology: The Einstein Tensor Cycle Transformation and Gravitational Invariance

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Abstract

We define the iterative map from the metric $g_{\mu\nu}$ to the Einstein tensor $G_{\mu\nu}$ as the *Einstein Tensor Cycle (ETC) transformation*, $g_{\mu\nu}^{(n+1)} := G_{\mu\nu}[g^{(n)}]$, and geometrically characterize Einstein spaces containing the cosmological constant Λ through its fixed points $G_{\mu\nu} = \lambda g_{\mu\nu}$. The FLRW metric's fundamental symmetries—spatial isotropy ($SO(3)$) and spacetime homogeneity—are preserved under the ETC transformation and manifest as a fixed-point structure. We apply the ETC transformation to the FLRW metric with curvature parameters $k = \pm 1, 0$, analyzing how distinct spatial geometries are uniformly derived through a single iteration procedure. For the de Sitter family ($H_0 = \sqrt{\Lambda/3}$), we confirm that $G_{00} = \Lambda$ and corresponding spatial components are realized in the first transformation and remain invariant in subsequent iterations for both $k = +1$ with $a(t) = a_0 \cosh(H_0 t)$ and $k = -1$ with $a(t) = a_0 \sinh(H_0 t)$. For the flat case ($k = 0$), the Friedmann equation $G_{00} = 8\pi G\rho/c^2$ is reproduced under exponential expansion. The ETC transformation functions as a unified framework that simultaneously provides solution identification and stability evaluation in cosmological models, clarifying the deep relationship between spacetime symmetry and fixed-point structure.

Keywords: fixed-point theory; FLRW cosmology; Einstein field equations; symmetry preservation; de Sitter spacetime; gravitational invariance; cosmological constant; tensor mappings

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1. Introduction

The FLRW metric, which serves as the standard framework for cosmology, is given by

$$ds^2 = dt^2 - a(t)^2\left(\frac{dr^2}{1 - k\,r^2} + r^2\,d\Omega^2\right), \quad d\Omega^2 = d\theta^2 + \sin^2\theta\,d\phi^2,\ k \in \{-1, 0, 1\}, \tag{1}$$

and describes the most general spacetime satisfying global isotropy and homogeneity[1–4]. Here, $a(t)$ is the scale factor and k is the curvature parameter. The FLRW metric possesses a high degree of symmetry in the form of spatial isotropy and homogeneity, and this symmetry structure can be rigorously characterized from the viewpoint of Killing vector fields and symmetry groups[5,6]. On the other hand, in the limit where the cosmological constant Λ is dominant, spacetime is approximated by de Sitter geometry, and the scale factor is given according to the value of curvature k by

$$a_{\rm dS}(t) = \begin{cases} H^{-1}\cosh(Ht), & k = 1, \\ e^{Ht}, & k = 0, \\ H^{-1}\sinh(Ht), & k = -1, \end{cases} \qquad H \equiv \sqrt{\Lambda/3} \tag{2}$$

[7–10]. Recent precision observations by the Planck satellite strongly support a flat universe ($k = 0$) with a positive cosmological constant $\Lambda > 0$[11], and it is believed that de Sitter-type expansion will govern the asymptotic behavior of the universe in the future. This paper presents an operation called

 doi:10.20944/preprints202510.2323.v1

the *ETC transformation* (Einstein Tensor Cycle Transform) with the aim of simplifying and unifying the computational procedure for deriving and verifying these known solutions, and systematically demonstrates its application to the FLRW system.

The basic idea of the ETC transformation is to generate the Einstein tensor

$$\mathcal{T}: \quad g_{\mu\nu} \longmapsto G_{\mu\nu}[g] \equiv R_{\mu\nu}[g] - \tfrac{1}{2}\, g_{\mu\nu}\, R[g] \tag{3}$$

from a given metric $g_{\mu\nu}$, and to directly and mechanically examine how the field equations are satisfied by applying Eq. (3) iteratively as needed:

$$g_{\mu\nu}^{(n+1)} = \mathcal{T}\Big[g_{\mu\nu}^{(n)}\Big] \quad (n = 0, 1, 2, \dots) \tag{4}$$

This iterative structure can be interpreted as a procedure for searching for fixed points (Einstein metrics) of curvature operations in Riemannian geometry[12], and has conceptual similarities to fixed-point analysis in asymptotic safety and functional renormalization group approaches in quantum gravity[13]. Since tensor calculations in a computer algebra environment are assumed, the work of manually expanding connection coefficients and curvature tensors step by step can be minimized, and the behavior of $G_{\mu\nu}$ defined by Eq. (3) can be immediately verified when a specific form of the scale factor is given (see Mathematica code).

Contributions of this paper

- **Clarification of the ETC transformation:** We formalize the mapping $g_{\mu\nu} \mapsto G_{\mu\nu}[g]$ shown in Eq. (3) as a computational procedure, and provide implementation guidelines for iterative application to the FLRW metric given by Eq. (1)[12].
- **Unified derivation:** By substituting the standard scale factors of de Sitter spacetime given by Eq. (2), we directly confirm that the G_{00} component is consistent with the cosmological constant Λ regardless of the curvature $k = \pm 1, 0$[7,8].
- **Simplification of calculations:** In the $k = 0$ (flat) case, under the exponential expansion $a(t) = e^{Ht}$ given by Eq. (2) and the setting of matter density ρ, we reproduce $G_{00} = 8\pi G\rho/c^2$ and confirm that the standard Friedmann relation is reproduced within the framework of this method[9,14,15].

Thus, the ETC transformation functions as a concise "computational protocol" that can verify and compare de Sitter solutions and FLRW dynamics through the same procedure via Eq. (3) and Eq. (4). In the remainder of this paper, we first rigorously present the definition and iteration rules of the ETC transformation (Sec. 3), then show the results of application to each case of $k = \pm 1, 0$ in Eq. (1) and their mutual comparison (Sec. 4). Finally, we discuss the extensibility and limitations of this method (Sec. 5)[10,16].

2. Conventions

In this section, we clearly establish the sign conventions, the form of Einstein's equation, and the unit system used in this paper.

2.1. Signature and Geometric Notation

The metric signature in this study is taken to be

$$(+, -, -, -)$$

which is widely used in general relativity. Thus, the time component g_{00} in Eq. (1) is positive, while the spatial components g_{ii} $(i = 1, 2, 3)$ are negative. For raising and lowering tensor indices, we use the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$, and the Ricci tensor and scalar curvature are defined as follows:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}, \tag{5}$$

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{6}$$

The Einstein tensor is defined by Eq. (3) as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \tag{7}$$

[1,2,12,17].

2.2. Form of Einstein's Equation

Einstein's equation in this paper is adopted in the following form including the cosmological constant Λ:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{8}$$

In the vacuum case, the right-hand side of Eq. (8) vanishes, yielding

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{9}$$

This is consistent with the adoption of the signature $(+, -, -, -)$[2–4]. Under this convention, note that the result $G_{00} = \Lambda$ for de Sitter spacetime corresponds to Eq. (9) with the sign convention[7]. That is, each component of $G_{\mu\nu}$ defined by Eq. (7) takes the form $\pm\Lambda$ reflecting the sign of the metric, thereby being consistent with Eq. (8).

When matter is included, the components of $T_{\mu\nu}$ are given by the energy density ρ and pressure p as

$$T^{\mu}{}_{\nu} = \mathrm{diag}(\rho c^2, -p, -p, -p) \tag{10}$$

[18,19]. Therefore, the $(0, 0)$ component of Eq. (8) gives

$$3\frac{\dot{a}^2 + kc^2}{a^2} = 8\pi G\rho + \Lambda c^2, \tag{11}$$

and for the spatial components (i, i),

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + kc^2}{a^2} = \frac{8\pi G}{c^2} p - \Lambda c^2, \tag{12}$$

[4,8,9,15]. Equations (11) and (12) serve as the basis for the analysis in the Friedmann universe and de Sitter universe described below.

2.3. Choice of Unit System

For computational convenience, we principally use geometrized units $(c = 1)$ in this paper. However, when it is necessary to explicitly show dimensional analysis or the introduction of physical constants, we restore SI units and explicitly retain c. For example, the form $G_{00} = 8\pi G\rho/c^2$ of the energy density term in the Friedmann universe represented by Eq. (11) is adopted as an expression in SI units[15].

Additionally, in symbolic computation, we set the differential variable as $x^0 = ct$, so c may be scaled internally by a parameter c_1 in the calculations. In such cases, the final results are also restored to physical units to confirm consistency (see Mathematica code).

2.4. Summary of Notation

We summarize the main symbols and definitions used in this paper below:

$$\begin{aligned} g_{\mu\nu} &: \text{ Metric tensor (Eq. (1))} \\ G_{\mu\nu} &: \text{ Einstein tensor (Eq. (7))} \\ R_{\mu\nu}, R &: \text{ Ricci tensor and scalar curvature (Eqs. (5), (6))} \\ T_{\mu\nu} &: \text{ Energy-momentum tensor (Eq. (10))} \\ \Lambda &: \text{ Cosmological constant} \\ a(t) &: \text{ Scale factor (time-dependent, Eqs. (1), (2))} \\ k &: \text{ Curvature parameter } (+1, 0, -1) \\ H_0 &: \text{ Hubble constant (in Eq. (2), } H_0^2 = \Lambda/3) \end{aligned}$$

Based on the above conventions, particularly Eqs. (1)–(12), all subsequent calculations and equation derivations are performed.

3. Method: ETC Transform

In this section, we formally define the Einstein Tensor Cycle Transform (hereafter, ETC transformation) and clarify its algorithmic structure.

3.1. Definition of the ETC Transformation

The ETC transformation is a mapping that takes a metric tensor $g_{\mu\nu}$ as input and recursively generates the corresponding Einstein tensor $G_{\mu\nu}[g]$ as a new metric[1,2,12]. That is, we define the mapping introduced in Eq. (3) as

$$\mathcal{T}: \; g_{\mu\nu} \longmapsto G_{\mu\nu}[g], \tag{13}$$

and by iteratively applying this, we obtain the tensor sequence

$$g_{\mu\nu}^{(n+1)} = \mathcal{T}\Big[g_{\mu\nu}^{(n)}\Big] = G_{\mu\nu}\Big[g^{(n)}\Big], \qquad n = 0, 1, 2, \ldots \tag{14}$$

In this iteration by Eq. (14), if there exists a metric $g^*_{\mu\nu}$ satisfying the fixed-point condition (Einstein metric)

$$G_{\mu\nu}[g^*] = \lambda\, g^*_{\mu\nu} \tag{15}$$

then it is an invariant metric (fixed point) of the ETC transformation, and the proportionality constant λ physically corresponds to the cosmological constant Λ[12].

The fixed-point condition of Eq. (15) is equivalent to the usual Einstein equation shown in Eq. (9)

$$G_{\mu\nu} = -\Lambda g_{\mu\nu}$$

[1,2], and therefore the ETC transformation defined by Eq. (13) can be viewed as an operation that reformulates the field equations as a fixed-point search problem. This fixed-point search framework can be interpreted as a procedure for finding fixed points of curvature operations in Riemannian geometry[12], and has conceptual similarities to asymptotic safety and fixed-point analysis in the functional renormalization group in quantum gravity theories[13].

3.2. Algorithmic Structure

In actual calculations, the ETC transformation defined by Eq. (13) and Eq. (14) is implemented by the following procedure.

 doi:10.20944/preprints202510.2323.v1

1. Set the initial metric $g_{\mu\nu}^{(0)}$. In this study, we adopt the FLRW metric given by Eq. (1)

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - k\, r^2} + r^2 d\Omega^2 \right) \tag{16}$$

[3,4,15]. This FLRW metric possesses symmetries of spatial isotropy and homogeneity, and these symmetries are expected to be preserved under iteration of the ETC transformation[5,6].

2. According to Eq. (7), calculate the Einstein tensor $G_{\mu\nu}^{(n)}$ from $g_{\mu\nu}^{(n)}$:

$$G_{\mu\nu}^{(n)} = R_{\mu\nu}^{(n)} - \tfrac{1}{2}\, g_{\mu\nu}^{(n)}\, R^{(n)}. \tag{17}$$

Here, $R_{\mu\nu}^{(n)}$ is the Ricci tensor defined by Eq. (5), and $R^{(n)}$ is the scalar curvature defined by Eq. (6)[1,2].

3. Following Eq. (14), substitute the obtained $G_{\mu\nu}^{(n)}$ as the metric in the next cycle:

$$g_{\mu\nu}^{(n+1)} := G_{\mu\nu}^{(n)}. \tag{18}$$

4. Perform the following simplification in each cycle:

$$g_{\mu\nu}^{(n+1)} \longleftarrow \mathrm{Chop}\Big[\mathrm{PowerExpand}\Big(\mathrm{FullSimplify}(g_{\mu\nu}^{(n+1)})\Big)\Big], \tag{19}$$

that is, remove redundant exponential/power expansions and small numerical errors for simplification. (This operation is automated in the Mathematica notebook in the appendix. See Mathematica code)

5. Convergence criterion:

$$\Delta^{(n)} = \left\| G_{\mu\nu}^{(n+1)} - G_{\mu\nu}^{(n)} \right\| < \varepsilon, \tag{20}$$

When this is satisfied, regard $G_{\mu\nu}^{(n)}$ as the fixed point $G_{\mu\nu}^{*}$.

As a result of this operation, in the de Sitter universe ($k = \pm 1, 0$) and Friedmann universe given by Eq. (2),

$$G_{\mu\nu}^{(n+1)} = G_{\mu\nu}^{(n)} = \Lambda g_{\mu\nu} \tag{21}$$

is confirmed, demonstrating that the ETC transformation defined by Eq. (13) functions as an invariant mapping[7,8].

3.3. Mathematical characteristics and meaning

The ETC transformation has a self-reproducing structure similar to the differential invariance of the exponential function

$$\frac{d}{dx} e^x = e^x,$$

and the multiplicative invariance of the identity matrix

$$I \cdot I = I.$$

That is, the geometric symmetry of the ETC transformation appears in the fact that the operation itself that generates the Einstein tensor shown in Eq. (17) recursively reproduces the same form by Eq. (14). This symmetry preservation mechanism means that the spatial isotropy and homogeneity possessed by the FLRW metric are kept invariant through the mapping from the metric tensor to the Einstein tensor[6,17].

Moreover, this iterative structure can also be interpreted as a procedure for searching for fixed points (Einstein metrics) of curvature operations in Riemannian geometry[12,13], and serves as a means to algebraically stabilize nonlinear field equations. In particular, in the study of asymptotic safety

 doi:10.20944/preprints202510.2323.v1

in quantum gravity theories, fixed-point searches in the function space of actions play an important role[13], and the ETC transformation can be positioned as its classical analogue.

Especially in the highly symmetric de Sitter space, its maximal symmetry (the $SO(4,1)$ or $SO(3,2)$ symmetry group with 10 Killing vector fields[3,5]) guarantees immediate convergence to the fixed point, and it was confirmed that Eq. (21) already holds at the first transformation. Furthermore, as shown by the attached Mathematica calculation results, $G_{\mu\nu} = \Lambda g_{\mu\nu}$ is strictly maintained in the second ($G^{(2)}_{\mu\nu}$) and third ($G^{(3)}_{\mu\nu}$) iterations, and the complete stability of the fixed point has been numerically verified[7]. This suggests that the high degree of symmetry of spacetime directly corresponds to the stability of the fixed-point structure[6].

From the above, it can be concluded that the ETC transformation formalized by Eqs. (13)–(20) is a unified algorithm that reconstructs Einstein's equation as a "fixed-point problem of tensor mappings that preserve symmetry."

4. Results: Application to FLRW

In this section, we apply the ETC transformation defined in the previous section to the Friedmann-Robertson-Walker (FLRW) metric and analyze the behavior of the Einstein tensor for each case of the curvature parameter $k = +1, 0, -1$[3,4,15]. Here, the metric is given by Eq. (1) and Eq. (16) as

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right), \quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2, \tag{22}$$

and the scale factor $a(t)$ is determined according to the de Sitter universe or Friedmann universe shown in Eq. (2)[8,9]. The FLRW metric possesses spatial isotropy (rotational symmetry at each point) and homogeneity (spatial translational symmetry), and it is expected that this symmetry structure is preserved under iteration of the ETC transformation[5,6].

4.1. Confirmation of Tensor Invariance by ETC Transformation

As a result of comparing the Einstein tensor components calculated in the first, second, and third cycles of the ETC transformation based on Eq. (14)

$$G^{(n)}_{\mu\nu} = R^{(n)}_{\mu\nu} - \frac{1}{2} g^{(n)}_{\mu\nu} R^{(n)} \tag{23}$$

it was confirmed that

$$G^{(1)}_{\mu\nu} = G^{(2)}_{\mu\nu} = G^{(3)}_{\mu\nu} = \Lambda g_{\mu\nu} \tag{24}$$

holds for all curvature parameters k. This means that the FLRW metric given by Eq. (22) has an invariant structure with respect to the ETC transformation defined by Eq. (13), and indicates that the de Sitter universe and the Friedmann universe under specific conditions correspond to fixed points of the ETC transformation[7,8]. This invariance is a consequence of the preservation through the ETC transformation of the high degree of symmetry (isotropy and homogeneity) possessed by the FLRW metric[5,6], and suggests that spacetimes with higher symmetry are more stable as fixed points.

4.2. Case I: $k = 0$ (Flat de Sitter/Friedmann solution)

In flat space ($k = 0$), substituting from Eq. (2)

$$a(t) = a_0 e^{H_0 t}, \qquad H_0 = \sqrt{\frac{\Lambda}{3}}, \tag{25}$$

 doi:10.20944/preprints202510.2323.v1

yields

$$G_{00} = 3\frac{\dot{a}^2}{a^2} = \Lambda, \tag{26}$$
$$G_{11} = -a(t)^2\Lambda, \tag{27}$$
$$G_{22} = -a(t)^2 r^2 \Lambda, \tag{28}$$
$$G_{33} = -a(t)^2 r^2 \sin^2\theta\, \Lambda. \tag{29}$$

Therefore, from Eqs. (26)–(29)

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \tag{30}$$

holds exactly[4,8]. This picture of a flat universe ($k = 0$) is strongly supported by precision observations from the Planck satellite[11] and has become the standard model representing the global structure of the present universe.

Moreover, under the condition for a Friedmann universe without cosmological constant in Eq. (8)

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{31}$$

when only the energy density ρ is considered, from Eq. (26)

$$G_{00} = \frac{8\pi G}{c^2}\rho \tag{32}$$

is reproduced, and the standard Friedmann equation

$$3\frac{\dot{a}^2}{a^2} = 8\pi G\rho \tag{33}$$

is obtained within the framework of the ETC transformation defined by Eq. (13)[4,14,15,18].

4.3. *Case II: $k = +1$ (Closed de Sitter Solution)*

In the case of closed space, substituting from Eq. (2)

$$a(t) = a_0 \cosh(H_0 t), \qquad a_0 = \frac{c}{H_0}, \tag{34}$$

yields

$$G_{00} = \Lambda, \tag{35}$$
$$G_{11} = -3a(t)^2 \frac{\cosh^2(H_0 t)}{1 - r^2}\Lambda, \tag{36}$$
$$G_{22} = -3a(t)^2 r^2 \cosh^2(H_0 t)\Lambda, \tag{37}$$
$$G_{33} = -3a(t)^2 r^2 \sin^2\theta\, \cosh^2(H_0 t)\Lambda. \tag{38}$$

From Eqs. (35)–(38), $G^{(1)}_{\mu\nu} = G^{(2)}_{\mu\nu} = G^{(3)}_{\mu\nu}$ holds in the first, second, and third cycles of the calculation, and the ETC invariance shown in Eq. (24) was confirmed[7,8]. This result shows that the maximal symmetry (the $SO(4,1)$ symmetry group) in closed space guarantees immediate convergence and stability of the fixed point[5,6], suggesting a deep relationship between the symmetry of spatial geometry and the invariance of the ETC transformation.

 doi:10.20944/preprints202510.2323.v1

4.4. *Case III: $k = -1$ (Open de Sitter Solution)*

In open space, substituting from Eq. (2)

$$a(t) = a_0 \sinh(H_0 t), \qquad a_0 = \frac{c}{H_0}, \tag{39}$$

yields

$$G_{00} = \Lambda, \tag{40}$$
$$G_{11} = -3a(t)^2 \frac{\sinh^2(H_0 t)}{1 + r^2} \Lambda, \tag{41}$$
$$G_{22} = -3a(t)^2 r^2 \sinh^2(H_0 t) \Lambda, \tag{42}$$
$$G_{33} = -3a(t)^2 r^2 \sin^2\theta \, \sinh^2(H_0 t) \Lambda. \tag{43}$$

Here too, from Eqs. (40)–(43), $G^{(1)}_{\mu\nu} = G^{(2)}_{\mu\nu} = G^{(3)}_{\mu\nu}$ holds, and it was confirmed that the ETC transformation is strictly invariant through three iterations[7,8]. Open space also possesses maximal symmetry (the $SO(3,2)$ symmetry group), and this high degree of symmetry realizes complete stability as a fixed point[5,6].

4.5. *Summary of results*

Summarizing the above three cases, namely Eq. (30), Eqs. (35)–(38), and Eqs. (40)–(43),

$$G_{\mu\nu} = \Lambda g_{\mu\nu} \tag{44}$$

holds in general. That is, even when the ETC transformation is applied one or more times by Eq. (14), the Einstein tensor maintains the form of Eq. (44), and it was shown that the metric itself is a fixed point of the ETC transformation defined by Eq. (15). This invariance suggests a self-mapping geometric symmetry hidden in the nonlinear structure of Einstein's equation given by Eq. (8)[6,17], and clarifies that the de Sitter expansion in the FLRW universe described by Eq. (22) is positioned as a fixed-point solution[3,10]. This fixed-point structure can also be interpreted as a classical analogue of asymptotic safety in quantum gravity theories[13], suggesting a universal relationship between spacetime symmetry and fixed-point stability.

5. Discussion

In this section, we discuss the results of the ETC transformation obtained so far from the perspectives of (1) aspects dependent on sign conventions, (2) potential for generalization of the method, and (3) computational resources and stability.

5.1. *Examination of sign conventions*

Under the signature $(+, -, -, -)$ adopted in this study and Einstein's equation shown in Eq. (8)

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{45}$$

the sign correspondence $G_{00} = \Lambda$ and $G_{ii} = -\Lambda g_{ii}$ appears for the de Sitter solutions shown in Eqs. (26)–(29), Eqs. (35)–(38), and Eqs. (40)–(43). That is, the geometric relationship given by Eq. (9)

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} \tag{46}$$

is maintained, but in the display of tensor components, the output appears with a positive sign, such as "$G_{00} = \Lambda$." This difference depends on the choice of signature and which side of the equation the cosmological constant is placed, and does not affect the physical content[1,2,4]. Therefore, the invariance in the ETC transformation based on Eq. (14)

$$G'_{\mu\nu} = G_{\mu\nu} \tag{47}$$

is consistent with the standard formulation of general relativity in that Eq. (47) is maintained regardless of which sign system is adopted[3].

5.2. Potential for Generalization of the Method

The ETC transformation defined by Eq. (13) and Eq. (14)

$$g_{\mu\nu}^{(n+1)} = G_{\mu\nu}[g^{(n)}] \tag{48}$$

holds generally as a mapping from the metric tensor to the Einstein tensor. Therefore, the ETC transformation represented by Eq. (48) is not limited to the FLRW spacetime given by Eq. (1), but can be extended to other static or rotationally symmetric systems such as the Schwarzschild metric and the Kerr-Newman metric[1,19,20]. The ETC transformation framework can also provide a new analytical method from the perspective of fixed-point search for particle motion and field propagation in curved backgrounds in these spacetimes[21].

In particular, by analyzing whether the proportional relationship shown in Eq. (15)

$$G_{\mu\nu}[g] = \lambda g_{\mu\nu} \tag{49}$$

is maintained, we can verify whether each spacetime has the property of being an ETC fixed point defined by Eq. (15). Geometrically, a metric satisfying the proportional relationship of Eq. (49) is an Einstein metric, which has been extensively studied in the context of Riemannian geometry[12,13]. Furthermore, this study has suggested a deep relationship between spacetime symmetry (the number of Killing vector fields and the structure of symmetry groups) and the convergence rate and stability toward ETC fixed points[5,6], and by systematically classifying this correspondence, it is expected that new characterizations of Einstein metrics will be possible.

Moreover, generalization to quantum gravity and gauge-theoretic frameworks is also conceivable. For example, if $G_{\mu\nu}$ defined by Eq. (7) is replaced with the gauge field strength $F_{\mu\nu}$ corresponding to the curvature form, the ETC transformation given by Eq. (13) can be generalized to a nonlinear mapping

$$F_{\mu\nu} \mapsto F_{\mu\nu}[F] \tag{50}$$

and self-reproducing structures (fixed-point properties) can be discussed through the correspondence between differential geometry and field theory (principal bundles, connections, curvature)[17,22]. From this perspective, the ETC transformation represented by Eq. (48) and Eq. (50) has a general meaning as a "fixed-point equation of nonlinear gravity." In particular, considering that fixed-point searches in the function space of actions play a central role in the framework of asymptotic safety in quantum gravity theories[13] and in non-perturbative approaches such as loop quantum gravity[23], the ETC transformation can be positioned as its classical counterpart and may provide a new perspective for exploring the consistency between spacetime symmetry and quantization.

5.3. Computational Resources and Numerical Stability

In the calculation of the ETC transformation based on Eq. (14), evaluation of the Ricci tensor and scalar curvature defined by Eq. (5) and Eq. (6) and application of iterative mappings are required, and in symbolic computation, expansion of enormous terms occurs. In this study, we used the Mathematica environment and automated the sequential simplification procedures such as `FullSimplify`, `PowerExpand`, and `Chop` shown in Eq. (19) to realize stable iteration. In particular, in the comparison of the Einstein tensors at the first, second, and third iterations shown in Eq. (24), the fact that perfectly matching results

$$G_{\mu\nu}^{(1)} = G_{\mu\nu}^{(2)} = G_{\mu\nu}^{(3)} \tag{51}$$

were obtained indicates that numerical rounding errors and algebraic expansion are sufficiently controlled, and that the stability of the de Sitter solution as a fixed point is strictly maintained through three iterations. Similar geometric calculations can also be implemented with packages/frameworks such as xAct and Cadabra[24–26]. In particular, Cadabra2[26] is optimized for field-theoretic calculations and enables efficient processing of symmetry preservation and tensor reduction. Additionally, recent developments in numerical relativity[27] provide a hybrid approach that integrates symbolic and numerical computation, enabling the application of ETC transformation to more complex spacetime structures.

However, since the ETC transformation itself defined by Eq. (13) is not provided as a ready-made function, the mapping $g \mapsto G[g]$ and the procedure for simplification and branch selection (including explicit specification of assumptions) need to be constructed as user-defined macros.

In the future, when introducing higher-order curvature terms or field fluctuations, algorithmic optimization and the introduction of high-precision arithmetic environments will be necessary to suppress exponential growth of tensor terms. Moreover, when treating the ETC transformation based on Eq. (14) as a numerical simulation, it is effective to apply methods established in numerical relativity, such as convergence analysis of eigenvalue spectra and monitoring of constraint conditions, for stability determination of $g_{\mu\nu}^{(n)}$ [27,28]. In particular, by numerically monitoring the preservation of symmetry (maintenance of isotropy and homogeneity)[6], we can confirm whether the ETC transformation correctly maintains the essential geometric structure of spacetime.

5.4. Summary

From the above, the ETC transformation formalized by Eqs. (13)–(20) has structural invariance that does not depend on sign conventions or the choice of initial metric, and its definition can be extended to arbitrary spacetime metrics as shown by Eq. (48)[1,12]. Furthermore, stable implementation in a symbolic computation environment using Eq. (19) is possible, and it has been confirmed that it can serve as a powerful computational method for geometrically analyzing the iterative structure of general relativistic equations given by Eq. (8)[24–26]. As shown in this study, spacetime symmetry (particularly the isotropy and homogeneity of the FLRW metric) is an important factor determining the convergence rate and stability toward fixed points of the ETC transformation[5,6], and this deep relationship between symmetry and fixed-point structure is expected to play a central role in applications to more general spacetimes.

6. Conclusions

In this study, we introduced the concept of the Einstein Tensor Cycle (ETC) transformation, and through its definition, computational procedure, and specific application to the FLRW universe, clarified the geometric invariance and symmetry preservation structure in general relativistic field equations.

6.1. Summary of this Study

The ETC transformation defined by Eq. (13) and Eq. (14) is defined as a recursive operation that maps the metric tensor $g_{\mu\nu}$ to the Einstein tensor $G_{\mu\nu}[g]$, and searches for a metric $g^*_{\mu\nu}$ satisfying the fixed-point condition shown in Eq. (15)

$$G_{\mu\nu}[g^*] = \lambda g^*_{\mu\nu} \tag{52}$$

through iteration of

$$g_{\mu\nu}^{(n+1)} = G_{\mu\nu}[g^{(n)}] \tag{53}$$

 doi:10.20944/preprints202510.2323.v1

[12,13]. In this study, for the FLRW metric given by Eq. (1) and Eq. (22), we showed that for each case of curvature parameter $k = +1, 0, -1$, from Eq. (24)

$$G^{(1)}_{\mu\nu} = G^{(2)}_{\mu\nu} = G^{(3)}_{\mu\nu} = \Lambda g_{\mu\nu} \tag{54}$$

holds exactly through three iterations, and clarified that the de Sitter universe described by Eq. (2) and the Friedmann universe shown by Eq. (33) are stable fixed points of the ETC transformation defined by Eq. (52)[4,7,8,14]. That is, under Einstein's equation given by Eq. (8), we have directly confirmed that highly symmetric spacetimes containing the cosmological constant possess a self-reproducing structure.

6.2. Theoretical Implications: Symmetry and Fixed-Point Structure

This result suggests that Einstein's equation shown in Eq. (8) can be interpreted as a *fixed-point equation of tensor mappings that preserve symmetry* [12,17]. The ETC transformation represented by Eq. (53) is characterized by reconstructing the nonlinear curvature structure defined by Eq. (5) and Eq. (6) as a repetition of sequential linear operations, and provides a new method for analyzing the non-perturbative structure of gravitational fields[1,2].

As a particularly important point, the ETC transformation strictly preserves at each iteration the spatial isotropy (rotational $SO(3)$ symmetry at each point) and homogeneity (spatial translational symmetry) possessed by the FLRW metric[5,6]. The perfect invariance over three iterations confirmed in Eq. (54) shows that the highly symmetric de Sitter spacetime (maximal symmetry: $SO(4,1)$ or $SO(3,2)$ symmetry group with 10 Killing vector fields[3,5]) is exceptionally stable as a fixed point. This correspondence between symmetry and fixed-point stability is deeply connected to the hierarchical recursive structure of the Ricci tensor and scalar curvature, and suggests the existence of *self-similar curvature spaces* in differential geometry[12,13].

Furthermore, the fact that different spatial geometries (positive, zero, and negative curvature) corresponding to curvature parameters $k = +1, 0, -1$ are uniformly derived from a single ETC iteration procedure can be interpreted as a structure in which discrete symmetry breaking (bifurcation of curvature sign) and continuous symmetry preservation (maintenance of isotropy and homogeneity) coexist[6], providing a new perspective in the hierarchical classification of symmetries.

6.3. Future Perspectives

Future research directions include the following.

1. **Extension to anisotropic metrics and symmetry breaking**: Apply the ETC transformation defined by Eq. (13) to anisotropic and rotationally symmetric spacetimes such as Bianchi-type universes or Kerr-Newman-type metrics, and verify how partial symmetry breaking affects the validity of the fixed-point condition $G_{\mu\nu} = \lambda g_{\mu\nu}$ shown in Eq. (49) and the convergence rate[16,20]. In particular, it is expected to clarify the quantitative relationship between the number of Killing vector fields and the convergence characteristics to fixed points[5,6].
2. **Quantum gravity and gauge-theoretic extension with internal symmetries**: By associating $G_{\mu\nu}$ defined by Eq. (7) with the gauge field strength $F_{\mu\nu}$, extend the ETC transformation to the self-mapping structure of gauge theory as shown in Eq. (50), and examine the correspondence through the framework of principal bundles, connections, and curvature[17,22]. This may open the way to a unified description of external symmetries of spacetime (Poincaré group, de Sitter group, etc.) and internal symmetries (gauge groups in Yang-Mills theory). Furthermore, by exploring connections with asymptotic safety in quantum gravity theories[13] and non-perturbative approaches such as loop quantum gravity[23], it is expected that a foundation for quantum-theoretic extension of the ETC transformation will be established.
3. **Symmetry indices in numerical stability and higher-order term analysis**: Using the convergence criterion based on Eq. (20), evaluate convergence when increasing the number of calculations, and analyze the asymptotic behavior of terms containing higher-order derivatives to quantitatively

identify the stability region of the ETC transformation[27,28]. In particular, introduce the degree of symmetry (number of Killing vector fields, isotropy parameters, etc.) as a numerical index and investigate the correlation with the convergence rate to fixed points[5,6].

4. **Dynamic introduction of energy-momentum tensor and dynamical symmetry breaking**: Incorporate the matter term $T_{\mu\nu}$ given by Eq. (10) into the iterative mapping of Eq. (53), and clarify the stable structure of the ETC transformation in dynamical universes (dark energy, scalar fields, etc.)[4,9,10]. This is expected to elucidate how spontaneous symmetry breaking and phase transitions accompanying cosmic evolution affect the ETC fixed-point structure[6].

6.4. Concluding Remarks

Through this study, the *recursive self-preservation property (ETC invariance)* of the Einstein tensor defined by Eq. (17) has been made explicit, and the deep symmetry structure of the equation system of general relativity represented by Eq. (8) has been depicted from a new perspective[3]. The ETC transformation formalized by Eqs. (13)–(20) functions as a geometric algorithm that searches for fixed points while preserving spacetime symmetry, and has clarified the deep relationship between the hierarchical structure of symmetry (from maximal symmetry to partial symmetry breaking) and fixed-point stability[5,6,13]. This framework extends existing geometric methods and has potential as a unified analytical framework in the intersection of gravity theory, cosmology, and gauge theory[17]. In the future, through application to more general spacetime structures as shown by Eq. (48), it is expected that the physical and geometric meaning of symmetry and ETC invariance will be further clarified.

Author Contributions: Conceptualization, H.M.; methodology, H.M.; software, H.M.; validation, H.M.; formal analysis, H.M.; investigation, H.M.; writing—original draft preparation, H.M.; writing—review and editing, H.M.; visualization, H.M. The author has read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: The Mathematica code and computational data supporting the findings of this study are openly available in Zenodo at DOI: 10.5281/zenodo.17451808.

Acknowledgments: During the preparation of this manuscript, the author used Claude Sonnet 4.5 (Anthropic) for the purposes of English editing, translation from Japanese to English, and manuscript formatting assistance. The author has reviewed and edited the output and takes full responsibility for the content of this publication.

Conflicts of Interest: The author declares no conflicts of interest.

7. Mathematica Code

The Mathematica notebook and PDF output containing detailed calculations are publicly available at:

- **Zenodo archive**: DOI: 10.5281/zenodo.17451808

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