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No–Inflation Under Pachner Moves for Teichmüller State–Integrals: A DSFL Residual Monotonicity Theorem

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Abstract

We establish a no-inflation (monotonicity) theorem for a residual that measures calibrated mismatch in Teichmüller state-integral models built from the Faddeev quantum dilogarithm. For the elementary $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ Pachner moves, cutting and gluing are recast in a single boundary Hilbert geometry, with "admissible" kernel maps that intertwine the calibration and do not expand norm. The pentagon identity then makes the induced gluing operator an isometry, so the residual cannot increase. Consequently, this residual is a Lyapunov functional for triangulation updates: pipelines of local moves are certified to be stable, and exact recursions inherit monotone decay, with quantitative conditioning controlled by principal–angle bounds between polarizations. The paper also provides a minimal, reproducible verification harness that enforces intertwining and nonexpansion and checks move-by-move monotonicity on standard examples. The result upgrades gluing stability from an assumption to a theorem and offers an algorithm-ready certification layer for Teichmüller TQFT computations.

Keywords: Teichmüller TQFT; state integrals; Faddeev quantum dilogarithm; pentagon identity; Pachner moves; residual monotonicity (no-inflation); DSFL; Lyapunov functional; cutting and gluing; triangulation stability; admissible (intertwining; contractive) kernels; pointer algebra (conditional expectation; MASA); principal/Friedrichs angles; modular double; Chern–Simons; quantum topology

1. Introduction

This paper turns cutting and gluing stability in Teichmüller TQFT into a theorem: Pachner moves act as calibration–intertwining contractions in a single comparison Hilbert geometry, so any triangulation change provably cannot worsen a calibrated error; we quantify robustness, polarization angles and conditioning, recursion decay, and provide a practical certification framework.

Cutting and gluing lie at the heart of low–dimensional quantum topology and of state–integral realizations of Chern–Simons/Teichmüller TQFT [1–4]. In practice, computations based on ideal triangulations or exact recursion repeatedly replace one local presentation by another via Pachner moves [5]. What has been missing is a certifiable stability law ensuring that these rearrangements do not amplify numerical or analytic mismatch at the gluing interface. We supply such a law: a no-inflation (monotonicity) theorem for a calibrated residual—the DSFL residual—under the elementary Pachner moves $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ in the Teichmüller state–integral class [3,6,7]. The mechanism is simple: unitarity of the edge operator together with the five-term (pentagon) identity yields a boundary isometry that intertwines the calibration, so the residual cannot increase. The result upgrades gluing from an assumption to a rate-certified, algorithm-ready invariant, with explicit angle and recursion bounds that make computations, proofs, and software in Teichmüller TQFT trustworthy, modular, and mechanically verifiable.

Primitives and the single observable.

Fix a boundary component Σ with a chosen polarization (e.g. shear or Fenchel–Nielsen). We attach: (i) a comparison Hilbert space H_Σ for boundary data, (ii) a statistical channel S_Σ (blueprint variables) and a physical channel $\mathsf{P}_\Sigma \subset \mathsf{H}_\Sigma$ (response amplitudes), and (iii) a calibration pair $(\mathcal{I}_\Sigma, \mathcal{J}_\Sigma)$ with

$$\mathcal{I}_\Sigma : \mathsf{S}_\Sigma \to \mathsf{P}_\Sigma, \qquad \mathcal{J}_\Sigma : \mathsf{P}_\Sigma \to \mathsf{S}_\Sigma, \qquad \mathcal{I}_\Sigma \mathcal{J}_\Sigma = \mathrm{id}_{\mathsf{P}_\Sigma}, \quad \mathcal{J}_\Sigma \mathcal{I}_\Sigma = P_{\mathsf{S}_\Sigma},$$

where P_{S_Σ} denotes the orthogonal projector in the comparison geometry. Concretely, the polarization determines a pointer (diagonal) MASA in $\mathcal{B}(\mathsf{H}_\Sigma)$; $\mathcal{J}_\Sigma$ is the orthogonal conditional expectation onto that MASA and $\mathcal{I}_\Sigma$ its adjoint realization on states [8,9]. In Teichmüller theory, H_Σ is the modular–double L^2 space of boundary coordinates [3,10].

Given boundary data $(s, p) \in \mathsf{S}_\Sigma \oplus \mathsf{P}_\Sigma$, we measure calibrated mismatch by

$$\mathcal{R}_\Sigma(s, p) := \| \, p - \mathcal{I}_\Sigma s \, \|^2_{\mathsf{H}_\Sigma}.$$

This residual is canonical once the boundary chart is fixed and enjoys three basic properties that we exploit throughout: (a) invariance under unitary reparametrizations of the boundary chart $U : \mathsf{H}_\Sigma \to \mathsf{H}_\Sigma$ and bounded changes of statistical coordinates $V : \mathsf{S}_\Sigma \to \mathsf{S}_\Sigma$,

$$\| \, Up - U\mathcal{I}_\Sigma V^{-1} V s \, \|^2 = \| \, p - \mathcal{I}_\Sigma s \, \|^2;$$

(b) convexity and continuity in each argument (Hilbert norm calculus); and (c) cylinder idempotence on collars, implemented by the DSFL projector $\Pi_{\mathrm{DSFL},\Sigma}$ onto the nullspace of $e := p - \mathcal{I}_\Sigma s$ in H_Σ, so $\Pi_{\mathrm{DSFL},\Sigma} \circ \Pi_{\mathrm{DSFL},\Sigma} = \Pi_{\mathrm{DSFL},\Sigma}$ and the cylinder acts as the identity on residual classes.

In this language, a gluing step is *admissible* when the induced boundary maps $(\tilde{\Phi}, \Phi)$ *intertwine* the calibration and are nonexpansive in H_Σ,

$$\Phi \, \mathcal{I}_\Sigma = \mathcal{I}_{\Sigma'} \tilde{\Phi}, \qquad \|\Phi\|_{\mathsf{H} \to \mathsf{H}} \le 1,$$

in which case the data–processing inequality holds:

$$\mathcal{R}_{\Sigma'}(\tilde{\Phi} s, \Phi p) \le \mathcal{R}_\Sigma(s, p).$$

All Teichmüller Pachner moves furnish such admissible pairs because the edge operator is unitary and satisfies the pentagon identity, making the boundary gluing operator a projective isometry that intertwines the calibration [3,6,7].

What we prove.

In the Teichmüller kernel class (Faddeev quantum dilogarithm), the edge operator T is unitary and satisfies the five–term (pentagon) identity [6,7]. These two facts imply that the boundary operator implementing each Pachner move is a (projective) *isometry* on H_Σ and that it *intertwines* the calibration [3,10]. Consequently, for any boundary datum (s, p),

$$\mathcal{R}_{\mathrm{glue}}(T', \Sigma) \;=\; \mathcal{R}_\Sigma(\tilde{\Phi} s, \Phi p) \;\le\; \mathcal{R}_\Sigma(s, p) \;=\; \mathcal{R}_{\mathrm{glue}}(T, \Sigma),$$

i.e. the DSFL residual cannot increase under a single move, hence along any move sequence. In short, *gluing in Teichmüller TQFT is residually non–expansive.*

Why it matters.

- **Triangulation robustness.** Residual monotonicity makes $\mathcal{R}_{\mathrm{glue}}$ a Lyapunov functional along triangulation updates, certifying that exact–recursion pipelines cannot worsen calibrated mismatch [3,5].
- **Algorithmic reliability.** The same inequality provides a priori control for state–sum/ state–integral compilers that simplify triangulations by local moves, and it interfaces with stepwise error accounting; see also exact/recursive implementations in related state–integral programs [4,11].

- **Clean separation of concerns.** Analytic special–function estimates are *not* needed: once maps are isometric/contractive and intertwine $(\mathcal{I}, \mathcal{J})$, the inequality is a two–line argument in H_Σ (Cauchy–Schwarz in the comparison geometry), with the calibration/expectation behaving as in [12].

Conceptual position.

The result is a concrete instance of the DSFL paradigm: formulate dynamics and sewing in a single comparison geometry, enforce *admissibility* (intertwining + non–expansion), and read all "axioms" (gluing stability, cylinder identity) as consequences for one quadratic functional. Here, unitarity + pentagon [7,13] ⇒ admissibility; DSFL then turns admissibility into *no–inflation*.

Scope and hypotheses.

We work in the Teichmüller state–integral setting where: (i) boundary spaces are L^2–type (modular double) Hilbert spaces attached to ideal triangulations [3,10], (ii) the edge operator T is unitary [7], (iii) the five–term identity implements $2 \leftrightarrow 3$ [6], (iv) unit/counit blocks implement $1 \leftrightarrow 4$ [3], and (v) calibration maps $(\mathcal{I}, \mathcal{J})$ are chosen compatibly with boundary restriction and orthogonal conditional expectation (pointer algebra) [12]. Within this envelope, every local move is an admissible map in the sense of (2).

Contributions (at a glance).

1. **Theorem 5:** No–inflation under Pachner moves for $\mathcal{R}_{\text{glue}}$ in the Teichmüller kernel class; $\mathcal{R}_{\text{glue}}$ is Lyapunov along any finite move sequence [5–7].
2. **Mechanism:** Unitarity + pentagon ⇒ boundary isometry and calibration intertwining ⇒ residual monotonicity (two–line proof) [3,7,10].
3. **Verification harness:** A minimal, reproducible testbed (mock admissible kernels) that checks move–by–move monotonicity on standard manifolds and reports conditioning proxies (cf. numerical pipelines in [4,14]).

Proof idea in one line.

Let $e := p - \mathcal{I}s$. Intertwining gives $\Phi p - \mathcal{I}\tilde{\Phi}s = \Phi e$; non–expansion gives $\|\Phi e\|_{\mathsf{H}} \le \|e\|_{\mathsf{H}}$; hence $\mathcal{R}(\tilde{\Phi}s, \Phi p) \le \mathcal{R}(s, p)$. "Pentagon + unitarity" provides the admissible Φ in the Teichmüller setting [3,6,7].

Organization.

Section 2 (renamed *Setting and standing hypotheses*) fixes the comparison geometry, calibration, and admissible class. Section 3 states and proves the no–inflation theorem. Section 4 recalls why Teichmüller kernels satisfy admissibility (unitarity and the pentagon identity). Section 5 outlines the verification harness and acceptance criteria.

2. Setting and Standing Hypotheses

Boundary comparison geometry.

For a boundary surface Σ with ideal triangulation, set the comparison Hilbert space $\mathsf{H}_\Sigma \cong L^2(\mathbb{R}^{E(\Sigma)})$ (or the Plancherel space for the modular double) as in the Teichmüller/quantum cluster literature [3,10]. We write $\langle \cdot, \cdot \rangle_{\mathsf{H}}$ and $\| \cdot \|_{\mathsf{H}}$ for its inner product and norm.

Interchangeability (calibration).

Fix bounded maps $(\mathcal{I}_\Sigma, \mathcal{J}_\Sigma)$ between a statistical channel S_Σ and a physical channel $\mathsf{P}_\Sigma \subset \mathsf{H}_\Sigma$ such that

$$\mathcal{I}_\Sigma \mathcal{J}_\Sigma = \mathrm{id}_{\mathsf{P}_\Sigma}, \qquad \mathcal{J}_\Sigma \mathcal{I}_\Sigma = P_{\mathsf{S}_\Sigma},$$

with P_{S_Σ} the orthogonal projector in the comparison geometry. At the operator–algebraic level, $\mathcal{J}_\Sigma$ is the orthogonal conditional expectation onto the pointer (diagonal) MASA for the chosen polarization [12,15,16].

DSFL residual on Σ.

For boundary data $(s,p) \in \mathsf{S}_\Sigma \oplus \mathsf{P}_\Sigma$ define

$$\mathcal{R}_\Sigma(s,p) \;:=\; \|\, p - \mathcal{I}_\Sigma s \,\|^2_{\mathsf{H}_\Sigma}. \tag{1}$$

This is the sector–natural quadratic mismatch (Residual of Sameness), expressed in the single comparison geometry attached to the boundary chart [3,10].

Admissible kernels.

A linear map on boundary data $(\tilde{\Phi},\Phi) : \mathsf{S}_\Sigma \oplus \mathsf{P}_\Sigma \to \mathsf{S}_{\Sigma'} \oplus \mathsf{P}_{\Sigma'}$ is *admissible* if it *intertwines* the calibration and is *non–expansive* in H:

$$\Phi\,\mathcal{I}_\Sigma = \mathcal{I}_{\Sigma'}\,\tilde{\Phi}, \qquad \|\Phi\|_{\mathsf{H}\to\mathsf{H}} \le 1, \quad \|\tilde{\Phi}\| \le 1. \tag{2}$$

(Optionally: one–budget/resource preserving and causal ceilings on collars; these play no role below.) Intertwining captures functoriality of boundary restriction under gluing [3].

Teichmüller kernel class.

For a cobordism (or an ideal tetrahedron) we represent the interior by a state–integral kernel built from Faddeev's quantum dilogarithm Φ_b in the modular–double Plancherel representation, combined via the standard edge operator T; the five–term *pentagon identity* and unitarity of T yield an isometric gluing operator on H_Σ [3,4,6,7]. Pachner moves $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ are implemented at the operator level by these identities [3,5,10].

Definition 1 (Gluing DSFL residual). *Let $T \mapsto T'$ be a single Pachner move on an ideal triangulation along a boundary Σ, implemented on H_Σ by a (projective) isometry G_Σ [6,7]. Given (s,p) on Σ, push forward by $(\tilde{\Phi},\Phi)$ induced by the move and define*

$$\mathcal{R}_{\mathrm{glue}}(T,\Sigma) \;:=\; \mathcal{R}_\Sigma(s,p), \qquad \mathcal{R}_{\mathrm{glue}}(T',\Sigma) \;:=\; \mathcal{R}_\Sigma(\tilde{\Phi}s, \Phi p).$$

2.1. Well–Posedness, Invariances, and Basic Calculus

Lemma 1 (Well–posedness and convexity). *$\mathcal{R}_\Sigma$ in (1) is finite on $\mathsf{S}_\Sigma \oplus \mathsf{P}_\Sigma$, jointly continuous, and convex in each argument. Moreover, for any $0 \le t \le 1$ and (s_i,p_i),*

$$\mathcal{R}_\Sigma\big(ts_1{+}(1{-}t)s_2,\; tp_1{+}(1{-}t)p_2\big) \;\le\; t\,\mathcal{R}_\Sigma(s_1,p_1) + (1{-}t)\,\mathcal{R}_\Sigma(s_2,p_2).$$

Proof. $\mathcal{I}_\Sigma$ is bounded and $\|\cdot\|^2_{\mathsf{H}}$ is continuous and strictly convex; composition preserves these properties. The final inequality follows from the convexity and unitary invariance of Hilbert norms (see, e.g., [17, Ch. II]). □

Proposition 1 (Reparametrization invariance). *Let $U : \mathsf{H}_\Sigma \to \mathsf{H}_\Sigma$ be unitary and suppose the boundary chart changes by $\mathcal{I}_\Sigma \mapsto U\mathcal{I}_\Sigma V^{-1}$ with a bounded, invertible $V : \mathsf{S}_\Sigma \to \mathsf{S}_\Sigma$ and $p \mapsto Up$, $s \mapsto Vs$. Then $\mathcal{R}_\Sigma$ is invariant:*

$$\|\, Up - U\mathcal{I}_\Sigma Vs \,\|^2 = \|\, p - \mathcal{I}_\Sigma s \,\|^2.$$

Proof. Unitary invariance of the Hilbert norm gives $\|Ux\| = \|x\|$; substitute and simplify [17, Ch. I]. □

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Lemma 2 (Cylinder identity via the DSFL projector). *Let* $\Pi_{\mathrm{DSFL},\Sigma}$ *be the (boundary) DSFL projector onto the nullspace of* $e := p - \mathcal{I}_\Sigma s$ *in the* H_Σ*–metric.*[1] *Then for any cylinder* $\mathrm{Cyl}(\Sigma) = \Sigma \times [0,1]$, $\Pi_{\mathrm{DSFL},\Sigma} \circ \Pi_{\mathrm{DSFL},\Sigma} = \Pi_{\mathrm{DSFL},\Sigma}$ *and the cylinder acts as the identity on the residual class.*

Proof. Idempotence is standard for orthogonal projections onto closed subspaces in Hilbert spaces [15,16, Ch. IV]; see also Halmos' two–subspace calculus [18]. The DSFL flow on the collar drives e to the residual kernel, hence one application of $\Pi_{\mathrm{DSFL},\Sigma}$ fixes the class and a second has no effect. ☐

2.2. Data–Processing Inequality and Composition

Theorem 1 (Data–processing for admissible maps). *If* $(\tilde{\Phi}, \Phi)$ *is admissible in the sense of* (2)*, then*

$$\mathcal{R}_{\Sigma'}(\tilde{\Phi}s, \Phi p) \;\le\; \mathcal{R}_\Sigma(s,p) \qquad \textit{for all } (s,p). \tag{3}$$

Proof. Set $e := p - \mathcal{I}_\Sigma s$. Intertwining gives $\Phi p - \mathcal{I}_{\Sigma'}\tilde{\Phi}s = \Phi(p - \mathcal{I}_\Sigma s) = \Phi e$. Taking norms and using $\|\Phi\| \le 1$, $\mathcal{R}_{\Sigma'}(\tilde{\Phi}s, \Phi p) = \|\Phi e\|^2 \le \|e\|^2 = \mathcal{R}_\Sigma(s,p)$. This is the Hilbert–space avatar of data processing/monotonicity under contractions; cf. the quantum information–theoretic data–processing principle for contractive distances [19, Sec. 9.2] and operator–algebraic monotonicity results of Petz [20]. ☐

2.3. Well–Posedness, Invariances, and Basic Calculus

Lemma 3 (Well–posedness and convexity). $\mathcal{R}_\Sigma$ *in* (1) *is finite on* $\mathsf{S}_\Sigma \oplus \mathsf{P}_\Sigma$*, jointly continuous, and convex in each argument. Moreover, for any* $0 \le t \le 1$ *and* (s_i, p_i)*,*

$$\mathcal{R}_\Sigma\big(ts_1 + (1-t)s_2,\ tp_1 + (1-t)p_2\big) \;\le\; t\,\mathcal{R}_\Sigma(s_1,p_1) + (1-t)\,\mathcal{R}_\Sigma(s_2,p_2).$$

Proof. $\mathcal{I}_\Sigma$ is bounded and $\|\cdot\|^2_{\mathsf{H}}$ is continuous and strictly convex; composition preserves these properties. The final inequality follows from the convexity and unitary invariance of Hilbert norms [17, Ch. I–II]. ☐

Proposition 2 (Reparametrization invariance). *Let* $U : \mathsf{H}_\Sigma \to \mathsf{H}_\Sigma$ *be unitary and suppose the boundary chart changes by* $\mathcal{I}_\Sigma \mapsto U\mathcal{I}_\Sigma V^{-1}$ *with a bounded, invertible* $V : \mathsf{S}_\Sigma \to \mathsf{S}_\Sigma$ *and* $p \mapsto Up$, $s \mapsto Vs$*. Then* $\mathcal{R}_\Sigma$ *is invariant:*

$$\|\, Up - U\mathcal{I}_\Sigma V s \,\|^2 = \|\, p - \mathcal{I}_\Sigma s \,\|^2.$$

Proof. Unitary invariance of the Hilbert norm gives $\|Ux\| = \|x\|$; substitute and simplify [17, Ch. I]. ☐

Lemma 4 (Cylinder identity via the DSFL projector). *Let* $\Pi_{\mathrm{DSFL},\Sigma}$ *be the (boundary) DSFL projector onto the nullspace of* $e := p - \mathcal{I}_\Sigma s$ *in the* H_Σ*–metric.*[2] *Then for any cylinder* $\mathrm{Cyl}(\Sigma) = \Sigma \times [0,1]$, $\Pi_{\mathrm{DSFL},\Sigma} \circ \Pi_{\mathrm{DSFL},\Sigma} = \Pi_{\mathrm{DSFL},\Sigma}$ *and the cylinder acts as the identity on the residual class.*

Proof. Idempotence is standard for orthogonal projections onto closed subspaces of a Hilbert space [16, Ch. IV]; see also Halmos' two–subspace calculus [18]. The DSFL flow on the collar drives e to the residual kernel; one application of $\Pi_{\mathrm{DSFL},\Sigma}$ fixes the class and a second has no effect. ☐

2.4. Data–Processing Inequality and Composition

Theorem 2 (Data–processing for admissible maps). *If* $(\tilde{\Phi}, \Phi)$ *is admissible in the sense of* (2)*, then*

$$\mathcal{R}_{\Sigma'}(\tilde{\Phi}s, \Phi p) \;\le\; \mathcal{R}_\Sigma(s,p) \qquad \textit{for all } (s,p). \tag{4}$$

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[1] In practice: the orthogonal projection onto $\ker(\mathrm{id} - \mathcal{I}_\Sigma \mathcal{J}_\Sigma)$ on collars; see the main text for the DSFL evolution justification.

[2] In practice: the orthogonal projection onto $\ker(\mathrm{id} - \mathcal{I}_\Sigma \mathcal{J}_\Sigma)$ on collars; see the main text for the DSFL evolution justification.

Proof. Set $e := p - \mathcal{I}_\Sigma s$. Intertwining gives $\Phi p - \mathcal{I}_{\Sigma'}\tilde{\Phi} s = \Phi(p - \mathcal{I}_\Sigma s) = \Phi e$. Taking norms and using $\|\Phi\| \le 1$, $\mathcal{R}_{\Sigma'}(\tilde{\Phi} s, \Phi p) = \|\Phi e\|^2 \le \|e\|^2 = \mathcal{R}_\Sigma(s,p)$. This is the Hilbert–space avatar of data processing under contractions; cf. [19, Sec. 9.2] and operator–algebraic monotonicity results of Petz [20]. □

Corollary 1 (Monotonicity under compositions). *If* $(\tilde{\Phi}_i, \Phi_i)$ *are admissible, then so is their composition and* $\mathcal{R}(\tilde{\Phi}_2\tilde{\Phi}_1 s,\ \Phi_2\Phi_1 p) \le \mathcal{R}(s,p)$.

Proof. Intertwining is preserved under composition; the operator norm is submultiplicative ($\|\Phi_2\Phi_1\| \le \|\Phi_2\|\,\|\Phi_1\|$), hence ≤ 1 [17, Thm. IV.2.2]. Apply Theorem 1 twice. □

Proposition 3 (Stochastic/ensemble admissibility). *Let* $\{(\tilde{\Phi}_x, \Phi_x)\}_{x\in X}$ *be admissible pairs and let* μ *be a probability measure on* X. *Define the averaged map* $\tilde{\Phi} := \int \tilde{\Phi}_x\, d\mu(x)$, $\Phi := \int \Phi_x\, d\mu(x)$. *If the integrals define bounded operators and intertwining holds pointwise, then* $(\tilde{\Phi}, \Phi)$ *is admissible and* (4) *holds.*

Proof. Convexity of the operator–norm unit ball implies $\|\Phi\| \le \int \|\Phi_x\|\, d\mu \le 1$; see [17, Ch. I]. Intertwining passes to Bochner integrals by linearity. Apply Theorem 1. □

2.5. Gluing and Pachner Moves

Theorem 3 (No–inflation under Pachner moves). *Let* $T \rightsquigarrow T'$ *be a* $2 \leftrightarrow 3$ *or* $1 \leftrightarrow 4$ *move along* Σ, *implemented on* H_Σ *by a (projective) isometry* G_Σ *and inducing an admissible pair* $(\tilde{\Phi}, \Phi)$. *Then*

$$\mathcal{R}_{\mathrm{glue}}(T',\Sigma) \ \le\ \mathcal{R}_{\mathrm{glue}}(T,\Sigma)$$

for the gluing residual of Definition 1. Consequently, along any finite move sequence, $\mathcal{R}_{\mathrm{glue}}$ *is nonincreasing.*

Proof. By Definition 1 and Theorem 1. In the Teichmüller class, the isometry G_Σ is provided by unitarity of the edge operator and the five–term identity [3,6,7]; Pachner's theorem ensures sufficiency of $2 \leftrightarrow 3$ and $1 \leftrightarrow 4$ moves [5]. □

Proposition 4 (Teichmüller admissibility criterion). *In the Teichmüller kernel class, the edge operator* T *(built from* Φ_b*) is unitary and satisfies the five–term identity* $\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23} = \mathsf{T}_{23}\mathsf{T}_{12}$. *Any Pachner move is implemented by replacing one ordered product of edge operators by the other. Thus* G_Σ *is (projectively) unitary on* H_Σ, *and* $(\tilde{\Phi}, \Phi)$ *is admissible.*

Idea. Unitary products are unitary; the pentagon identity identifies the two sides of the move [6,7]. Boundary restriction and calibration intertwiners commute with these replacements in the Teichmüller setting [3,10], yielding (2). □

2.6. Quantitative Refinements and Robustness

Definition 2 (Principal–angle constant). *Let* $P_{\mathrm{in}}, P_{\mathrm{out}}$ *be the orthogonal projectors onto two Lagrangian polarizations on* Σ. *Define* $\gamma := \|P_{\mathrm{in}} P_{\mathrm{out}}\| = \cos\theta_F \in [0,1]$.

Proposition 5 (Through–seam bound). *For any contraction* K *on* H_Σ, $\|P_{\mathrm{out}} K P_{\mathrm{in}}\| \le \|P_{\mathrm{out}} P_{\mathrm{in}}\| = \gamma$.

Proof. Use $\|AB\| \le \|A\|\,\|B\|$ and $\|P_{\mathrm{out}}\| = \|P_{\mathrm{in}}\| = 1$; the sharper $\|P_{\mathrm{out}} P_{\mathrm{in}}\|$ description follows from the principal–angle (Friedrichs) calculus for two closed subspaces [17,18]. □

Corollary 2 (Angle–controlled uncertainty and conditioning). *For any decomposition* $e = e_{\mathrm{in}} + e_{\mathrm{out}}$ *with* $e_{\mathrm{in}} \in \mathrm{ran}(P_{\mathrm{in}})$, $e_{\mathrm{out}} \in \mathrm{ran}(P_{\mathrm{out}})$,

$$(1-\gamma)\big(\|e_{\mathrm{in}}\|^2 + \|e_{\mathrm{out}}\|^2\big) \ \le\ \|e\|^2 \ \le\ (1+\gamma)\big(\|e_{\mathrm{in}}\|^2 + |e_{\mathrm{out}}\|^2\big).$$

Hence the condition number for resolving the split is $\kappa(\theta_F) = \frac{1+\gamma}{1-\gamma}$.

Lemma 5 (Davis–Kahan stability). *If the two polarizations arise from spectral subspaces of nearby self–adjoint operators with gap* $\delta > 0$ *and perturbation* E*, then* $\|P_{\mathrm{in}} - P_{\mathrm{out}}\| = \sin\theta_{\max} \lesssim \|E\|/\delta$*; hence* γ *and* κ *are Lipschitz in* $\|E\|$ *while the gap persists.*

Proof sketch. Apply the Davis–Kahan $\sin\Theta$ theorem in the projector form; see [17,21]. Let P and $\tilde{P}$ denote the spectral projectors onto the target (unperturbed vs. perturbed) invariant subspaces, and let $\gamma := \mathrm{dist}\big(\mathrm{spec}(A|_{\mathrm{target}}), \mathrm{spec}(A|_{\mathrm{complement}})\big)$ be the spectral separation. Davis–Kahan yields

$$\|(I-P)\tilde{P}\| \;=\; \|\sin\Theta\| \;\le\; \frac{\|E\|}{\gamma},$$

so the largest principal angle $\theta_{\max}$ between the two subspaces satisfies $\sin\theta_{\max} \le \|E\|/\gamma$, which is the stated bound. □

2.7. Consequences for Algorithms and Exact Recursion

Definition 3 (Move–by–move Lyapunov profile). *Given a sequence of moves with induced admissible pairs* $\{(\tilde{\Phi}_k, \Phi_k)\}_{k=1}^{n}$*, define* $\mathcal{R}^{(0)} := \mathcal{R}(s,p)$*,* $\mathcal{R}^{(k)} := \mathcal{R}(\tilde{\Phi}_k \cdots \tilde{\Phi}_1 s,\ \Phi_k \cdots \Phi_1 p)$*.*

Proposition 6 (Uniform nonincrease and certification). $\mathcal{R}^{(k+1)} \le \mathcal{R}^{(k)}$ *for all* k*. Moreover, if each step admits a principal–angle constant* γ_k *on the seam, then any through–seam contraction satisfies* $\|P^{(k)}_{\mathrm{out}} K^{(k)} P^{(k)}_{\mathrm{in}}\| \le \gamma_k$ *and the cumulative leakage is controlled by* $\prod_{k=1}^{n} \gamma_k$*.*

Proof. Iterate Theorem 1; use the principal–angle (Friedrichs) calculus and the bound of Proposition 5 at each seam [17,18]. □

Theorem 4 (Stability of exact recursion). *Consider an exact recursion on boundary amplitudes* $A_{g+1} = \mathcal{K}_g(A_g)$ *with each* $\mathcal{K}_g$ *admissible (intertwining,* $\|\mathcal{K}_g\| \le 1$*). Then the residuals decay monotonically:* $\mathcal{R}(A_{g+1}) \le \mathcal{R}(A_g) \le \cdots \le \mathcal{R}(A_0)$*. If additionally* $\|\mathcal{K}_g\| \le \rho < 1$ *uniformly, then* $\mathcal{R}(A_g) \le \rho^{2g}\mathcal{R}(A_0)$*.*

Proof. Apply Theorem 1 at each recursion step; submultiplicativity of the operator norm and the contraction principle yield the uniform exponential bound [17, Thm. I.2.3], [22]. □

2.8. Teichmüller Specialization: Why Admissibility Holds and What It Buys

Proposition 7 (Unitarity and pentagon ⇒ admissibility). *In the Teichmüller class, each local move is implemented by a product of edge operators* T*. Unitarity of* T *implies* $\|\Phi\| = 1$*; the pentagon identity guarantees the replacement across* $2 \leftrightarrow 3$ *moves. Boundary restriction and pointer dephasing commute with unitary conjugation, hence* (2)*.*

Proof. Unitarity of T and the operator pentagon identity are classical for the Faddeev quantum dilogarithm [6,7]; functoriality of boundary restriction in Teichmüller TQFT is described in [3,10]. These facts ensure Φ is (projectively) unitary and intertwines the calibration, i.e. admissible. □

Corollary 3 (No–inflation in Teichmüller TQFT). *All conclusions of Theorem 3 through Theorem 4 hold for Teichmüller state–integrals, with equality in* (4) *iff* e *lies in the isometric invariant subspace of the move.*

Consequences (scientific takeaways).

- **Certified triangulation robustness.** The DSFL residual $\mathcal{R}_{\mathrm{glue}}$ is a Lyapunov functional along Pachner sequences; any pipeline of local simplifications cannot inflate calibrated mismatch.
- **Quantitative seam control.** Principal angles between boundary polarizations bound leakage and algorithmic conditioning by $\gamma = \cos\theta_F$ and $\kappa(\theta_F) = \frac{1+\gamma}{1-\gamma}$.

- **Recursion reliability.** Exact recursion steps modeled by admissible kernels inherit monotone residual decay; uniform contractivity yields exponential envelopes in the recursion depth.
- **Chart independence.** By Proposition 1, reparametrizations/changes of boundary chart by unitary/metaplectic transforms preserve the value of $\mathcal{R}$.
- **Perturbation stability.** Under spectral gaps, Davis–Kahan (Lemma 5) gives Lipschitz robustness of γ and κ against boundary perturbations.

3. Main Results

Theorem 5 (No–inflation under Pachner moves). *Assume the Teichmüller kernel class with pentagon identity and unitary edge operator* T, *so that the induced gluing operator* G_Σ *is an isometry on* H_Σ *and the pair* $(\tilde{\Phi}, \Phi)$ *implementing a single Pachner move is admissible* in the sense of (2). *Then for any boundary datum* (s, p),

$$\mathcal{R}_{\text{glue}}(T', \Sigma) \;\le\; \mathcal{R}_{\text{glue}}(T, \Sigma). \tag{5}$$

In particular, $\mathcal{R}_{\text{glue}}$ *is a Lyapunov (nonincreasing) functional along any sequence of* $2 \leftrightarrow 3$ *and* $1 \leftrightarrow 4$ *moves.*

Proof. By intertwining, $\Phi\mathcal{I}_\Sigma = \mathcal{I}_\Sigma\tilde{\Phi}$, hence for $e := p - \mathcal{I}_\Sigma s$,

$$\Phi p - \mathcal{I}_\Sigma\tilde{\Phi}s \;=\; \Phi(p - \mathcal{I}_\Sigma s) \;=\; \Phi e.$$

Taking H–norms and using $\|\Phi\| \le 1$,

$$\mathcal{R}_\Sigma(\tilde{\Phi}s, \Phi p) \;=\; \|\Phi e\|_{\mathsf{H}}^2 \;\le\; \|e\|_{\mathsf{H}}^2 \;=\; \mathcal{R}_\Sigma(s, p).$$

This is exactly (5). In the Teichmüller class, the move operator is realized by compositions of T's; unitarity of T and the pentagon identity ensure the boundary map is a (projective) isometry, hence admissible with $\|\Phi\| = 1$. The same argument applies to $1 \leftrightarrow 4$ via unit/counit blocks (capping/uncapping), which are also isometric on H_Σ. ☐

Corollary 4 (Triangulation independence up to nonincrease). *Let* $T \rightsquigarrow T^{(n)}$ *be any finite sequence of Pachner moves along* Σ. *Under the hypotheses of Theorem 5,*

$$\mathcal{R}_{\text{glue}}\big(T^{(n)}, \Sigma\big) \;\le\; \mathcal{R}_{\text{glue}}(T, \Sigma) \quad \textit{for all } n.$$

3.1. Strengthenings, Equality Cases, and Strictness

Proposition 8 (Equality characterization). *In Theorem 5 one has equality* $\mathcal{R}_\Sigma(\tilde{\Phi}s, \Phi p) = \mathcal{R}_\Sigma(s, p)$ *iff the mismatch* $e := p - \mathcal{I}_\Sigma s$ *lies in the* isometric invariant subspace $\mathsf{Fix}(\Phi) := \{x \in \mathsf{H}_\Sigma : \|\Phi x\| = \|x\|\}$ *and* Φe *is orthogonal to* $\ker(\Phi)$. *In the Teichmüller class (unitary boundary operator) this reduces to* $e \in \mathsf{H}_\Sigma$ *with* $\Phi e = Ue$ *for a unitary* U *determined by the move.*

Proof. $\|\Phi e\| = \|e\|$ is necessary and sufficient for equality in (5); the orthogonality condition eliminates degenerate directions when Φ has nontrivial kernel. For Teichmüller moves, Φ is (projectively) unitary on the boundary, hence equality iff e belongs to an eigenspace of the implementing unitary. ☐

Corollary 5 (Strict decrease off invariants). *If* $\|\Phi\| < 1$ *or if* $e \notin \mathsf{Fix}(\Phi)$ *for a unitary* Φ, *then* $\mathcal{R}_\Sigma(\tilde{\Phi}s, \Phi p) < \mathcal{R}_\Sigma(s, p)$.

3.2. Quantitative and Robust Variants

Definition 4 (Almost–admissible (toleranced) move). *We say* $(\tilde{\Psi}, \Psi)$ *is* (ε, δ)–admissible *if*

$$\|\Psi\mathcal{I}_\Sigma - \mathcal{I}_{\Sigma'}\tilde{\Psi}\|_{\mathsf{H}\leftarrow(\mathsf{S}\oplus\mathsf{P})} \le \varepsilon, \qquad \|\Psi\|_{\mathsf{H}\to\mathsf{H}} \le 1 + \delta, \quad \|\tilde{\Psi}\| \le 1 + \delta.$$

Theorem 6 (Stability under small violations). *If $(\tilde{\Psi}, \Psi)$ is (ε, δ)–admissible, then for any (s, p)*

$$\mathcal{R}_{\Sigma'}(\tilde{\Psi} s, \Psi p) \;\le\; (1+\delta)^2\, \mathcal{R}_{\Sigma}(s, p) \;+\; \varepsilon^2\, \|(s, p)\|^2.$$

In particular, for $\delta, \varepsilon \ll 1$ the violation in monotonicity is second order in the tolerances.

Proof. Write $\Psi p - \mathcal{I}_{\Sigma'} \tilde{\Psi} s \;=\; \Psi(p - \mathcal{I}_{\Sigma} s) \;+\; (\Psi \mathcal{I}_{\Sigma} - \mathcal{I}_{\Sigma'} \tilde{\Psi}) s$. Apply the triangle inequality and $(a + b)^2 \le (1+\eta) a^2 + (1 + 1/\eta) b^2$ with $\eta > 0$, then use the operator–norm bounds. □

Proposition 9 (Angle–refined through–seam bound). *Let $P_{\text{in}}, P_{\text{out}}$ be orthogonal projectors onto Lagrangian polarizations with principal cosine γ. For any contraction K on H_{Σ} and any e decomposed as $e = e_{\text{in}} + e_{\text{out}}$ with $e_{\text{in}} \in \operatorname{ran} P_{\text{in}}$, $e_{\text{out}} \in \operatorname{ran} P_{\text{out}}$,*

$$\|P_{\text{out}} K P_{\text{in}} e_{\text{in}}\| \;\le\; \gamma \, \|e_{\text{in}}\|, \qquad (1-\gamma)\big(\|e_{\text{in}}\|^2 + \|e_{\text{out}}\|^2\big) \le \|e\|^2 \le (1+\gamma)\big(\|e_{\text{in}}\|^2 + \|e_{\text{out}}\|^2\big).$$

Proof. The first bound follows from $\|AB\| \le \|A\|\|B\|$ and $\|P_{\text{out}} P_{\text{in}}\| = \gamma$. The two–sided inequality is the classical principal–angle estimate (cf. Friedrichs angle). □

3.3. Compositions, Randomization, and Certification

Theorem 7 (Composition law and supermartingale property). *Let $\{(\tilde{\Phi}_k, \Phi_k)\}_{k \ge 1}$ be admissible (possibly random) with $\mathbb{E}[\|\Phi_k\|^2 \mid \mathcal{F}_{k-1}] \le 1$ almost surely. Set $E_k := \mathcal{R}(\tilde{\Phi}_k \cdots \tilde{\Phi}_1 s,\; \Phi_k \cdots \Phi_1 p)$. Then $(E_k)_{k \ge 0}$ is a supermartingale: $\mathbb{E}[E_k \mid \mathcal{F}_{k-1}] \le E_{k-1}$. Consequently, E_k converges almost surely and in L^1 to a limit E_∞ with $\mathbb{E} E_\infty \le E_0$.*

Proof. Condition on $\mathcal{F}_{k-1}$ and apply Theorem 1 in expectation together with $\mathbb{E}\|\Phi_k e\|^2 \le \mathbb{E}\|\Phi_k\|^2 \|e\|^2 \le \|e\|^2$. Doob's convergence theorem yields the claim. □

Corollary 6 (Move–by–move certificate). *Given a finite sequence of admissible moves, the log–residual profile $k \mapsto \log E_k$ is nonincreasing. If, in addition, each $\|\Phi_k\| \le \rho < 1$, then $E_k \le \rho^{2k} E_0$ and a linear fit of $\log E_k$ vs. k has slope bounded above by $2 \log \rho$.*

3.4. Consequences for Exact Recursion and DSFL Envelopes

Theorem 8 (Monotone decay in exact recursion). *Let $A_{g+1} = \mathcal{K}_g(A_g)$ be an exact recursion with each $\mathcal{K}_g$ admissible (intertwining, $\|\mathcal{K}_g\| \le 1$). Then $\mathcal{R}(A_{g+1}) \le \mathcal{R}(A_g)$ $\forall g$. If $\|\mathcal{K}_g\| \le \rho < 1$ uniformly, then $\mathcal{R}(A_g) \le \rho^{2g} \mathcal{R}(A_0)$.*

Proof. Iterate (4) and use $\|Kx\| \le \rho \|x\|$. □

Theorem 9 (Gluing–stable exponential envelopes (abstract form)). *Suppose a recursion or evaluation scheme admits a coercive DSFL identity on the tail $\mathfrak{R}$, $\frac{d}{dN}\|\mathfrak{R}\|^2 \le -2\alpha\|\mathfrak{R}\|^2 + 2\langle \mathfrak{R}, g\rangle$ with $|\langle \mathfrak{R}, g\rangle| \le C\|\mathfrak{R}\|^2$ uniformly. Then for the optimally truncated order N, the remainder obeys $\|\mathfrak{R}_N\| \;\le\; C' e^{-(\alpha - C)N}$. If the computation is decomposed into admissible glued blocks, the rate is stable: $\alpha_{\text{glue}} \ge \min_i \alpha_i$.*

Proof. Grönwall's inequality gives the envelope; admissible composition preserves the minimum coercivity margin. □

3.5. Categorical and Structural Corollaries

Proposition 10 (Naturality square and projective unitarity)**.** *Let $\mathcal{M}$ be a reference modular functor and $\eta_\Sigma := \Pi_{\mathrm{DSFL},\Sigma} \circ \mathbb{Q}_\Sigma$ the DSFL comparison map. For a bordism $M : \Sigma \to \Sigma'$ with admissible boundary kernels,*

$$\begin{tikzcd}
\mathcal{M}(\Sigma) \arrow[r, "\mathcal{M}(M)"] \arrow[d, "\eta_\Sigma"'] & \mathcal{M}(\Sigma') \arrow[d, "\eta_{\Sigma'}"] \\
V(\Sigma) \arrow[r, "\mathcal{R}(M)"'] & V(\Sigma')
\end{tikzcd} \qquad \textit{commutes up to phase,}$$

and the vertical maps are nonexpansive. Hence mapping–class actions are projectively unitary on $V(\Sigma)$.

Proof. Intertwining ensures commutation after applying Π_{DSFL}; pentagon–unitarity supplies the projective unitary action; nonexpansivity follows from admissibility. ☐

Scientific consequences (summary).

- *Gluing robustness:* $\mathcal{R}_{\mathrm{glue}}$ is a Lyapunov functional under Pachner moves (Theorem 5); triangle–free simplification pipelines cannot inflate calibrated mismatch.
- *Quantitative interfaces:* principal angles control seam leakage and conditioning (Proposition 9); Davis–Kahan stability (Sec. 2) transfers to robustness of γ and κ.
- *Noisy pipelines:* small violations of admissibility produce controlled slack (Theorem 6), useful for discretized numerics and floating–point implementations.
- *Probabilistic guarantees:* residuals form a supermartingale for randomized admissible updates (Theorem 7), yielding convergence and certification criteria.
- *Recursive reliability:* exact recursion inherits monotone decay and may exhibit exponential envelopes; rates are glue–stable (Theorems 8, 9). [23,24]
- *Categorical packaging:* DSFL provides a natural transformation to a residual modular functor with projective unitarity (Proposition 10).

Remark 1 (Interpretation and scope)**.** *The mechanism behind Theorem 5 is deliberately minimal:*

1. *a* single *comparison Hilbert geometry* H_Σ *in which both channels live and the residual* $\mathcal{R}_\Sigma(s,p) = \|p - \mathcal{I}_\Sigma s\|^2_{\mathsf{H}_\Sigma}$ *is measured;*
2. admissibility, *i.e. boundary/interior maps* intertwine *the calibration* $(\mathcal{I}, \mathcal{J})$ *and are* nonexpansive *in* H_Σ*;*
3. *for the concrete Teichmüller class, two structural identities: the* unitarity *of the edge operator* T *and the* pentagon (five–term) identity*, which together force the gluing operator* G_Σ *to be (projectively) unitary and to intertwine the calibration.*

Because residual monotonicity needs only these three items, we never appeal to delicate asymptotics of special functions: no saddle–point estimates, no growth bounds for Φ_b*—just unitarity and the pentagon relation.*

4. Teichmüller Kernels: Why Admissibility Holds

We recall the standard modular–double representation of the Heisenberg pair and the edge operator, state the pentagon identity at the operator level, and prove admissibility of Pachner moves.

4.1. Operators and Identities

Heisenberg pair and modular double.

Let $(\hat{q}, \hat{p})$ act self–adjointly on $L^2(\mathbb{R})$ with $[\hat{q}, \hat{p}] = \frac{1}{2\pi i}\,\mathrm{id}$, and write $e^{2\pi b\hat{q}}$, $e^{2\pi b\hat{p}}$ for the Weyl generators. The *modular double* uses both b and b^{-1}, $b \in \mathbb{R}_{>0}$ (or on the unit circle), to ensure self–duality. The Hilbert space for a collection of edges $E(\Sigma)$ is $\mathsf{H}_\Sigma \cong L^2(\mathbb{R}^{E(\Sigma)})$; the copies of $(\hat{q}_e, \hat{p}_e)$ act on each factor.

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Faddeev quantum dilogarithm and edge operator.

Let Φ_b denote Faddeev's quantum dilogarithm; on $L^2(\mathbb{R})$ one defines the (unitary) edge operator

$$\mathsf{T} \;:=\; \Phi_b(\hat{p})\, e^{i\pi\hat{q}^2}\,, \qquad \mathsf{T}^*\mathsf{T} = \mathsf{T}\mathsf{T}^* = \mathrm{id}. \tag{6}$$

On tensor factors we use the leg notation, e.g. T_{12} acts nontrivially on factors $1, 2$. The fundamental *pentagon identity* is the operator equality

$$\mathsf{T}_{12}\,\mathsf{T}_{13}\,\mathsf{T}_{23} \;=\; \mathsf{T}_{23}\,\mathsf{T}_{12}\,, \tag{7}$$

encoding the $2 \leftrightarrow 3$ Pachner move at the level of edge operators.

Lemma 6 (Unitarity and pentagon)**.** *For $b \in \mathbb{R}_{>0}$ (or $|b| = 1$) the operators $\Phi_b(\hat{p})$ and $e^{i\pi\hat{q}^2}$ are unitary on $L^2(\mathbb{R})$, hence T in (6) is unitary; moreover (7) holds as a strong operator identity on the natural domain core.*

Sketch. Unitarity of Φ_b and the operator Gaussian is standard in the Plancherel representation for the modular double. The pentagon identity is a well–known consequence of the integral kernel identity for Φ_b and the Weyl relations; see any of the standard references on Teichmüller TQFT. As we never leave the von Neumann algebra generated by the Weyl operators, the common Schwartz core suffices to justify strong operator equalities. ☐

4.2. Isometry and Intertwining for Pachner Moves

We now prove the two admissibility properties required by (2): *(i)* boundary isometry (nonexpansion) and *(ii)* calibration intertwining.

Proposition 11 (Isometry of the boundary gluing operator)**.** *Let T be as above. Every $2 \leftrightarrow 3$ Pachner move along Σ is implemented on H_Σ by replacing a product $\mathsf{T}_{12}\mathsf{T}_{13}\mathsf{T}_{23}$ by $\mathsf{T}_{23}\mathsf{T}_{12}$ (or the inverse). Hence the induced boundary operator G_Σ is (projectively) unitary on H_Σ, i.e. $\|\mathsf{G}_\Sigma x\|_{\mathsf{H}_\Sigma} = \|x\|_{\mathsf{H}_\Sigma}$ for all x and up to a central phase depending on the normalization of measures/framing.*

Proof. By Lemma 6, each T is unitary and (7) holds. Thus both sides of the move are unitary operators mapping $\mathsf{H}_\Sigma \to \mathsf{H}_\Sigma$. Any central phase coming from product ordering or measure normalization multiplies the operator by a scalar on the unit circle, which does not affect norms. Therefore G_Σ is unitary up to phase, i.e. an isometry on H_Σ. ☐

Proposition 12 (Intertwining of the calibration)**.** *Let $\mathcal{I}_\Sigma$ be the boundary restriction of the interior state–integral transform (the comparison map), and let $\mathcal{J}_\Sigma$ be the H_Σ–orthogonal conditional expectation onto the pointer algebra of a chosen polarization. Then the Pachner move operator G_Σ satisfies*

$$\mathsf{G}_\Sigma\,\mathcal{I}_{before} = \mathcal{I}_{after}, \qquad \mathcal{J}_{after} \circ \mathsf{G}_\Sigma = \mathsf{G}_\Sigma \circ \mathcal{J}_{before}. \tag{8}$$

Proof. *First identity:* At the kernel level, a boundary amplitude is obtained by convolving interior tetrahedral kernels and restricting to the boundary variables. The $2 \leftrightarrow 3$ move replaces a triple of kernels by a double via (7). Because both sides of (7) represent the same integral transform on the shared boundary variables, restricting to the boundary after the move equals first restricting and then applying G_Σ: $\mathsf{G}_\Sigma\,\mathcal{I}_{\mathrm{before}} = \mathcal{I}_{\mathrm{after}}$.

Second identity: The pointer algebra for a polarization is generated by commuting functions of either a position–type or momentum–type family of boundary coordinates (shear/Fenchel–Nielsen charts), i.e. a maximal abelian von Neumann algebra. For any unitary U implementing a change of presentation at the boundary, the orthogonal conditional expectation E onto such a MASA satisfies $E \circ \mathrm{Ad}_U = \mathrm{Ad}_U \circ E$ precisely when U normalizes the MASA or when we pass to its image MASA under U (change of chart). In our situation G_Σ sends one triangulation chart to another and therefore maps the pointer algebra to the pointer algebra of the new chart; orthogonal projection (being characterized by minimization in H_Σ) commutes with this unitary conjugation, giving the second equality in (8). □

Corollary 7 (Admissibility of Pachner moves)**.** *With $\mathcal{I}, \mathcal{J}$ as above, the pair $(\tilde{\Phi}, \Phi) := (\mathsf{G}_\Sigma, \mathsf{G}_\Sigma)$ implementing a single $2 \leftrightarrow 3$ (or $1 \leftrightarrow 4$) move is* admissible *in the sense of* (2)*:*

$$\Phi\, \mathcal{I}_{before} = \mathcal{I}_{after}, \qquad \mathcal{J}_{after}\Phi = \Phi\mathcal{J}_{before}, \qquad \|\Phi\|_{\mathsf{H}\to\mathsf{H}} = 1.$$

Proof. Combine Propositions 11 and 12. □

4.3. Consequences and Refinements

Equality and strictness.

By the proof of Theorem 5, equality $\mathcal{R}(\tilde{\Phi}s, \Phi p) = \mathcal{R}(s, p)$ holds iff the mismatch $e := p - \mathcal{I}s$ lies in the isometric invariant subspace of $\Phi = \mathsf{G}_\Sigma$. For generic e and for any move with $\|\Phi\| < 1$ (e.g. with admissible damping), the inequality is strict.

Projective phases and anomaly line.

The projective ambiguity in G_Σ (a central phase) does not affect admissibility or residuals. Such phases organize into a flat line bundle over auxiliary choices (framing/Teichmüller parameters), i.e. an *anomaly line*. All results above are phase–insensitive.

Robustness to discretization/regularization.

If a numerical implementation replaces G_Σ by a near–isometry Ψ and the intertwining identities by (ε, δ)–violations (Definition 4), the stability bound of Theorem 6 shows that residual monotonicity degrades only quadratically in (ε, δ).

Caps/cups and $1 \leftrightarrow 4$.

Unit/counit blocks (adding/removing a tetrahedron) are realized by partial isometries obtained from T by fixing or integrating out one leg. Their boundary action is again an isometry on H_Σ and intertwines the calibration, hence they are admissible and satisfy the same no–inflation bounds.

Seam conditioning via principal angles.

When a move changes the boundary polarization (e.g. shear ↔ FN), the principal cosine $\gamma = \|P_{\text{out}} P_{\text{in}}\|$ controls the *through–seam* operator norm $\|P_{\text{out}} \mathsf{G}_\Sigma P_{\text{in}}\| \le \gamma$ (Proposition 9), yielding quantitative uncertainty/conditioning bounds for polarization changes and hence for numerical stability. [25]

No special–function estimates.

All arguments above rely solely on operator unitarity and the pentagon identity—both algebraic/representation–theoretic facts of the Teichmüller kernel class—and on basic properties of orthogonal conditional expectations. No asymptotic analysis of Φ_b is required.

Bottom line. In the Teichmüller state–integral realization, *every* Pachner move is an admissible, isometric, calibration–intertwining map on the boundary Hilbert space. Therefore the DSFL residual is a bona fide Lyapunov functional for cutting & gluing pipelines, with quantitative robustness under polarization changes and numerical perturbations.

Detailed proof and consequences. We make precise the two commuting statements in the idea and record their implications.

(A) Boundary restriction commutes with pentagon rearrangements. Let $\mathcal{K}_t$ denote the (tempered) integral kernel of a single tetrahedral block built from Faddeev's quantum dilogarithm Φ_b [6,7], and let $\star$ denote

the boundary convolution/integration prescribed by the state–integral gluing rule in the Teichmüller TQFT models [3,4,10,26,27]. A 2↔3 move replaces

$$\mathcal{K}_{t_1}\star\mathcal{K}_{t_2}\star\mathcal{K}_{t_3}\quad\longleftrightarrow\quad\mathcal{K}_{t_1'}\star\mathcal{K}_{t_2'}$$

with the five–term (pentagon) identity holding at the *operator* level [4,6,7]. Denote by $\mathcal{R}_\Sigma$ the boundary restriction (partial integration) to the variables living on the seam Σ. Under standard temperateness assumptions on the kernels (ensuring Fubini/Tonelli applies) one has

$$\mathcal{R}_\Sigma\big[\mathcal{K}_{t_1}\star\mathcal{K}_{t_2}\star\mathcal{K}_{t_3}\big]\;=\;\big(\mathcal{R}_\Sigma\mathcal{K}_{t_1}\big)\star\big(\mathcal{R}_\Sigma\mathcal{K}_{t_2}\big)\star\big(\mathcal{R}_\Sigma\mathcal{K}_{t_3}\big),$$

and the same for the right–hand side. Therefore boundary restriction commutes with the pentagon rearrangement; in operator notation this is exactly $\mathsf{G}_\Sigma\,\mathcal{I}_{\text{before}}=\mathcal{I}_{\text{after}}$.

(B) Orthogonal conditional expectations commute with conjugation of MASAs. Let $\mathcal{N}\subset\mathcal{B}(\mathsf{H}_\Sigma)$ be a MASA (pointer algebra) associated to a boundary polarization (e.g. shear or Fenchel–Nielsen); let $E_\mathcal{N}:\mathcal{B}(\mathsf{H}_\Sigma)\to\mathcal{N}$ be the *orthogonal* conditional expectation for the Hilbert–Schmidt inner product. If U is unitary and $U\mathcal{N}U^*=\mathcal{N}'$ is the MASA for the new polarization, then

$$E_{\mathcal{N}'}\big(UXU^*\big)\;=\;U\,E_\mathcal{N}(X)\,U^*\qquad\big(\forall\,X\in\mathcal{B}(\mathsf{H}_\Sigma)\big).\tag{9}$$

Indeed, (i) $E_{\mathcal{N}'}$ is characterized by orthogonality: $\langle X-E_{\mathcal{N}'}X,\,Y\rangle_{\text{HS}}=0$ for all $Y\in\mathcal{N}'$; (ii) for $Z\in\mathcal{N}$ and $Y'=UZU^*\in\mathcal{N}'$,

$$\langle UXU^*-UE_\mathcal{N}(X)U^*,\,Y'\rangle_{\text{HS}}\;=\;\langle X-E_\mathcal{N}(X),\,Z\rangle_{\text{HS}}\;=\;0,$$

hence $UE_\mathcal{N}(X)U^*$ is the unique $\mathcal{N}'$–orthogonal projection of UXU^*, proving (9). Taking X to be the boundary density/observable associated to the amplitude yields $\mathcal{J}_{\text{after}}\circ\text{Ad}_U=\text{Ad}_U\circ\mathcal{J}_{\text{before}}$.

Combining (A) and (B) with unitarity of the pentagon implementer $U=\mathsf{G}_\Sigma$ in the Teichmüller class [3,6,7] gives admissibility: $\Phi\,\mathcal{I}_{\text{before}}=\mathcal{I}_{\text{after}}$ and $\mathcal{J}_{\text{after}}\Phi=\Phi\mathcal{J}_{\text{before}}$ with $\|\Phi\|=1$. The no–inflation inequality then follows from a single H–norm estimate.

Consequences.

- **Axioms ⇒ Theorems.** In Teichmüller TQFT, gluing stability becomes a *Theorem* (unitarity + pentagon + calibration) rather than an axiom.
- **Chart–independence of certification.** Because $E_{\mathcal{N}'}\circ\text{Ad}_U=\text{Ad}_U\circ E_\mathcal{N}$, residual certification (*no inflation*) is invariant under unitary changes of boundary charts (metaplectic transforms).
- **Numerical robustness.** The proof is purely operator–theoretic; it carries over to finite–dimensional discretizations once we enforce (approximate) intertwining and spectral–norm ≤ 1 by construction (see next section).

□

5. Minimal Verification Harness (Reproducible Tests)

Aim.

Given discretized boundary maps we *certify* (5) empirically without ever evaluating oscillatory special functions. The harness isolates two properties: (i) *intertwining* $B_{\text{out}}.I\;T_{\text{stat}}=T_{\text{phys}}\;B_{\text{in}}.I$ and (ii) *nonexpansion* $\|T_{\text{phys}}\|_2\le 1$. Both are enforced by projection (*spectral clipping*) and checked by property–based sampling.

A. Repository Skeleton (Annotated)

```
residual-tqft/
  README.md
  pyproject.toml
```

```
  residual_tqft/
    boundary.py         # Boundary spaces, calibration (I,J), DSFL residual; principal angles
    kernels.py          # Mock admissible kernels; norm-clipping; intertwining projection
    moves.py            # 2<->3 and 1<->4 compositions on a seam (as operator words)
    tests/
      test_pachner.py   # Monotonicity tests (property-based sampling + confidence bounds)
      test_angles.py    # Principal-angle (Friedrichs) bounds and conditioning checks
  examples/
    fig8_move.ipynb     # Figure-eight triangulation move check (mock operators)
    run_moves.py        # CLI: check-pachner --samples N --tol EPS --seed S
```

B. Core API (Pythonic Pseudocode with Enforcement)

```
class Boundary:
    def __init__(self, S_dim, P_dim, I, J):
        self.S_dim, self.P_dim = S_dim, P_dim
        self.I = I      # shape (P_dim, S_dim)
        self.J = J      # shape (S_dim, P_dim)
    def rsameness(self, s, p):
        e = p - self.I @ s
        return float(e.T @ e)

def spectral_clip(M):
    # Return M / max(1, ||M||_2) to enforce ||.||_2 <= 1 (power iteration for ||.||_2)
    sigma = op_norm(M)
    return M / max(1.0, sigma)

def intertwine_project(Bin, Bout, T_stat_guess, T_phys_guess):
    # Enforce Bout.I @ T_stat = T_phys @ Bin.I in least-squares sense
    # Solve for T_stat, T_phys minimizing ||Bout.I @ T_stat - T_phys @ Bin.I||_F
    # then clip both to spectral norm <= 1
    T_stat, T_phys = solve_coupled_least_squares(Bin, Bout, T_stat_guess, T_phys_guess)
    return spectral_clip(T_stat), spectral_clip(T_phys)

class Kernel:
    def __init__(self, B_in, B_out, T_stat, T_phys):
        # Validate intertwining
        assert np.allclose(B_out.I @ T_stat, T_phys @ B_in.I, atol=1e-12)
        # Validate nonexpansion
        assert op_norm(T_phys) <= 1 + 1e-12
        self.B_in, self.B_out = B_in, B_out
        self.T_stat, self.T_phys = T_stat, T_phys
    def push(self, s, p):
        return self.T_stat @ s, self.T_phys @ p

def pachner_move_2to3(B):
    # Compose admissible blocks to emulate the 2->3 rearrangement on the seam
    T_stat, T_phys = build_operator_word_for_2to3()
    T_stat, T_phys = intertwine_project(B, B, T_stat, T_phys)
    return Kernel(B, B, T_stat, T_phys)

def check_monotonicity(B, move, samples=10000, tol=1e-10, seed=0):
    rng = np.random.default_rng(seed)
    K = move(B)
    worst = -np.inf; ok = True
    for _ in range(samples):
```

```
        s = rand_unit(rng, B.S_dim); p = rand_unit(rng, B.P_dim)
        R0 = B.rsameness(s,p)
        s1,p1 = K.push(s,p)
        R1 = B.rsameness(s1,p1)
        worst = max(worst, R1 - R0)
        if R1 > R0 + tol: ok=False
    return ok, worst
```

C. Acceptance Criterion and Statistical Confidence

A run *passes* if `ok=True` and the maximal defect $\delta_{\max} = \max(R_{\text{after}} - R_{\text{before}}) \le \varepsilon_{\text{tol}}$ across N independent samples. To report confidence, model the indicator $Z_i = \mathbf{1}\{R_{1,i} > R_{0,i} + \varepsilon_{\text{tol}}\}$ $(i = 1, \dots, N)$. If $\sum_i Z_i = 0$, Hoeffding gives, for any $\eta \in (0, 1)$,

$$\mathbb{P}(\mathbb{E}[Z_1] \ge \eta) \le e^{-2N\eta^2}.$$

Thus with $N = 10^4$ and $\eta = 10^{-2}$, observing no violations certifies that the true violation rate is $< 1\%$ with probability at least $1 - e^{-200} \approx 1$; larger N pushes the bound lower. (If some violations occur, report $\widehat{p} = \frac{1}{N}\sum Z_i$ with an exact binomial CI.)

Numerical stability notes.

- *Intertwining enforcement.* Use `intertwine_project` once per move instance to project a guessed pair $(T_{\text{stat}}, T_{\text{phys}})$ onto the affine subspace $\{(X, Y) : B_{\text{out}}.I\, X = Y\, B_{\text{in}}.I\}$, then clip both by `spectral_clip`.
- *Operator norm.* Estimate $\|T_{\text{phys}}\|_2$ via the power method with a stringent stopping tolerance (e.g. 10^{-12}); add a safety factor in the assertion.
- *Principal angles (optional).* Provide `principal_cosine(Pin,Pout)` via SVD of $P_{\text{in}} P_{\text{out}}|_{\text{ran}(P_{\text{out}})}$ to log the conditioning proxy γ.

D. What the harness proves *in practice*

Under exact admissibility, Theorem 5 guarantees $R_{\text{after}} \le R_{\text{before}}$ pointwise. Under discretization, the harness verifies the (ε, δ)–admissible variant (Thm. 6) by construction—intertwining error ε and norm inflation δ are made *as small as numerically attainable* by projection and clipping, and any remaining slack is detected in the empirical certificate.

Remark 2 (Referencing the kernel facts)**.** *The only nontrivial analytic inputs we used are (i) the unitarity of the edge operator in the modular–double Plancherel representation and (ii) the operator pentagon identity, both classical for the Faddeev quantum dilogarithm [3,4,6,7,27]. The operator–algebraic identity* (9) *is standard for orthogonal conditional expectations onto MASAs (see, e.g., Kadison–Ringrose).*

6. Afterword

The results above isolate what is *universal*: once every local gluing step acts as an *intertwining* contraction in a single comparison Hilbert geometry, the DSFL residual (1) is a bona fide Lyapunov functional for cutting & gluing. In the Teichmüller state–integral class, this hypothesis is realized by two structural facts—unitarity of the edge operator and the pentagon identity—so no special–function asymptotics are needed. We conclude by making this universality precise, recording robustness and limits, and sketching scientific consequences.

6.1. Universality Principle and a Converse

Theorem 10 (Universality of residual Lyapunov law)**.** *Let* H_Σ *be a fixed comparison Hilbert space and* $(\mathcal{I}_\Sigma, \mathcal{J}_\Sigma)$ *a calibration. Suppose every elementary gluing step along any seam* Σ *is implemented by a linear pair*

$(\tilde{\Phi}, \Phi)$ satisfying the admissibility conditions (2). Then for any composite gluing pipeline $\mathsf{G} = \prod_k(\tilde{\Phi}_k, \Phi_k)$ and any boundary data (s,p) one has the chain of inequalities

$$\mathcal{R}(\tilde{\Phi}_n \cdots \tilde{\Phi}_1 s,\ \Phi_n \cdots \Phi_1 p) \ \le\ \cdots\ \le\ \mathcal{R}(\tilde{\Phi}_1 s, \Phi_1 p)\ \le\ \mathcal{R}(s,p),$$

i.e. $\mathcal{R}$ is Lyapunov along the pipeline, independent of triangulation choices.

Proof. Iterate the data–processing inequality (Theorem 1). No additional structure is used. ☐

Theorem 11 (A converse: Lyapunov ⇒ admissibility)**.** *Let $(\tilde{\Psi}, \Psi)$ be a bounded pair such that $\mathcal{R}(\tilde{\Psi}s, \Psi p) \le \mathcal{R}(s,p)$ for* all *(s,p). Then necessarily:*

1. *Ψ is nonexpansive on the* residual directions*: $\|\Psi e\| \le \|e\|$ for all e;*
2. *Intertwining holds: $\Psi\,\mathcal{I}_\Sigma = \mathcal{I}_{\Sigma'}\,\tilde{\Psi}$.*

If, moreover, $\mathcal{I}_\Sigma$ has dense range and $\mathcal{J}_\Sigma$ is the orthogonal projector dual to $\mathcal{I}_\Sigma$ ($\mathcal{I}\mathcal{J} = \mathrm{id}_{\mathsf{P}}$, $\mathcal{J}\mathcal{I} = P_{\mathsf{S}}$), then $\|\Psi\| \le 1$ on H.

Proof. Setting $p = \mathcal{I}s + e$ gives $\mathcal{R}(\tilde{\Psi}s, \Psi p) = \|\Psi e\|^2$ and $\mathcal{R}(s,p) = \|e\|^2$, hence $\|\Psi e\| \le \|e\|$ for all e. Taking $e = 0$ forces $\Psi\mathcal{I}s = \mathcal{I}\tilde{\Psi}s$ for all s, i.e. intertwining. If $\operatorname{ran}\mathcal{I}$ is dense, any $x \in \mathsf{H}$ is a limit of residual directions $e_n = p_n - \mathcal{I}s_n$, hence $\|\Psi x\| \le \|x\|$ by continuity. ☐

Remark 3 (Equivalence class viewpoint)**.** *Theorems 10–11 show that the Lyapunov property is* equivalent *to admissibility, once the calibration is fixed. Thus the DSFL residual upgrades "axioms of sewing" to an* if and only if *criterion:* gluing is stable precisely when it is an intertwining contraction in the comparison geometry.

6.2. Robustness, Limits, and Failure Modes

Small violations.

Section 3 (Thm. 6) shows (ε,δ)–admissible steps produce at most quadratic slack. Hence discretizations that enforce intertwining up to ε and clip spectral norms at $1+\delta$ inherit a *controlled* near–Lyapunov property.

Failure modes.

If either condition in (2) fails badly, monotonicity can break:

- *Nonintertwining.* If $\Psi\mathcal{I} - \mathcal{I}\tilde{\Psi}$ has large operator norm, the residual can inflate by $O(\|\Psi\mathcal{I} - \mathcal{I}\tilde{\Psi}\|^2\|s\|^2)$ even when $\|\Psi\| \le 1$.
- *Expansion.* If $\|\Psi\| > 1$, then directions e in the top singular subspace inflate the residual by $\sigma_{\max}(\Psi)^2$.

Both effects are observable and certifiable by the verification harness (§5).

6.3. Categorical Consequences and Anomaly Bookkeeping

Proposition 13 (Residual naturality and anomaly line)**.** *Let $\eta_\Sigma := \Pi_{\mathrm{DSFL},\Sigma} \circ \mathbb{Q}_\Sigma$ be the comparison map from a reference modular functor $\mathcal{M}$ to the residual functor $\mathcal{R}$. For any bordism $M : \Sigma \to \Sigma'$ implemented by admissible kernels, the naturality square commutes up to a central phase:*

$$\eta_{\Sigma'} \circ \mathcal{M}(M) \ =\ e^{i\vartheta(M)}\,\mathcal{R}(M) \circ \eta_\Sigma, \qquad \vartheta : \mathbf{Cob} \to \mathbb{R}/2\pi\mathbb{Z}.$$

The phases assemble into a flat line bundle (the anomaly line). All DSFL inequalities are phase–insensitive.

Proof. Intertwining gives strict commutation before projecting; the DSFL projector is idempotent and bounded; phases multiply both sides by a unit scalar. ☐

6.4. Algorithmic and Numerical Consequences

Certification by supermartingales.

For randomized pipelines $(\tilde{\Phi}_k, \Phi_k)$ with $\mathbb{E}[\|\Phi_k\|^2 | \mathcal{F}_{k-1}] \leq 1$, the residual sequence $E_k = \mathcal{R}(\tilde{\Phi}_k \cdots \tilde{\Phi}_1 s, \Phi_k \cdots \Phi_1 p)$ is a supermartingale (Thm. 7). This yields:

- almost–sure convergence of E_k;
- tail bounds for empirical violation rates via Hoeffding/Bernstein, enabling reproducible pass/fail certificates;
- linear semi–log decay when the mean contraction is uniform ($\mathbb{E}\|\Phi_k\|^2 \leq \rho^2 < 1$).

Seam conditioning.

Principal angles between polarizations bound through–seam operator norms by $\|P_{\text{out}} K P_{\text{in}}\| \leq \gamma = \cos\theta_F$ (Prop. 9), giving: (i) sharp two–sided uncertainty for residual splits; (ii) a conditioning number $\kappa(\theta_F) = \frac{1+\gamma}{1-\gamma}$ to guide chart choices in numerics.

6.5. Teichmüller Specialization: Why It Works "for Free"

In the Teichmüller class, admissibility is guaranteed "from the start": [14,28–30]

- **Unitarity.** The edge operator T (built from $\Phi_b(\hat{p})$ and the metaplectic Gaussian) is unitary on the modular–double Plancherel space.
- **Pentagon identity.** The $2 \leftrightarrow 3$ move holds as a strong operator identity (no approximation).
- **Pointer compatibility.** Orthogonal conditional expectations onto MASAs commute with unitary conjugations of those MASAs (change of polarization).

Therefore every elementary move is an *exact* intertwining isometry on H_Σ, and all of the above consequences apply verbatim.

6.6. Outlook: Envelopes, Modularity, and Testable Conjectures

DSFL envelopes and resurgence.

When a coercive DSFL identity is available for tails $\mathfrak{R}$ (e.g. along steepest–descent thimbles), Grönwall yields exponential envelopes $\|\mathfrak{R}_N\| \leq C' e^{-(\alpha - C)N}$ (Thm. 9). Because admissible gluing preserves the minimum coercivity margin, *rates are glue–stable*: $\alpha_{\text{glue}} \geq \min_i \alpha_i$.

Quantum modularity link (program).

In state–integral models for hyperbolic 3–manifolds, the nearest Borel singularity is the smallest real part among relative Chern–Simons actions. It is therefore natural to conjecture (and verify case by case) that the DSFL envelope rate equals that real part, and that it is preserved under JSJ gluing. This identifies a concrete bridge from DSFL Lyapunov rates to quantum modularity exponents.

Practical takeaway.

From the DSFL vantage point, *topology is what survives the flow*: admissibility collapses metric minutiae and forces residual monotonicity. This gives a unifying, verifiable stability layer for cutting & gluing pipelines, exact recursion, and categorical sewing—together with quantitative, phase–insensitive error bars and robust numerical certification.

Author's Note.

This paper develops a sector-neutral Lyapunov–residual framework—the Deterministic *Statistical* Feedback Law (DSFL)—aimed at recovering standard equilibrium relations as dynamical attractors under explicit hypothesis gates. The intention is not to replace established formalisms, but to clarify their restoration mechanisms by isolating a minimal quadratic residual and its propagation–gap structure.

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Appendix A. Notation

Table A1. Symbols and conventions used throughout. The comparison Hilbert space is $(\mathsf{H}, \langle\cdot,\cdot\rangle)$ with norm $\|x\|_{\mathsf{H}} := \sqrt{\langle x, x\rangle}$.

Symbol	Type / Domain	Meaning / Assumptions
Spaces and geometry		
H	Hilbert space	Comparison geometry for both channels; inner product $\langle x, y\rangle$, norm $\|x\|_{\mathsf{H}} = \sqrt{\langle x, x\rangle}$.
S	Linear space	Statistical channel space (e.g., vacuum/constraint objects).
$\mathsf{P} \subset \mathsf{H}$	Closed subspace	Physical channel space (e.g., observables/fields inside H).
$P_{\mathsf{S}} : \mathsf{S} \to \mathsf{S}$	Projector	Metric projection onto the admissible statistical subspace; encodes statistical gauge.
Channels and maps		
$s \in \mathsf{S}$	State (stat.)	Statistical channel. In one-budget model: $s(x) = w(x)s_0$, $\int_V w = 1, w \geq 0$.
$p \in \mathsf{P}$	State (phys.)	Physical channel.
$\mathcal{I} : \mathsf{S} \to \mathsf{P}$	Linear map	*Interchangeability* (calibration/embedding) of s into P.
$\mathcal{J} : \mathsf{P} \to \mathsf{S}$	Linear map	Statistical representative of p; satisfies $\mathcal{J} \circ \mathcal{I} = P_{\mathsf{S}}$.
$C : \mathsf{S} \to \mathsf{P}$	Linear map	Calibration operator (units/indices/gauge); often $C \equiv \mathcal{I}$.
Interchangeability identities		
$\mathcal{I} \circ \mathcal{J} = \mathrm{id}_{\mathsf{P}}$	Identity	Pushing p to S then back gives p.
$\mathcal{J} \circ \mathcal{I} = P_{\mathsf{S}}$	Identity	Pushing s to P then back gives the *projected* s.
Residuals (mismatch measures)		
$\mathcal{R}_{\mathrm{phys}}(s, p)$	Scalar	Physical-side residual: $\| p - \mathcal{I}(s) \|_{\mathsf{H}}^2$.
$\mathcal{R}_{\mathrm{stat}}(s, p)$	Scalar	Statistical-side residual: $\| s - \mathcal{J}(p) \|_{\mathsf{S}}^2$.
$R_{\mathrm{sameness}}(s, p)$	Scalar	Canonical residual $\| p - Cs \|_{\mathsf{H}}^2$ (often $C = \mathcal{I}$).
$R^{(D)}_{\mathrm{sameness}}$	Scalar	Differential residual $\| Dp + D(Cs) \|_{\mathsf{H}}^2$ (e.g., $D = \nabla, \nabla^2$).
Propagation and DSFL parameters (optional, when dynamics are used)		
$e := p - Cs$	Element of P	Residual vector in H.
$K = K^* \succeq 0$	Operator on P	Dissipative/elliptic part (Dirichlet/Lichnerowicz/constitutive).
g	Element of P	Controlled remainder (lower orders, background drift).
$\kappa > 0$	Scalar	Gap/coercivity constant: $\langle Ke, e\rangle \geq \kappa\|e\|^2$.
$\varepsilon \geq 0$	Scalar	Remainder bound: $|\langle e, g\rangle| \leq \varepsilon\|e\|^2$.
$\alpha = 2\kappa - 2\varepsilon$	Scalar	DSFL rate in $\dot{R} \leq -\alpha R$ (when dynamics are present).

Table A1. *Cont.*

Symbol	Type / Domain	Meaning / Assumptions
Angles and subspace geometry		
$U = \mathsf{P},\ V = \overline{\operatorname{ran}\mathcal{I}}$	Subspaces of H	Physical subspace and calibrated statistical range.
P_U, P_V	Projectors	Orthogonal projectors onto U and V.
$\theta_F \in [0, \pi/2]$	Angle	Friedrichs angle: $\lVert P_U P_V \rVert = \cos\theta_F$.
Q_U, Q_V	Matrices/bases	Orthonormal bases spanning U and V; CS/SVD: $Q_U^* Q_V = W\Sigma Z^*, \Sigma = \operatorname{diag}(\cos\theta_k)$.
Admissible ("entanglement-like") redistribution		
$\tilde{\Phi} : \mathsf{S} \to \mathsf{S}$	Linear map	Statistical operation (Markov/coherent/CPTP marginal).
$\Phi : \mathsf{P} \to \mathsf{P}$	Linear map	Physical operation (contractive in H).
Intertwining	Identity	$\Phi \circ C = C \circ \tilde{\Phi},\ \tilde{\Phi} \circ \mathcal{J} = \mathcal{J} \circ \Phi$.
Contractivity	Inequality	$\lVert \Phi x \rVert_{\mathsf{H}} \le \lVert x \rVert_{\mathsf{H}},\ \lVert \tilde{\Phi} y \rVert_{\mathsf{S}} \le \lVert y \rVert_{\mathsf{S}}$.
Residual monotonicity	Inequality	$R_{\mathrm{sameness}}(\tilde{\Phi}s, \Phi p) \le R_{\mathrm{sameness}}(s, p)$.
One-budget (statistical resource) model		
$s_0 \in \mathsf{S}$	Fixed template	Global statistical prototype (primordial sameness), $\lVert s_0 \rVert$ normalized.
$w(x)$	Nonnegative weight	Share field, $\int_V w = 1$; $s(x) = w(x)s_0$.
$K(x,y)$	Kernel	Markov kernel: $K \ge 0$, $\int K(x,y)\,dx = 1$; preserves $\int w = 1$.
Budget/causality constraints		
$\mathfrak{d}(\cdot)$	Counter	Local complexity/effective rank/energy counter; monotone & subadditive.
$v_\star$	Speed	Carrier/relay speed (e.g., wave speed, Lieb–Robinson velocity).
ℓ_{corr}	Length	Correlation diameter/interaction range.
Causal ceiling	Bound	$\dfrac{d}{dt}\mathfrak{d}(p_{U(t)}) \lesssim \kappa \dfrac{v_\star}{\ell_{\mathrm{corr}}}$ for a moving volume $U(t)$.
Sector shorthands (used in mini-cases)		
PDE	—	$u = P - \nabla\rho$, $B \succeq \beta I$, Helmholtz split $u = \nabla\phi + w$, Poincaré λ_1.
OA/QMS	—	$L^2(\omega)$ GNS space; $\mathsf{E}_{\mathcal{N}}$ conditional expectation (orthogonal projector).
OU/free	—	$A = -\Delta + m^2$, covariance Σ_τ, gap $\lambda_* := \inf\sigma(A\vert_{\ker A^\perp})$.
Constants frequently used		
$\beta > 0$	Scalar	Uniform ellipticity margin (PDE).
λ, λ_1	Scalars	Poincaré/spectral constants (domain/semigroup).
λ_*	Scalar	Hamiltonian/spectral gap (OU/free field).
κ, ε	Scalars	Coercivity/remainder (DSFL template).
α	Scalar	Dissipation rate ($\alpha = 2\kappa - 2\varepsilon$ when used dynamically).

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