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The Infinite Field of Rational-Conforming Integer Sets

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ABSTRACT

This paper explores special prime classes, S, such that if q is a prime in S, then there exists rational functions, $\alpha_q = \frac{f(q)}{g(q)}$, such that $S = \{q \in \wp : \exists f, g \in \mathbb{Z}, g(q) \neq 0, \alpha_q \in \mathbb{Z}, \}$ have asymptotic distribution with a prime density coefficient, $C_\wp(\alpha_q, \alpha_p) = \frac{{\alpha_q}^2}{{\alpha_p}^2} \frac{[A(\alpha_p)]^2}{[A(\alpha_q)]^2}$, where, the primes p, Prime classes include Sophie primes. The rational functions, $\frac{f(q)}{g(q)}$, form a prime index class for $\wp$. I demonstrate that the set of primes conforming to a rational mapping $\frac{f(q)}{g(q)} \in \mathbb{Z}$ and the associated cotangent–tangent phase condition $\tan\left(\frac{f(q)}{g(q)}\right)\cot\beta_p^m = 1$ constitutes an infinite field of phase-locked primes, p. Each prime corresponds to a unique angular phase, β_q^m on a cotangent lattice, and the iteration among successive phases defines a self-referential, recursively stable prime ladder. The distribution of phased lock sets of primes is not unique to Sophie primes. Indeed, any sets of integers that have the rational function property $\tan\left(\frac{f(q)}{g(q)}\right)\cot\beta_p^m = 1$, for which the phase is a rational set of values $\frac{f(q)}{g(q)} = constant \in \mathbb{Z}$ including Perfec numbers also follow the successive phases defines a self-referential, recursively stable infinite ladder.

Keywords: Mersenne primes; Sophie Germain Primes; Perfect numbers; trigonometric functions,Pahse locked primes sets, Primes; cotangent; trigonometry; Sums of divisors; invariance. P

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1. Introduction

Ordinary numbers including prime sequences can form an infinite arithmetic sequence. However, not all such numbers contribute equally to field symmetries governed by rational ratios of arithmetic functions. Here I define a subset of primes S_ϕ that satisfy both a rational functional relation and a geometric phase constraint. I show that this subset is infinite and structurally distinct from the classical sequence of progression. The first sensor used to determine this fact is the Gauss Gamma product formula [1]. The second sensor used is cotangent relation as an expanded series separated into two sets. The first being the set of desired primes in the sequence that satisfies

$$\alpha_p = \frac{f(q)}{g(q)} (mod\ \pi) \in \mathbb{Z}, \qquad \text{where } \mathbb{Z} = \frac{f(p)}{g(p)} \in \mathbb{Q} \tag{1.}$$

The second being the integers in the series that do not satisfy the condition (1).

Invariance of phase lock sequences

The product-form of the Γ-function due to Gauss, provides further insights into many relations that will be developed in this paper. The product form is given by, [4, p. 896]:

$$\Gamma(y \cdot n) = (2\pi)^{\frac{1-n}{2}} n^{(n\cdot y)-\frac{1}{2}} \prod_{k=0}^{n-1} \Gamma\left(y + \frac{k}{n}\right) \tag{2.}$$

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LEMMA 1: *[4]. Let* $f_j(z) > 0$, *represent one integer factor of* $g(z)$, *then, if,* $f_j(z)$ *is a number theoretic function, the functional relation:*

$$\Gamma\big(g(z)\big) = (2\pi)^{\frac{1-f_j(z)}{2}}\left(f_j(z)\right)^{g(z)-\frac{1}{2}}\prod_{k=0}^{f_j(z)-1}\Gamma\left(\frac{g(z)+k}{f_j(z)}\right) \quad (3.)$$

is invariant with respect to choices of any other factors of $g(z)$. *Hence* $F(z)$ *is a phase-locked function.*

The significance of the LEMMA 1, is its consequences for prime numbers, and their relations to sets of primes such as the Sophie primes and the Mersenne primes.

PROOF:

Let $\boldsymbol{f_j(z)}$, be some $\boldsymbol{j^{th}}$ integer factor of $\boldsymbol{k}$-factors a real or complex function $\boldsymbol{g(z)}$. Then,

$$g(z) = \prod_{n=1}^{k} f_n(z) = f_j(z)\left(\prod_{n=1}^{j-1} f_n(z)\prod_{n=j+1}^{k} f_n(z)\right) \quad (4.)$$

The Gauss gamma product formula is a simple relation given by:

$$\Gamma(n\cdot y) = (2\pi)^{\frac{1-n}{2}} n^{(n\cdot y)-\frac{1}{2}}\prod_{k=0}^{n-1}\Gamma\left(y+\frac{k}{n}\right) \quad (5.)$$

Then, since $f_j(z)$, is an integer-factor of $g(z)$, we have, putting $n = f_j(z)$, $y = \prod_{n=1}^{j-1} f_n(z)\prod_{n=j+1}^{k} f_n(z)$ in

$g(z) = f_j(z)\left(\prod_{n=1}^{j-1} f_n(z)\prod_{n=j+1}^{k} f_n(z)\right)$. Then,

$$\Gamma\big(g(z)\big) = (2\pi)^{\frac{1-f_j(z)}{2}}\left(f_j(z)\right)^{g(z)-\frac{1}{2}}\prod_{k=0}^{f_j(z)-1}\Gamma\left(\frac{g(z)+k}{f_j(z)}\right) \quad (6.)$$

If there any other integer factor labelled here $f_v(z), \in Z$, then, the substitution $f_j(z) \to f_v(z)$ leaves $\Gamma\big(g(z)\big)$ invariant. However this is also true for any integer factors, m of $f_v(z)$, then, for any m,

$$\Gamma\big(g(z)\big) = (2\pi)^{\frac{1-m}{2}}(m)^{g(z)-\frac{1}{2}}\prod_{k=0}^{m-1}\Gamma\left(\frac{g(z)+k}{m}\right) \quad (7.)$$

remains invariant to the substitutions $f_j(z) \to f_v(z) \to m$.

It is clear that inside the product terms, we have a different set of rules. Hence the Corollary applies in the sense that if $f_j(z)$ is a number theoretic function, then the *invariance applies*. However, when we consider rational functions outside of the product, we find that simple arithmetic operations do apply, hence (7).

Equation (7) can be formed as a series:

$$\log\Gamma\big(g(z)\big) = \left(\frac{1-f_j(z)}{2}\right)\log(2\pi)\left(g(z) - \frac{1}{2}\right)\log\left(f_j(z)\right)\sum_{k=0}^{f_j(z)-1}\log\left(\frac{g(z)+k}{f_j(z)}\right) \quad (8.)$$

By using the well know asymptotyic expansion (Sirling-Bernoulli expansion) we get:

$$\log\Gamma(x) = \left(x-\frac{1}{2}\right)\log(x) - \frac{x}{2}\log(2\pi)\sum_{n=1}^{\infty}\log\left(\frac{B_{2n}}{2n(2n-1)}\right) \quad |x| \to \infty \quad (9.)$$

where, B_{2n} are the Bernoulli numbers.

Hence

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$$\log\Gamma\big(g(z)\big) = \left(\frac{1-f_j(z)}{2}\right)\log(2\pi)\left(g(z)-\frac{1}{2}\right)\log\left(f_j(z)\right) + \sum_{k=0}^{f_j(z)-1}\left[\left(\frac{g(z)+k}{f_j(z)}-\frac{1}{2}\right)\log\left(\frac{g(z)+k}{f_j(z)}\right)-\left(\frac{g(z)+k}{f_j(z)}\right)\frac{1}{2}\log(2\pi)\sum_{n=1}^{\infty}\frac{B_{2n}}{2n(2n-1)}\left(\frac{f_j(z)}{g(z)+k}\right)^{2n-1}\right] \quad (10.)$$

Note also that $\sum_{k=0}^{f_j(z)-1}\left(\frac{f_j(z)}{g(z)+k}\right)^{2n-1}$ is invariant to the labelling of factors $f_j(z)$, since each factor just permutates the same rational fractions.

THEOREM: *Every sequence of primes for which there exists rational functions* $\frac{f(q)}{g(q)}$ *such that*

$$\alpha_p = \frac{f(q)}{g(q)}(mod\ \pi) \in \mathbb{Z}, \qquad \text{where } \mathbb{Z} = \frac{f(p)}{g(p)} \in \mathbb{Q}$$

forms an infinite sequence.

Proof:

2. The Rational Mapping

Let $f, g\colon \mathbb{N} \to \mathbb{R}$ be arithmetic functions for example the divisor sums, Totient, and other multiplicative forms, with $g(x) \neq 0$.

Define

$$\alpha_p = \frac{f(q)}{g(q)}(\text{mod } \pi) \in \mathbb{Z}, \mathbb{Z} = \frac{f(p)}{g(p)} \in \mathbb{Q}$$

The mapping $p \mapsto \mathbb{Z}$, assigns to each prime a rational "phase parameter." The set of conforming primes $p \in \mathbf{P}$ (**primes**) is

$$S_{\mathbb{Z}} = \left\{p \in \mathbf{P}\colon \frac{f(p)}{g(p)} = \mathbb{Z} \in \mathbb{Q}\right\}$$

Define the first phase map (reduced modulo π into the convergence interval)

$$\phi_{q_0} := \beta_{q_0} \equiv \frac{f(q_0)}{g(q_0)}\ (\text{mod } \pi), \beta_{q_0} \in (0, \pi)$$

For a fixed first prime q_0, that defines the first phase β_{q_0}, impose the condition between two successive primes, $\tan(\beta_{q_0})\tan(\beta_{q_1}) = 1$. This guarantees that q_0, and q_1 belong to the same phase-field F_q.The indicator field is

$$1_{F_{q_0}}(q_0) = \begin{cases} 1, & \tan(\beta_{q_0})\cot(\beta_{q_1}) = 1 \\ 0, & \text{otherwise} \end{cases} \quad (11.)$$

Then, construct a first field partition S_{ϕ_1}:

$$S_{\phi_0} = \sum_p 1_{F_{q_0}}(p), S_k = \sum_p \left(1 - 1_{F_{q_0}}(p)\right) \quad (12.)$$

Then, in the general infinite sequence of integers, $n = 1, 2, 3, 4, 5, \ldots$, that satisfy the cotangent relation from [4, p.42, 1.411 (7)] using the first phase, β_{q_1}.

$$\cot(\beta_{q_0}) = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|}{(2n)!} \beta_{q_0}{}^{2n-1} \quad , \qquad [\alpha_q{}^2 < \pi^2] \tag{13.}$$

The expansion of the function (2) can be partitioned using the field:

$$\cot(\alpha_{q_0}) = \frac{1}{2} - S_k - \tan(\beta_{q_0}) S_{\phi_0} \cot(\beta_{q_1}) \tag{14.}$$

$$S_{\phi_0} \cot(\beta_{q_1}) \tan^2(\beta_{q_0}) + \left(S_k - \frac{1}{2}\right) \tan(\beta_{q_0}) + 1 = 0 \tag{15.}$$

The quadratic governs the equilibrium between primes that belong to the field and those that do not belong.

Since the tangent of $\tan(\beta_{q_0})$ is single valued for specified phase condition that governs the selected set of primes, the discrimant must vanish.

Setting the discriminant of (15) to zero gives the resonant relation

$$\left(\left(S_n - \frac{1}{2}\right)\right)^2 = 4S_{\phi_0} \cot(\beta_{q_1}) \tag{16.}$$

If, in addition, $\tan(\beta_{q_0}) = \tan(\beta_{q_1})$, (the resonance occurs at the field phase), and

$$S_n = \frac{1}{2} - 2S_{\phi_0} \cot(\beta_{q_1}) \ , \quad S_{\phi_0} = \cot(\beta_{q_1}) \tag{17.}$$

Hence the in-field amplitude equals the cotangent of the defining first phase-locked prime, q_1, by β_{q_1}. This leads to a recursive phenomenon.

3. Recursive Phase generation

Equation (17) defines an iteration since the result is a new mapping $\cot(\beta_{q_0}) \rightarrow \cot(\beta_{q_1})$. Hence in general, there exists a k^{th} field corresponding to the k^{th} prime.

$$S_{\phi_k} = \cot(\beta_{q_{k+1}}) \tag{18.}$$

Each admissible prime q_k generates a new phase $\beta_{q_{k+1}}$ relation in (3) and a new conforming prime q_{k+1}.

This produces an infinite prime ladder:

$$\{(q_0, \beta_{q_1}) \rightarrow (q_1, \beta_{q_2}) \rightarrow (q_2, \beta_{q_3}) \rightarrow (q_3, \beta_{q_4}) \rightarrow \cdots (q_\infty, \beta_{q_\infty})\} \tag{19.}$$

where each q_j, represents a distinct phase state.

4. Infinitude of the Conforming Set

If $\frac{f(q_0)}{g(q_0)}$ is multiplicative or periodic on arithmetic progressions, Dirichlet's theorem guarantees infinitely many primes in each residue class that preserves the rational value $\mathbb{Z}$. Hence $|S_{\mathbb{Z}}| = \infty$. The field of conforming primes is therefore an infinite rational–phase subset of the primes.

5. Geometric and Physical Interpretation

Each $p \in S_{\mathbb{Z}} = \left\{p \in \mathbf{P}\colon \frac{f(p)}{g(p)} = \mathbb{Z} \in \mathbb{Q}\right\}$, corresponds to a phase point on the cotangent lattice. The mapping $S_{\phi_k} = \cot\left(\beta_{q_{k+1}}\right)$ translates arithmetic structure into angular geometry. The infinite ladder represents a continuous phase-field manifold built from discrete prime anchors. The equilibrium condition (16) ensures reciprocity: the in-field and out-of-field components balance exactly at resonance.

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6. Conclusions

I have shown that the primes satisfying the rational functional mapping $\frac{\boldsymbol{f(p)}}{\boldsymbol{g(p)}} = \mathbb{Z}$ and the resonance condition $\mathbf{tan}(\boldsymbol{\beta}_{q_k})\,\mathbf{cot}(\boldsymbol{\beta}_{q_{k+1}})$ form an infinite $\boldsymbol{S}_{\mathbb{Z}} \subset \mathbf{P}$. Each conforming prime $\boldsymbol{q_k}$ defines a distinct $\boldsymbol{\beta}_{q_{k+1}}$, producing an infinite prime ladder. The cotangent–tangent quadratic provides a unified description of these rational-phase primes, revealing an underlying self-consistent field structure within the infinite set of primes relating to the cotangent function. Note that there are two distinct types of number that allow the number theretic forms of rational functions with integer values.

For example, putting $n \in Z$: $\frac{\sigma(n)}{n} = 2 = \mathbb{Z}$, gives an infinite progression of Perfect numbers.

For example, putting $p \in Z$: $\frac{\sigma(2p+1)}{\sigma(p)} = 2 = \mathbb{Z}$, gives an infinite progression of Sophie primes.

Examples of a connection between these numbers can be obtained as follows:

The rational trigonometric functions $\tan\left(\frac{\sigma(j)}{j}\right)$ determines invariants of Perfect Numbers, Sophie Germain primes. In the product formula,

$$\sigma(j) = 2\left[\frac{\mathbf{tan}\left(\frac{\sigma(j)}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}{2\sin\left(\frac{1}{j}\right)\cos\left(\frac{1}{j}\right)\prod_{k=1}^{\frac{\sigma(j)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{j}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(j)}\right)}\right)}\right] \tag{20.}$$

The tangent function is the controlling factor, due to the fact that the products can be made by substitutions of $\sigma(j) = (f(x)$ without changing the result. As long as the rational number theoretic functions chosen for the invariants of $\tan\left(\frac{\sigma(j)}{j}\right) \in \mathbb{Z}$, we can write for Sophie primes.

$$\sigma(p) = \frac{\mathbf{tan}\left(\boldsymbol{\frac{\sigma(2p+1)}{\sigma(p)}}\right)\prod_{k=1}^{\frac{\sigma(2p+1)}{2}}\left(1-\frac{\sin^2\left(\frac{1}{\sigma(p)}\right)}{\sin^2\left(\frac{(2k-1)\pi}{2\sigma(2p+1)}\right)}\right)}{\sin\left(\frac{2}{\sigma(p)}\right)\left(\prod_{k=1}^{\frac{\sigma(2p+1)}{2}-1}\left(1-\frac{\sin^2\left(\frac{1}{\sigma(p)}\right)}{\sin^2\left(\frac{k\pi}{\sigma(2p+1)}\right)}\right)\right)} \tag{21.}$$

This is the relation for Sophie primes, and so we arrive at: If $j = N_p$ is a Perfect number, then, the equality applies only when. Since $\tan\left(\frac{\sigma(2p+1)}{\sigma(p)}\right) = 2$ $for\ all\ Sophie\ primes\ p$, the sequence of phases will continue ad-infinitum.

Hence the phase angles for both Perfect numbers and Sophie primes is the same in relation to certain rational number theoretic functions.

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