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Technical Note

Generalized Relational Expression and Its Application

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Abstract

We propose a generalized relational expression (GRE) through dimensional analysis, which serves to unify a broad class of dimensional uncertainty relations (URs). We derive a general form of UR wherein the product of two or more non-commutative physical quantities (PQs) of specific dimensions is equated to a power product of the fundamental constants: the reduced Planck constant ħ, gravitational constant G, speed of light in vacuum c, Boltzmann constant k, and elementary charge e. Our analysis reveals that every dimensioned PQ is associated with a characteristic Planck scale. PQs sharing the identical dimensions consequently share identical Planck scales. These Planck scales are categorized into two primary types: one comprising the basic and derived Planck scales, the other including Fermi-Planck, Bose-Planck, and other scales. We demonstrate that the Planck scale corresponding to any PQ can be expressed as a power product of the Planck length, Planck time, Planck mass, Planck temperature, and the elementary charge (or Planck charge). The GRE is then established by equating the power product of non-commutative PQs to the one of their corresponding Planck scales. Applying the GRE, we derive a Big Bang UR relating the temperature and volume of the Big Bang, and a Schwarzschild black hole (SBH) UR connecting the mass and volume of a SBH. These URs, when quantum effects are incorporated, suggest no singularities in both the Big Bang and SBH scenarios. The functional relationships between PQs are inherently governed by the GRE. By selecting sets of two, three, and four PQs within the GRE framework, we obtain corresponding general formulae. Under specific constraints, such as setting the exponents of the fundamental constants to zero or to empirically fitted values, these general formulae reduce to numerous famous factorless equations. These include the Einstein's mass-energy relation, the SBH horizon temperature formula, Casimir effect equation, Planck blackbody radiation law, Stefan-Boltzmann law, Einstein field equations, Newton's law of gravitation, Schrödinger equation, Coulomb's law, Newton's second law, acceleration of holographic dark energy (HDE), Clapeyron equation, superconducting thin-film power law, and formulas for the critical temperature of LSCO cuprates, among others. Furthermore, several novel relationships are proposed, such as those connecting the square of the SBH energy to its density, sixth power of the SBH radius to its energy density, and SBH pressure to its central entropy density. We conclude that the proposed GRE is a generalized, insightful, and potent tool with significant theoretical utility and broad applicability in theoretical physics.

Keywords: generalized relational expression (GRE); Heisenberg uncertainty principle; dimensional analysis; power product; Planck scale; physical quantity (PQ)

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1. Introduction

The Heisenberg uncertainty principle [1] has led to significant advances in applications [2-4], theoretical developments [5-30], and experimental verifications [31-40]. These contributions have reinforced its foundational status and expanded its conceptual scope. A variety of uncertainty relations (URs) have since been proposed

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$\Delta p\Delta r \geq \hbar$ [1]; $\Delta E\Delta t \geq \hbar$ [1]; $\Delta x > \hbar / \Delta p + \alpha L_P^2 \Delta p / \hbar$ [41-50]; $\Delta R\Delta S \geq |\langle\psi|[R,S]|\psi\rangle| / 2$ [51, 52]; $H(R) + H(Q) \geq \log_2 1 / c$ [53-63]; $S(Q|B) + S(R|B) \geq \log_2 1 / c + S(A|B)$ [64-74]; $\delta t = \beta t_P^{2/3} t^{1/3}$ [75]; $\eta / s \geq 4\pi\hbar / \kappa$ [76]; $\Delta T\Delta X \sim L_S^2 \sim L_P^2 / c$ [72-74]; $\delta x\delta y\delta t \sim L_P^3 / c$ [75-80]; $L_{\mu\nu} \sim \sqrt{L_P L}$ [81-87]; $\varepsilon(Q)\eta(P) + \varepsilon(Q)\sigma(P) + \sigma(Q)\eta(P) \geq \hbar / 2$ [35, 36]; $(\delta t)(\delta r)^3 \geq \pi r^2 L_P^2 / c$ [88], etc.

where Δp is the momentum fluctuation, Δr the position momentum, $\hbar$ the reduced Planck constant; ΔE the energy fluctuation, Δt the time fluctuation; Δx the position momentum, αa dimensionless constant; $L_P = \sqrt{\hbar G/c^3}$ Planck length,G the gravitational constant, c the speed of light in vacuum;ΔR and ΔS the standard deviation of two arbitrary observables R and S; δt the time fluctuation, β an order one constant, $t_P = \sqrt{\hbar G/c^5}$ Planck time, t the time; η the ratio of shear viscosity of a given fluid perfect, s its volume density of entropy, κ the Boltzmann constant; ΔT the time-like, ΔX its space-like, L_S the string scale; δx, δy, δt are the position fluctuation and time fluctuation separately; $L_{\mu\nu}$ the transverse length, L the radial length; Q the position of a mass, $\varepsilon(Q)$ the root-mean-square error, P its momentum, $\eta(P)$ the root-mean-square disturbance, $\sigma(P)$ the standard deviation; δt and δr the sever space-time fluctuations of the constituents of the system at small scales, and r the radius of globular computer.

Observing these URs, we can classify them to four types

I URs

$\Delta p\Delta r \geq \hbar$; $\Delta E\Delta t \geq \hbar$; $\Delta R\Delta S \geq |\langle\psi|[R,S]|\psi\rangle| / 2$; $\delta t = \beta t_P^{2/3} t^{1/3}$; $\eta / s \geq 4\pi\hbar / \kappa$; $\Delta T\Delta X \sim L_S^2 \sim L_P^2 / c$; $L_{\mu\nu} \sim \sqrt{L_P L}$; $\delta x\delta y\delta t \sim L_P^3 / c$; $(\delta t)(\delta r)^3 \geq \pi r^2 L_P^2 / c$;

II URs

$\varepsilon(Q)\eta(P) + \varepsilon(Q)\sigma(P) + \sigma(Q)\eta(P) \geq \hbar / 2$;

III URs

$\Delta x > \hbar / \Delta p + \alpha L_P^2 \Delta p / \hbar$;

IV URs (dimensionless)

$H(R) + H(Q) \geq \log_2 1 / c$; $S(Q|B) + S(R|B) \geq \log_2 1 / c + S(A|B)$.

Etc.

We only research the I URs, II URs and III URs with dimensions. Two natural questions arise: (i) Why does the gravitational constant G not appear on the right-hand side of certain URs? (ii) Can these relations be unified within a single framework? In this work, we address these questions by demonstrating that the absence of G results from appropriate dimensional reduction, and we propose a unified formulation in the form of a generalized relational expression (GRE). Regarding the origin and development of Planck units，such as the Planck length, Planck time, Planck mass $M_P = \sqrt{\hbar c/G}$, Planck energy $E_P = \sqrt{\hbar c^5/G}$ and Planck temperature $T_P = \sqrt{\hbar c^5/\kappa^2 G}$, please refer to the literature [89-96].

This paper is organized as follows. In Sec. 2, the general form of URs for two and n physical quantities (PQs) is derived, and the underlying foundational relationship is established. Sec. 3 presents the concept of the Planck scale and provides a classification scheme for different types of Planck scales. In Sec. 4, it is shown that the Planck scale corresponding to any PQ can be expressed as a power product of the basic Planck scales; the GRE is formulated and rigorously proven, and the URs introduced in Sec. 1 are subsequently verified. Sec. 5 applies the GRE to deduce several significant results, including the Big Bang UR, the Schwarzschild black hole (SBH) UR, and a number of well-known factor-free equations, such as Einstein's mass-energy relation, the SBH horizon temperature formula, and the Casimir effect equation, among others. Additionally, several new physical relationships are proposed. Finally, concluding remarks and a summary are provided in Sec. 6.

2. General Expression of URs and Basic Relationship

In this section, we discover the normal form of URs; derive the general expression of URs for two PQs, basic relationship, and general expression of URs for n PQs.

2.1. General Expression of URs for two PQs

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For the I URs and II URs (III URs can be regarded as the recombination of I and II), we discover the physical constants such as ħ, G, c and κ on the right hand, and the PQs on left hand. We rewrite them as

$\Delta p\Delta r \geq \hbar$; $\Delta E\Delta t \geq \hbar$; $\delta t / \beta t^{1/3} = t_P^{2/3} = \hbar^{1/3}G^{1/3}c^{-5/3}$; $\eta / 4\pi s \geq \hbar\kappa^{-1}$; $\Delta T\Delta X \sim L_S^2 \sim L_P^2 / c = \hbar G c^{-4}$; $L_{\mu\nu} / \sqrt{L} \sim \sqrt{L_P} = \hbar^{1/4}G^{1/4}c^{-3/4}$;

$\Delta x\delta y\delta t \sim L_P^3 / c = \hbar^{3/2}G^{3/2}c^{-11/2}$; $(\delta t)(\delta r)^3 / \pi r^2 \geq L_P^2 / c = \hbar G c^{-4}$,

$2[\varepsilon(Q)\eta(P)+\varepsilon(Q)\sigma(P)+\sigma(Q)\eta(P)] \geq \hbar$;

Etc.

Therefore, the right-hand side of such relations naturally takes the form of a power product of fundamental physical constants. This represents their canonical form. Considering two non-commutative dimensional PQs, we derive the general form of the URs

$$AB \sim \hbar^x G^y c^z \kappa^w e^u \quad (1)$$

Where A and B are non-commutative PQs, x, y, z, w and u the unknown number, and e the elementary charge. Applying the dimensional analysis (here we use the LMTΘQ units [97][1]), the dimensions of A and B are expressed as

$$[A]=[L]^{\alpha_1}[M]^{\beta_1}[T]^{\gamma_1}[\Theta]^{\delta_1}[Q]^{\varepsilon_1},\ [B]=[L]^{\alpha_2}[M]^{\beta_2}[T]^{\gamma_2}[\Theta]^{\delta_2}[Q]^{\varepsilon_2} \quad (2)$$

where L, M, T, Θ and Q are the dimensions of length, mass, time, temperature and electric charge separately, α_1, α_2, β_1, β_2, γ_1, γ_2, δ_1, δ_2, ε_1 and ε_2 the known real number. The dimensions of $\hbar^x G^y c^z \kappa^w e^u$ is

$$[\hbar^x G^y c^z \kappa^w e^u] = \{[L^2][M][T^{-1}]\}^x\{[L^3][M^{-1}][T^{-2}]\}^y\cdot\{[L][T^{-1}]\}^z\{[L^2][M][T^{-2}][\Theta^{-1}]\}^w\{[Q]\}^u \quad (3)$$

Then we obtain

$$[L]^{\alpha_1}[M]^{\beta_1}[T]^{\gamma_1}[\Theta]^{\delta_1}[Q]^{\varepsilon_1}[L]^{\alpha_2}[M]^{\beta_2}[T]^{\gamma_2}[\Theta]^{\delta_2}[Q]^{\varepsilon_2} = \{[L^2][M][T^{-1}]\}^x\{[L^3][M^{-1}][T^{-2}]\}^y \cdot\{[L][T^{-1}]\}^z\{[L^2][M][T^{-2}][\Theta^{-1}]\}^w\{[Q]\}^u \quad (4)$$

Solving the Eq. (4), we gain

$$x = [(\alpha_1+\alpha_2)+(\beta_1+\beta_2)+(\gamma_1+\gamma_2)+(\delta_1+\delta_2)] / 2,$$
$$y = [(\alpha_1+\alpha_2)-(\beta_1+\beta_2)+(\gamma_1+\gamma_2)-(\delta_1+\delta_2)] / 2,$$
$$z = -[3(\alpha_1+\alpha_2)-(\beta_1+\beta_2)+5(\gamma_1+\gamma_2)-5(\delta_1+\delta_2)] / 2,$$
$$w = -(\delta_1+\delta_2),\ u = (\varepsilon_1+\varepsilon_2) \quad (5)$$

Thus we find the general expression of URs for two PQs

$$AB \sim [\hbar^{((\alpha_1+\alpha_2)+(\beta_1+\beta_2)+(\gamma_1+\gamma_2)+(\delta_1+\delta_2))}]^{\frac{1}{2}}\cdot[G^{((\alpha_1+\alpha_2)-(\beta_1+\beta_2)+(\gamma_1+\gamma_2)-(\delta_1+\delta_2))}]^{\frac{1}{2}} \cdot [c^{-(3(\alpha_1+\alpha_2)-(\beta_1+\beta_2)+5(\gamma_1+\gamma_2)-5(\delta_1+\delta_2))}]^{\frac{1}{2}}\cdot\kappa^{-(\delta_1+\delta_2)}e^{(\varepsilon_1+\varepsilon_2)} \quad (6)$$

This indicates that the product of two non-commutative dimensional PQs is equivalent to a power product of the reduced Planck constant, gravitational constant, speed of light, Boltzmann constant, and elementary charge.

2.2. Basic Relationship

Assuming $\alpha_1=\alpha_2=\alpha$, $\beta_1=\beta_2=\beta$, $\gamma_1=\gamma_2=\gamma$, $\delta_1=\delta_2=\delta$, and $\varepsilon_1=\varepsilon_2=\varepsilon$ in the general expression of URs (6), that is A and B having the identical dimensions

$$[A]=[B]=[L]^{\alpha}[M]^{\beta}[T]^{\gamma}[\Theta]^{\delta}[Q]^{\varepsilon} \quad (7)$$

We obtain

$$\hbar^{(\alpha+\beta+\gamma+\delta)}G^{(\alpha-\beta+\gamma-\delta)}c^{-(3\alpha-\beta+5\gamma-5\delta)}\kappa^{-2\delta}e^{2\varepsilon} = A_P B_P = A_P^2 = B_P^2 \quad (8)$$

where A_P and B_P indicatethe corresponding Planck scale of A and B separately. Here we assume the Planck scales being identical because of their identical dimensions. Extracting the square root, we find the basic relationship

$$A \sim A_P = [\hbar^{(\alpha+\beta+\gamma+\delta)}G^{(\alpha-\beta+\gamma-\delta)}c^{-(3\alpha-\beta+5\gamma-5\delta)}\kappa^{-2\delta}e^{2\varepsilon}]^{\frac{1}{2}} \quad (9)$$

This relationship indicates that any PQ with dimension has a corresponding Planck scale, expressible as a power product of ħ, G, c, κ and e, that is PQs and Planck scales having the supersymmetry [98-104].

[1] Chien Wei-Zang used L, M, T, θ and Q indicated the dimensions of length, mass, time, temperature and electric charge separately in [97].

If $A_P \neq B_P$, we assume $A_P = \lambda B_P$, where λ is a fitted coefficient. Substituting it and Eq. (7) into Eq. (6), we get

$$\hbar^{(\alpha+\beta+\gamma+\delta)} G^{(\alpha-\beta+\gamma-\delta)} c^{-(3\alpha-\beta+5\gamma-5\delta)} \kappa^{-2\delta} e^{2\varepsilon} = A_P B_P = \lambda A_P^2$$

Extracting the square root, we obtain

$$A \sim A_P = [\hbar^{(\alpha+\beta+\gamma+\delta)} G^{(\alpha-\beta+\gamma-\delta)} c^{-(3\alpha-\beta+5\gamma-5\delta)} \kappa^{-2\delta} e^{2\varepsilon}/\lambda]^{\frac{1}{2}} \sim [\hbar^{(\alpha+\beta+\gamma+\delta)} G^{(\alpha-\beta+\gamma-\delta)} c^{-(3\alpha-\beta+5\gamma-5\delta)} \kappa^{-2\delta} e^{2\varepsilon}]^{\frac{1}{2}}$$

That is Eq. (9) omitting the coefficient. Same deduction applies to Eq. (13).

2.3. General Expression of URs for n PQs

Extending the analysis to n non-commutative dimensional PQs, we write

$$\prod_{i=1}^{n} A_i \sim \hbar^x G^y c^z \kappa^w e^u, \; i = 1, 2, 3 \ldots n \qquad (10)$$

where A_i is a PQ, A_i and A_{i+1} are non-commutative. The dimensions of $\prod_{i=1}^{n} A_i$ are

$$[\prod_{i=1}^{n} A_i] = [L]^{\sum_i^n \alpha_i} [M]^{\sum_i^n \beta_i} [T]^{\sum_i^n \gamma_i} [\Theta]^{\sum_i^n \delta_i} [Q]^{\sum_i^n \varepsilon_i} \qquad (11)$$

where α_i, β_i, γ_i, δ_i and ε_i are known real number. Applying the dimensional analysis again, we find the general expression for n PQs

$$\prod_{i=1}^{n} A_i \sim [\hbar^{\left((\sum_i^n \alpha_i)+(\sum_i^n \beta_i)+(\sum_i^n \gamma_i)+(\sum_i^n \delta_i)\right)}]^{\frac{1}{2}} \cdot [G^{\left((\sum_i^n \alpha_i)-(\sum_i^n \beta_i)+(\sum_i^n \gamma_i)-(\sum_i^n \delta_i)\right)}]^{\frac{1}{2}} \cdot [c^{-\left(3(\sum_i^n \alpha_i)-(\sum_i^n \beta_i)+5(\sum_i^n \gamma_i)-5(\sum_i^n \delta_i)\right)}]^{\frac{1}{2}} \cdot \kappa^{-(\sum_i^n \delta_i)} e^{(\sum_i^n \varepsilon_i)} \qquad (12)$$

For $n = 2$, it reduces to Eq. (6). Ordering $\alpha_i = \alpha_{i+1} = \alpha$, $\beta_i = \beta_{i+1} = \beta$, $\gamma_i = \gamma_{i+1} = \gamma$, $\delta_i = \delta_{i+1} = \delta$ and $\varepsilon_i = \varepsilon_{i+1} = \varepsilon$ in Eq. (12), A_i and A_{i+1} having identical dimensions, we obtain

$$[\hbar^{n(\alpha+\beta+\gamma+\delta)}]^{\frac{1}{2}} [G^{n(\alpha-\beta+\gamma-\delta)}]^{\frac{1}{2}} [c^{-n(3\alpha-\beta+5\gamma-5\delta)}]^{\frac{1}{2}} \kappa^{-n\delta} e^{n\varepsilon} \sim A_P^n \qquad (13)$$

Extracting the nth-root, we gain Eq. (9) again.

3. Planck Scale

In this section, we derive various Planck scales and present a systematic classification.

3.1. Basic Planck Scale

By assigning specific values to the dimensional exponents in Eq. (7) and applying Eq. (9), the basic Planck scales are obtained as follows

Ordering $\alpha = 1$, $\beta = \gamma = \delta = \varepsilon = 0$, we obtain Planck length immediately

$L_P = \sqrt{\hbar G/c^3}$

Instructing $\gamma = 1$, $\alpha = \beta = \delta = \varepsilon = 0$, obtain Planck time

$t_P = \sqrt{\hbar G/c^5}$

Ordering $\beta = 1$, $\alpha = \gamma = \delta = \varepsilon = 0$, obtain Planck mass

$M_P = \sqrt{\hbar c/G}$

Instructing $\delta = 1$, $\alpha = \beta = \gamma = \varepsilon = 0$, obtain Planck temperature

$T_P = \sqrt{\hbar c^5/\kappa^2 G}$

Ordering $\varepsilon = 1$, $\alpha = \beta = \gamma = \delta = 0$, obtain elementary charge

$Q_e = e$

If the dimension of electric charge is expressed as $[Q]^2 = [L]^3[M][T]^{-2}$, the Planck charge is obtained as

$Q_P = \sqrt{\hbar c} \sim e$

These constitute the basic Planck scales [88].

3.2. Derived Planck Scale

Using Eqs. (7) and (9), additional derived Planck scales [88] can be obtained. For example

Planck energy E_P with $[E_P] = [L]^2[M][T]^{-2}$

$E_P = \sqrt{\hbar c^5/G}$

Planck momentum P_P with $[P_P] = [L][M][T]^{-1}$

$P_P = \sqrt{\hbar c^3/G}$

Planck curvature tensor $R_{\mu\nu P}$ with $[R_{\mu\nu P}] = [L]^{-2}$

$$R_{\mu\nu P} = c^3 / \hbar G$$

Etc.

Many PQs share the same dimensions and therefore correspond to the same Planck scale. For instance

Planck energy density ρ_P, Planck pressure p_P, Planck energy-momentum tensor $T_{\mu\nu P}$ all have dimensions $[L]^{-1}[M][T]^{-2}$, and share the Planck scale

$\rho_P = p_P = T_{\mu\nu P} = c^7/\hbar G^2$

And so on.

3.3. Classifications

All the Planck scales can be categorized into two types. The first includes the basic and derived Planck scales [88]. The second category comprises

The Femi-Planck scale, with half-integer exponents, such as L_P, t_P, M_P, T_P, E_P, P_P etc;

Bose-Planck scale, with integer exponents, such as Q_e, ρ_P, p_P, $R_{\mu\nu P}$, $T_{\mu\nu P}$, etc;

Other-Planck scale, such as the Planck wave function ψ_P, $[\psi_P] = [L]^{-3/2}$, $\psi_P = (\hbar G/c^3)^{-3/4}$

4. GRE

In this section, we demonstrate that the basic relation (9) can be expressed as a power product of basic Planck scales. We then introduce and prove the GRE, and use it to verify the URs presented in Sec. 1.

4.1. Proof of Basic Relationship

Basic relationship (9) can be rewritten as

$$A_P = L_P^{\alpha} M_P^{\beta} t_P^{\gamma} T_P^{\delta} Q_e^{\varepsilon} \tag{14}$$

From Eq. (9), we have

$$A_P = [\hbar^{\alpha} G^{\alpha} c^{-3\alpha}]^{\frac{1}{2}}[\hbar^{\beta} G^{-\beta} c^{\beta}]^{\frac{1}{2}}[\hbar^{\gamma} G^{\gamma} c^{-5\gamma}]^{\frac{1}{2}}[\hbar^{\delta} G^{-\delta} c^{5\delta}]^{\frac{1}{2}}\kappa^{-\delta} e^{\varepsilon} = [\sqrt{\hbar G/c^3}]^{\alpha}[\sqrt{\hbar c/G}]^{\beta}[\sqrt{\hbar G/c^5}]^{\gamma}[\sqrt{\hbar c^5/\kappa^2 G}]^{\delta} e^{\varepsilon} = L_P^{\alpha} M_P^{\beta} t_P^{\gamma} T_P^{\delta} Q_P^{\varepsilon}$$

Therefore, the Planck scale corresponding to any PQ can be expressed as a power product of the Planck length, Planck time, Planck mass, Planck temperature, and elementary charge.

4.2. GRE

Considering all the non-commutative PQs with dimension, we find the GRE

$$\prod_{i=1}^{n} A_i^{a_i} \sim \prod_{i=1}^{n} A_{iP}^{a_i}; \quad i = 1, 2, 3 \ldots n \tag{15}$$

where A_i is a PQ, A_i and A_{i+1} are non-commutative, a_i the real number, and A_{iP} the corresponding Planck scale of A_i. This indicates that the power product of non-commutative PQs is equivalent to the one of their respective Planck scales.

4.3. Proving GRE

The proof follows the same dimensional analysis approach as in Section 2.3. For n non-commutative PQs raised to powers a_i power, we write

$$\prod_{i=1}^{n} A_i^{a_i} \sim \hbar^x G^y c^z \kappa^w e^u \tag{16}$$

The dimensions of $\prod_{i=1}^{n} A_i^{a_i}$ are expressed as

$$[\prod_{i=1}^{n} A_i^{a_i}] = [L]^{\sum_i^n a_i\alpha_i}[M]^{\sum_i^n a_i\beta_i}[T]^{\sum_i^n a_i\gamma_i}[\Theta]^{\sum_i^n a_i\delta_i}[Q]^{\sum_i^n a_i\varepsilon_i} \tag{17}$$

Using the dimensional analysis also, we obtain the general form

$$\prod_{i=1}^{n} A_i^{a_i} \sim [\hbar^{\left((\sum_i^n a_i\alpha_i)+(\sum_i^n a_i\beta_i)+(\sum_i^n a_i\gamma_i)+(\sum_i^n a_i\delta_i)\right)}]^{\frac{1}{2}} \cdot [G^{\left((\sum_i^n a_i\alpha_i)-(\sum_i^n a_i\beta_i)+(\sum_i^n a_i\gamma_i)-(\sum_i^n a_i\delta_i)\right)}]^{\frac{1}{2}}$$
$$\cdot [c^{-\left(3(\sum_i^n a_i\alpha_i)-(\sum_i^n a_i\beta_i)+5(\sum_i^n a_i\gamma_i)-5(\sum_i^n a_i\delta_i)\right)}]^{\frac{1}{2}} \cdot \kappa^{-(\sum_i^n a_i\delta_i)} e^{(\sum_i^n a_i\varepsilon_i)}$$
$$= [\sqrt{\hbar G/c^3}]^{\sum_i^n a_i\alpha_i}[\sqrt{\hbar c/G}]^{\sum_i^n a_i\beta_i}[\sqrt{\hbar G/c^5}]^{\sum_i^n a_i\gamma_i} \cdot [\sqrt{\hbar c^5/\kappa^2 G}]^{\sum_i^n a_i\delta_i} e^{\sum_i^n a_i\varepsilon_i} = L_P^{\sum_i^n a_i\alpha_i} M_P^{\sum_i^n a_i\beta_i} t_P^{\sum_i^n a_i\gamma_i} T_P^{\sum_i^n a_i\delta_i} Q_e^{\sum_i^n a_i\varepsilon_i}$$
$$= \prod_{i=1}^{n} L_P^{a_i\alpha_i} M_P^{a_i\beta_i} t_P^{a_i\gamma_i} T_P^{a_i\delta_i} Q_e^{a_i\varepsilon_i} = \prod_{i=1}^{n} A_{iP}^{a_i} \quad (18)$$

where $A_{iP} = L_P^{\alpha_i} M_P^{\beta_i} t_P^{\gamma_i} T_P^{\delta_i} Q_e^{\varepsilon_i}$, which confirms the GRE.

4.4. Proving URs

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Applying the GRE (15), we can prove the URs in Sec.1.

$\Delta p\Delta r\sim \mathrm{P_P L_P} = \sqrt{\hbar c^3/G}\sqrt{\hbar G/c^3} = \hbar$; $\Delta E\Delta t\sim \mathrm{E_P t_P} = \sqrt{\hbar c^5/G}\sqrt{\hbar G/c^5} = \hbar$; $\delta t / t^{1/3} \sim \mathrm{t_P} / \mathrm{t_P^{1/3}} = \mathrm{t_P^{2/3}}$; $\eta / s\sim \eta_P / s_P = \sqrt{c^9/\hbar G^3} / \sqrt{c^9\kappa^2/\hbar^3 G^3} = \hbar / \kappa$; $\Delta T\Delta X\sim \mathrm{t_P L_P}\sim \hbar G / c^4 = \mathrm{L_P^2} / c\sim \mathrm{L_S^2}$; $L_{\mu\nu} / \sqrt{L}\sim \mathrm{L_P} / \sqrt{\mathrm{L_P}} = \sqrt{\mathrm{L_P}}$; $\delta x\delta y\delta t\sim \mathrm{L_P^2 t_P} = \mathrm{L_P^3} / c$; $(\delta t)(\delta r)^3 / r^2\sim \mathrm{t_P L_P^3} / \mathrm{L_P^2} = \mathrm{L_P^2} / c$; $\varepsilon(Q)\eta(P)+\varepsilon(Q)\sigma(P)+\sigma(Q)\eta(P)\sim\sqrt{\hbar G/c^3}\sqrt{\hbar c^3/G} = \hbar$, etc.

where $\eta_P = \sqrt{c^9/\hbar G^3}$ is the Planck ratio of shear viscosity of a given fluid perfect, and $s_P = \sqrt{c^9\kappa^2/\hbar^3 G^3}$ its Planck volume density of entropy (from basic relationship (9)). This demonstrates that the gravitational constant G does not appear on the right-hand side of certain URs due to appropriate dimensional reduction.

5. Application

A central goal in theoretical physics is to develop a universal framework from which established physical laws can be derived. The Standard Model [105], represents a major achievement in this direction, successfully unifying the electromagnetic, weak, and strong interactions. With the recent experimental confirmation of the Higgs boson [106-112], all 62 predicted elementary particles have been observed. However, the model does not incorporate gravity. Numerous beyond Standard Model theories, including supersymmetry [98-104], supergravity [98-104], superstring/M-theory [41-50], loop quantum gravity [44, 45, 47, 48], the causal set approach [113-117], the holographic principle [118], the asymptotic safety scenario [119], causal dynamical triangulation [120-123], an exceptionally simple theory of everything [124], unified field equations [125, 126], SQS theory [127], Quantum Field Theory of Gravity and Hyperunified Field Theory [128], have been proposed to describe all four fundamental forces. Nonetheless, experimental validation remains elusive [129].

5.1. Bing Bang UR and SBH UR

In this section, we derive URs for the Big Bang and SBH applying the GRE.

5.1.1. Big Bang UR

S.W. Hawking and R. Penrose established that the universe originated from a Big Bang singularity [130, 131]. Subsequent studies have explored the possibility of avoiding this and other singularities in black holes by incorporating quantum effects [98-129]. A key characteristic of the Big Bang singularity is a spacetime point of zero volume and infinitely high temperature.

Applying the GRE (15), we derive a relation between the temperature and volume of the Big Bang

$$T_B V_B\sim \mathrm{T_P V_P} = \mathrm{T_P L_P^3} = \hbar^2 G / \kappa c^2 \tag{19}$$

where T_B is the Big Bang temperature, V_B itsvolume, and $\mathrm{V_P} = \mathrm{L_P^3}$ the Planck volume. This constitutes the Big Bang UR. It implies that the temperature and volume of the Big Bang cannot be simultaneously determined with arbitrary precision. When $\hbar \to 0$, we obtain

$$T_B V_B\sim 0 \tag{20}$$

Because $T_B > 0$ [133], it follows that $V_B \sim 0$, indicating a singular Big Bang origin when quantum effects are neglected. This supports the view that the inclusion of quantum mechanics may resolve the initial singularity.

Substituting $a = c\kappa T / 2\pi\hbar$ [134] into Eq. (19), we obtain

$$a_B V_B\sim a_p \mathrm{V_p} = \hbar G / 2\pi c \tag{21}$$

where a_B is the Big Bang acceleration, and $a_p = \sqrt{c^7/\hbar G}$ the Planck acceleration. It is the UR for Big Bang acceleration and its volume.

5.1.2. SBH UR

Similarly, for a SBH of mass and volume, we find

$$M_H V_H\sim \mathrm{M_P V_P} = \mathrm{M_P L_P^3} = \hbar^2 G / c^4 \tag{22}$$

Where M_H is the SBH mass, and V_H its volume. It is the SBH UR, indicating that the mass and volume of a SBH cannot be simultaneously measured precisely also. When ħ → 0, we obtain

$$M_H V_H \sim 0 \tag{23}$$

Since $M_H > 0$, this implies $V_H \sim 0$, the volume is zero, suggesting a singularity emerges in the classical limit. Therefore, we suggest that quantum effects may also prevent the formation of a singularity in SBH.

Expressing the mass as $M = \rho V$, Eq. (22) leads to

$$M_H^2 / \rho_H \sim \hbar^2 G / c^4, \ \rho_H V_H^2 \sim \hbar^2 G / c^4 \tag{24}$$

where ρ_H is the mass density of SBH. These relations describe the uncertainty between the density and mass or volume of a SBH.

5.2. Power Product Relationship Between Two PQs

In this section, we derive power product relations for the case where n = 2 within the GRE. Corresponding general formulas are established, leading to the recovery of many fundamental physical laws, including the Einstein's mass-energy relation, event horizon temperature of a SBH [3], observed density of dark energy [135, 136], Casimir effect equation, Planck's blackbody radiation formula, Stefan– Boltzmann law, and Einstein field equations [138], and so on.

5.2.0. For the GRE (15), when n = 2, We Obtain

$$A_1^{a_1} A_2^{a_2} \sim A_{1P}^{a_1} A_{2P}^{a_2} \tag{25}$$

Instructing $a_1 = 1$, $a_2 = b$, $A_1 = A$ and $A_2 = B$, we gain

$$AB^b \sim A_P B_P^b \tag{26}$$

Especially when $b = 1$, we obtain

$$A \sim A_P B_P / B \tag{27}$$

When $b = -1$, we gain

$$A \sim A_P B / B_P \tag{28}$$

Therefore, we can determine the power product relationship between two PQs. For example

5.2.1. Assuming that Energy E Has Relations with Mass M Only, We Find

$$EM^b \sim E_P M_P^b = (\hbar c^5/G)^{1/2} (\hbar c/G)^{a/2} = \hbar^{(1+b)/2} G^{-(1+b)/2} c^{(5+b)/2} \tag{29}$$

Above is the general formulae for energy and mass.

5.2.1.1. Ordering 1+b = 0, → b = −1, We Obtain

$$E \sim Mc^2$$

That is the Einstein's mass-energy relation.

5.2.1.2. Instructing 5+b = 0, → b = −5,We Have

$$E \sim G^2 M^5 / \hbar^5 \ ?$$

5.2.1.3. Ordering b = 1, We Gain

$$E \sim \hbar c^3 / GM$$

Substituting $E \sim \kappa T$ into above formula, we obtain

$$T \sim \hbar c^3 / \kappa G M$$

where T is the temperature. Above is the SBH event horizon temperature formula [3], but it hasn't 1 / 8π.

5.2.2. Supposing That Energy E Has Relations with Frequency ω Merely, We Find

$$E\omega^b \sim E_P \omega_P^b = \hbar^{(1-b)/2} G^{-(1+b)/2} c^{5(1+b)/2} \tag{30}$$

where $\omega_P=\sqrt{c^5/\hbar G}$ is the Planck frequency.

5.2.2.1. Instructing 1+b = 0, → b =−1, We Gain

$E \sim \hbar\omega$

Above formula is the light quantum relation.

5.2.2.2. Ordering 1−b = 0, → b = 1, We Obtain

$\omega \sim c^5 / GE$

Substituting $E \sim Mc^2$ into above formula, we gain

$\omega \sim c^3 / GM$

where $\omega \sim v_G$. That is the inverse correlation between high-frequency quasi-periodic oscillation and black hole mass [139-147].

5.2.2.3. Instructing b =−3, We Have

$E \sim \hbar^2 G\omega^3 / c^5$?

5.2.3. Assuming that Energy E Has Relations with Energy Density ϱ Only, We Find

$$E\rho^b \sim E_P\rho_P^b = \hbar^{(1-2b)/2}G^{-(1+4b)/2}c^{(5+14b)/2} \tag{31}$$

5.2.3.1. Ordering 1−2b = 0, → b = 1 / 2, We Obtain

$E^2 \sim c^{12} / G^3\rho$

So the above formula is the relativistic gravitational energy.

5.2.3.2. Instructing 1+4b = 0, → b =−1 / 4, We Obtain

$E^4 \sim \hbar^3 c^3 \rho$

That is the relationship between biquadratic quanta energy and its density [139-147].

5.2.3.3. Ordering b =−1 / 2, We Gain

$E^2 \sim \hbar^2 G\rho / c^2$

From $MV \sim \hbar^2 G / c^4$, $E = \rho V$ and $E = Mc^2$, where M is the mass of SBH, and V its volume, we obtain the above formula with square of energy and its density of SBH.

5.2.4. Supposing That Distance R Has Relations with Mass M Merely, We Find

$$RM^b \sim L_P M_P^b = \hbar^{(1+b)/2}G^{(1-b)/2}c^{-(3-b)/2} \tag{32}$$

5.2.4.1. Instructing 1+b = 0, → b =−1, We Obtain

$R \sim GM / c^2$

Above is the radius of event horizon of stationary black holes [131].

5.2.4.2. Ordering 1−b = 0, → b = 1, we gain

$R \sim \hbar / Mc$

That is A.H. Compton wavelength formula.

5.2.4.3. Instructing 3−b = 0, → b = 3, we have

$R \sim \hbar^2 / GM^3$?

5.2.4.4. Ordering b =−3, we obtain

$R \sim G^2M^3 / \hbar c^3$

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Substituting $R = ct$ into above formula, we gain
$t \sim G^2M^3 / \hbar c^4 \propto M^3$
Above is the age of SBH [3].
From $R \sim GM / c^2$, we obtain $V \sim R^3 \sim G^3M^3 / c^6$, substituting $t \sim G^2M^3 / \hbar c^4$, we gain
$V \sim \hbar Gt / c^2$
That is the relation between the volume of event horizon of stationary black holes and its age. For the SBH, $R= 2GM / c^2$, $V=32\pi G^3M^3 / 3c^6$ and $t \approx 15360\pi G^2M^3 / \hbar c^4$, we have $V \approx \hbar Gt / 1440c^2$.

5.2.5. Assuming That Energy Density ϱ Has Relations with Distance R Only, We Find

$$\rho R^b \sim \rho_P L_P^b = \hbar^{-(2-b)/2} G^{-(4-b)/2} c^{(14-3b)/2} \tag{33}$$

5.2.5.1. Instructing 2—b = 0, → b = 2, We Obtain

$\rho \sim c^4 / GR^2$
This is the gravitational energy density.

5.2.5.2. Ordering 4—b = 0, → b = 4, We Gain

$\rho \sim \hbar c / R^4 \rightarrow R \sim \sqrt[4]{\hbar c / \rho}$
where $R \sim \lambda_d$ is the length scale associated with dark energy and $\rho \sim \rho_d$ the observed density of dark energy [135, 136].

5.2.5.3. Instructing 14—3b = 0, → b = 14 / 3, We Have

$\rho^3 \sim \hbar^4 G / R^{14}$?

5.2.5.4. Ordering b = 6, We Obtain

$\rho \sim \hbar^2 G / c^2R^6$
From $MV \sim \hbar^2G / c^4$, $E = \rho V$, $E = Mc^2$ and $V \sim R^3$, we gain the above formula. That is the energy density with sixth power radius of SBH.

5.2.6. Supposing That Per Area Force f Has Relations with Distance R Merely, We Find

$$fR^b \sim f_P L_P^b = \hbar^{-(2-b)/2} G^{-(4-b)/2} c^{(14-3b)/2} \tag{34}$$

where $f_P = c^7 / \hbar G^2$ is the Planck per area force.

5.2.6.1. Instructing 4—b = 0, → b = 4, We Gain

$f \sim \hbar c / R^4$
That is Casimir effect formula, hasn't $-\pi^2 / 240$.

5.2.6.2. Ordering 2—b = 0, → b = 2, We Obtain

$f \sim c^4 / GR^2 = F_P / R^2$
where $F_P = c^4/G$ is the Planck force. It is the relativistic gravitational pressure or holographic dark energy (HDE) negative pressure [137, 148-152].

5.2.6.3. Instructing 14—3b = 0, → b =14 / 3, We Have

$f^3 \sim \hbar^4 G / R^{14}$?

5.2.6.4. Ordering b = 6, We Obtain

$f \sim \hbar^2 G / c^2R^6$
From 2.5.4 and $p = \omega\rho$, we gain
$p \sim \omega\hbar^2 G / c^2R^6$

That is the pressure $p \sim f$ in SBH centre.

5.2.7. Assuming that Radiation Density ρ_r Has Relations with Frequency γ Only, We Find

$$\rho_r \gamma^b \sim \rho_{rP} \gamma_P^b = \hbar^{-(1+b)/2} G^{-(3+b)/2} c^{(9+5b)/2} \quad (35)$$

where $\rho_{rP} = \sqrt{c^9/\hbar G^3}$ is the Planck radiation density, and $\gamma_P = \sqrt{c^5/\hbar G}$ the Planck frequency.

5.2.7.1. Instructing $3+b=0, \to b=-3$, We Obtain

$\rho_r \sim \hbar\gamma^3 / c^3$

Comparing M. Planck blackbody radiation formula, it hasn't $8\pi / (e^{\hbar\gamma/\kappa T} - 1)$.

5.2.7.2. Ordering $1+b=0, \to b=-1$, We Gain

$$\rho_r \sim c^2\gamma / G \ ?$$

5.2.7.3. Instructing $9+5b=0, \to b=-9/5$, We Have

$\rho_r^5 \sim \hbar^2\gamma^9 / G^3$?

5.2.7.4. Ordering $b=-5$, We Get

$\rho_r \sim \hbar^2 G\gamma^5 / c^8$?

5.2.8. Supposing That Energy Density ρ Has Relations with Temperature T Merely, We Find

$$\rho T^b \sim \rho_P T_P^b = \hbar^{-(2-b)/2} G^{-(4+b)/2} c^{(14+5b)/2} \kappa^{-b} \quad (36)$$

5.2.8.1. Instructing $4+b=0, \to b=-4$, We Obtain

$\rho \sim \kappa^4 T^4 / \hbar^3 c^3$

That is Stefan-Boltzmann law, hasn't $\pi^2 / 15$.

5.2.8.2. Ordering $2-b=0, \to b=2$, We Gain

$\rho \sim c^{12} / G^3 \kappa^2 T^2$

This is the relativistic gravitational energy density with square temperature.

5.2.8.3. Instructing $14+5b=0, \to b=-14/5$, We Obtain

$\rho^5 \sim \kappa^{14} T^{14} / \hbar^6 G^3$?

5.2.8.4. Ordering $b=-2$, We Get

$\rho \sim c^2\kappa^2 T^2 / \hbar^2 G$

It is the gravitational energy density far from the horizon inside SBH [137].

5.2.9. Assuming That Acceleration a Has Relations With Temperature T Only, We Find

$$a T^b \sim a_P T_P^b = \hbar^{-(1-b)/2} G^{-(1+b)/2} c^{(7+5b)/2} \kappa^{-b} \quad (37)$$

5.2.9.1. Instructing $1+b=0, \to b=-1$, We Gain

$a \sim c\kappa T / \hbar$

That is Unruh formula [134], hasn't $1/2\pi$.

5.2.9.2. Ordering $1-b=0, \to b=1$, We Obtain

$a \sim c^6 / \kappa G T$

That is the relativistic gravitational temperature.

5.2.9.3. Instructing 7+5b = 0, → b =−7 / 5, We Have

$b^5 \sim G\kappa^7 T^7 / \hbar^6$?

5.2.9.4. Ordering b =−3, We Obtain

$a \sim G\kappa^3 T^3 / \hbar^2 c^4$?

5.2.10. Supposing That Entropy Density s Has Relations with Temperature T Merely, We Find

$$sT^b \sim s_P T_P^b = \hbar^{-(3-b)/2} G^{-(3+b)/2} c^{(9+5b)/2} \kappa^{(1-b)} \quad (38)$$

where $s_P = \sqrt{\kappa^2 c^9/\hbar^3 G^3}$ is the Planck entropy density.

5.2.10.1. Instructing 3+b = 0, → b =−3, We Obtain

$s \sim \kappa^4 T^3 / \hbar^3 c^3$

That is entropy density with cube of temperature [144].

5.2.10.2. Ordering 3−b = 0, → b = 3, We Have

$s \sim c^{12} / G^3 \kappa^2 T^3$

This is the relativistic gravitational entropy density.

5.2.10.3. Instructing 9+5b = 0, → b =−9 / 5, We Get

$s^5 \sim \kappa^{14} T^9 / \hbar^{12} G^3$ **?**

5.2.10.4. Ordering 1−b = 0, → b = 1, We Gain

$s \sim c^7 / \hbar G^2 T$?

5.2.10.5. Instructing b =−1, We Obtain

$s \sim \kappa^2 c^2 T / \hbar^2 G$

Above is the entropy density of SBH center [137].

5.2.11. Assuming That Energy Density ϱ Has Relations with Acceleration a Only, We Find

$$\rho a^b \sim \rho_P a_P^b = \hbar^{-(2+\alpha)/2} G^{-(4+\alpha)/2} c^{7(2+\alpha)/2} \quad \textbf{(39)}$$

5.2.11.1. Instructing 2+b = 0, → b = −2, We Obtain

$\rho \sim a^2/G \rightarrow a \sim \sqrt{G\rho}$

That is the relativistic gravitational acceleration.

5.2.11.2. Ordering 4+b = 0, → b = −4, We Gain

$\rho \sim \hbar a^4/c^7 \rightarrow a \sim c\sqrt[4]{\rho c^3/\hbar}$

Substituting Unruh formula $T = \hbar a/2\pi c\kappa$ [134] and Stefan-Boltzmann law $\rho = \pi^2 \kappa^4 T^4/15\hbar^3 c^3$, we get $\rho = \hbar a^4/16\pi^4 c^7$. So it is the quantized acceleration.

5.2.11.3. Instructing b =−6, We Obtain

$\rho \sim \hbar^2 G a^6/c^{14} \rightarrow a \sim c^2 \sqrt[6]{\rho c^2/\hbar^2 G}$

Taking $a \sim c^2/r$ (confer to 3.4.3) to 2.5.4 $\rho \sim \hbar^2 G/c^2 R^6$, where $r \sim R$, we get the above equation, therefore it is the acceleration far from the horizon inside SBH.

5.2.12. Assuming that curvature tensor $R_{\mu\nu}$ has relations with energy-momentum tensor $T_{\mu\nu}$ only, we find

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$$R_{\mu\nu}T_{\mu\nu}^{b}\sim \mathrm{R}_{\mu\nu P}\mathrm{T}_{\mu\nu P}^{b}=\hbar^{-(1+b)}\mathrm{G}^{-(1+2b)}\mathrm{c}^{(3+7b)} \quad (40)$$

5.2.12.1. Ordering 1+b = 0, → b =−1, We Gain

$R_{\mu\nu}\sim GT_{\mu\nu}/c^4$

Above is Einstein field equation [138], hasn't $-Rg_{\mu\nu}/2$ and -8π.

5.2.12.2. Instructing 1+2b = 0, → b =−1 / 2, We Obtain

$R_{\mu\nu}^2\sim T_{\mu\nu}/\hbar c$? or $R_{\mu\nu}R^{\mu\nu}\sim T_{\mu\nu}/\hbar c$?

5.2.12.3. Ordering 3+7b = 0, → b =−3 / 7, We Have

$R_{\mu\nu}^7\sim T_{\mu\nu}^3/\hbar^4 G$?

5.2.13. Supposing that Lagrange Density Function φ Has Relations with Electromagnetic Field tensor $F_{\mu\nu}$ Merely, We Find

$$\varphi F_{\mu\nu}^{b}\sim \varphi_{P}\mathrm{F}_{\mu\nu P}^{b}=\hbar^{-(1+b)}\mathrm{G}^{-(2+b)}\mathrm{c}^{(7+3b)}\mathrm{e}^{b}\sim\hbar^{-(2+b)/2}\mathrm{G}^{-(2+b)}\mathrm{c}^{7(2+b)/2} \quad (41)$$

where $\varphi_P = c^7/\hbar G^2$ is the Planck Lagrange density function, $F_{\mu\nu} = e c^3/\hbar G$ the Planck electromagnetic field tensor, and $e\sim\sqrt{\hbar c}$.

5.2.13.1. Instructing 2+b = 0, → b =−2, We Obtain Only

$\varphi\sim F_{\mu\nu}^2\sim F_{\mu\nu}F^{\mu\nu}$

Above is electromagnetic Lagrange density function under Lorentz gauge [153], hasn't −1 / 4 and $-(\partial_\mu A^\mu)^2/2$.

5.2.14. Assuming that Superfluid Density n_{sf} Has Relations with Voltage V Only, We Find

$$n_{sf}V^{b}\sim \mathrm{n}_{sfP}\mathrm{V}_{P}^{b}=\hbar^{-1}\mathrm{G}^{-(2+b)/2}\mathrm{c}^{(6+5b)/2} \quad (42)$$

where $n_{sfP}=c^3/\hbar G$ is the Planck superfluid density, and $V_P = ec^2/\sqrt{\hbar G}\sim\sqrt{c^5/G}$ the Planck voltage.

5.2.14.1. Ordering 2+b = 0, → b =−2, We Obtain

$n_{sf}\sim V^2/\hbar c^2$

That is $n_{sf}\propto(I_C R_N)^2$, where $V=I_C R_N$, I_C is the critical current intensity of the iron-based superconductor $F_eT_{e0.55}S_{e0.45}$, R_N the normal state resistance [154].

5.2.14.2. Instructing 6+5b = 0, → b =−6 / 5, We Obtain

$n_{sf}^5\sim V^6/\hbar G^2$?

5.3. Power Product Relationship Between Three PQs

This section extends the analysis to systems of three PQs n = 3 under the GRE framework. General formulas are formulated and applied to derive multiple canonical equations, such as the Newton’s law of universal gravitation, Schrödinger equation, Coulomb’s law, Newton’s second law, Clapeyron equation, power law for superconducting films [155-156], and two expressions for the critical temperature of LSCO superconductors [157, 158], etc.

5.3.0. Similarly when n = 3, We Obtain

$$A_1^{a_1}A_2^{a_2}A_3^{a_3}\sim \mathrm{A}_{1P}^{a_1}\mathrm{A}_{2P}^{a_2}\mathrm{A}_{3P}^{a_3} \quad (43)$$

Ordering $a_1=1$, $a_2=b$, $a_3=j$, $A_1=A$, $A_2=B$, and $A_3=C$, we give

$$AB^bC^j\sim \mathrm{A_P B_P^b C_P^j} \quad (44)$$

when $j = 0$, Eq. (26) is recovered. Thus we can determine the power product relationship between three PQs. For example

5.3.1. Assuming That Energy E Has Relations with Mass M and Distance r, We Find

$$EM^b r^j \sim \hbar^{(1+b+j)/2} G^{-(1+b-j)/2} c^{(5+b-3j)/2} \quad (45)$$

5.3.1.1. Instructing 1+b+j = 0, and 5+b−3j = 0 → b =−2 and j = 1, We Obtain

$E \sim GM^2 / r \sim GMm / r$
Above is Newton's law, hasn't −1.

5.3.1.2. Ordering 1+b−j = 0, and 5+b−3j = 0 → b = 1 and j = 2, We Gain

$E \sim \hbar^2 / Mr^2$
Substituting $E \to i\hbar\partial / \partial t$ and $1 / r^2 \to \nabla^2$ into above formula, we obtain
$i\hbar\partial\psi / \partial t \sim \hbar^2 \nabla^2 \psi / M$
where ψ is wave function. That is Schrödinger equation, hasn't −1 / 2.

5.3.1.3. Instructing 1+b+j = 0, and 1+b−j = 0 → b =−1 and j = 0, We Obtain

$E \sim Mc^2$
Above is Einstein's mass-energy relation again.

5.3.1.4. Ordering b =−1 and j =2, We Gain

$E \sim \hbar GM / cr^2$
From Unruh formula $T = 2\pi\hbar a / c\kappa$ [134], $a \sim g$ and $g = GM / r^2$, we have
$T = 2\pi\hbar GM / c\kappa r^2$
So above is the temperature $T \sim E / \kappa$ in Newtonian attraction, hasn't 2π.

5.3.2. Supposing that energy E has relations with electric charge Q and distance r, we find

$$EQ^b r^j \sim \hbar^{(1+j)/2} G^{-(1-j)/2} c^{(5-3j)/2} e^b \sim \hbar^{(1+b+j)/2} G^{-(1-j)/2} c^{(5+b-3j)/2} \quad (46)$$

5.3.2.1 Ordering $1+b+j = 0$, and $1 - j = 0 \to b = -2$ and $j = 1$, also $5+b-3j = 0$, we gain only
$E \sim Q^2 / r \sim Q_1 Q_2 / r$
That is Coulomb law.

5.3.3. Assuming That Acceleration a Has Relations with Force F and Mass M, We Find

$$aF^b M^j \sim \hbar^{-(1-j)/2} G^{-(1+2b+j)/2} c^{(7+8b+j)/2} \quad (47)$$

5.3.3.1. Instructing 1−j = 0, and 1+2b+j = 0 → b =−1 and j = 1, also 7+8b+j = 0, We Obtain Merely

$a \sim F / M$
That is Newton's second law.
Only ordering $1 - j = 0$, $\to j = 1$, we gain
$a \sim G^{-(1+b)} c^{4(1+b)} F^{-b} / M$
when $b = -2$, we have
$a \sim GF^2 / Mc^4$
this is the relativistic gravity acceleration modifier.

5.3.4. Supposing That Acceleration a Has Relations with Mass M and Distance r, We Find

$$aM^b r^j \sim \hbar^{-(1-b-j)/2} G^{-(1+b-j)/2} c^{(7+b-3j)/2} \quad (48)$$

5.3.4.1. Ordering 1−b−j = 0, and 7+b−3j = 0 → b =−1 and j = 2, We Gain

$a\sim GM/r^2$

That is Newtonian gravitational acceleration.

5.3.4.2. Instructing 1+b−j = 0, and 7+b−3j = 0 → b = 2 and j = 3, We Have

$$a\sim\hbar^2/M^2r^3\rightarrow r\sim\sqrt[3]{\hbar^2/bM^2}$$

Above is $h_n=\sqrt[3]{9[(n-\frac{1}{4})\pi\hbar/m]^2/8g}$ [159] probably, where $h_n\sim r$ is the height of the nth energy level, $m\sim M$ the neutron mass and $g\sim a$ the Earth's gravitational acceleration.

5.3.4.3. Ordering 1−b−j = 0, and 1+b−j = 0 → b = 0 and j = 1, We Obtain

$$a\sim c^2/r$$

From $\rho_{de}=3c_L^2c^3M_{pl}^2L^{-2}$, $p=\omega\rho$, $F\sim pL^2$, $a\sim F/M$, $Mc^2=\rho V$ and $V\sim L^3$, we gain

$$a\sim 3w_{de}c^2/8\pi L$$

where $r\sim L$. Above is the acceleration of HDE, hasn't $3w_{de}/8\pi$.

5.3.5. Assuming That Pressure p Has Relations with Volume V and Temperature T, We Find

$$pV^bT^j\sim p_PV_P^bT_P^j=\hbar^{-(2-3b-j)/2}G^{-(4-3b+j)/2}c^{(14-9b+5j)/2}\kappa^{-j} \quad (49)$$

5.3.5.1. Instructing 2−3b−j = 0, and 4−3b+j = 0 → b = 1 and j =−1, also 14−9b+5j = 0, we obtain only

$$pV\sim\kappa T$$

That is Clapeyron equation, hasn't WN_A/M, where W is the gaseous mass, N_A the Avogadro constant and M the mass of gaseous mole molecule.

5.3.6. Assuming That Thickness d Has Relations with Temperature T and Resistance R, We Find

$$dT^bR^j\sim L_PT_P^bR_P^j=\hbar^{(1+b)/2}G^{(1-b)/2}c^{-(3-5b+2j)/2}\kappa^{-b} \quad (50)$$

where $R_P=\hbar/e^2\sim 1/c$ is the Planck resistance.

5.3.6.1. Ordering 1−b = 0 → b = 1, We Obtain

$$dT\sim\hbar c^{(1-j)}\kappa^{-1}R^{-j}$$

Above is the superconducting thin film power law $dT_c=AR_S^{-B}$ [155-156], where T_c is critical temperature, R_S sheet resistance, A and B are fitting parameters. When $j=1$,we get $dT\sim\hbar\kappa^{-1}R^{-1}$.

5.3.6.2. Instructing 1−b = 0, and 3−5b+2j = 0 → b = 1 and j = 1, We Gain Also

$$dT\sim\hbar\kappa^{-1}R^{-1}$$

5.3.6.3. Ordering 1+b = 0 and 3−5b+2j = 0→ b =−1 and j =−4, We Obtain

$d\sim G\kappa TR^4$?

5.3.7. Supposing that Temperature T Has Relations with Superfluid Density ρ_s and Mass m, We Find

$$T\rho_s^bm^j\sim T_P\rho_{sP}^bM_P^j=\hbar^{(1-2b+j)/2}G^{-(1+2b+j)/2}c^{(5+6b+j)/2}\kappa^{-1} \quad (51)$$

where $\rho_{sP}=c^3/\hbar G$ is the Planck superfluid density.

5.3.7.1. Ordering 1+2b+j = 0→j =−(1+2b), We Get

$T\sim\hbar^{-2b}c^{2(1+b)}\kappa^{-1}\rho_s^{-b}m^{(1+2b)}$

(1) Instructing $1+b=0\rightarrow b=-1$, we obtain

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$T \sim \hbar^2 \rho_s / \kappa m$

That is the Uemura's law $T_c \propto n_{so} / m^*$ [160] or one of the two formulas of critical temperature of LSCO [157, 158] and its superfluid density $T_c = T_0 + \alpha \rho_{s0}$, where n_{so} is the densityof superconductingelectrons, m^* the electron effective mass, T_0= (7.0±0.1)K and α= 0.37±0.02 [161].

(2) Ordering $1+2b=0 \to b=-1/2$,wegain

$T \sim \hbar c \sqrt{\rho_s} / \kappa$

Above is the other one of the two formulas of LSCO [157, 158] $T_c = \gamma \sqrt{\rho_{so}}$, where γ= (4.2±0.5) $K^{1/2}$ [161].

5.3.7.2. Ordering 1−2b+j = 0 and 5+6b+j = 0, → b =−1 / 2, j =−2, We Gain

$T \sim Gm^2 \sqrt{\rho_s} / \kappa$?

5.3.8. Assuming that Conductivity ρ_R Has Relations with Temperature T and Carrier Density n_C, We Find

$$\rho_R T^b n_C^j \sim \rho_{RP} T_P^b n_{CP}^j = \hbar^{(1+b-3j)/2} G^{-(1-b-3j)/2} c^{-(5-5b-9j)/2} \kappa^{-b} \quad (52)$$

where $\rho_{RP} = \sqrt{\hbar G / c^5}$ is the Planck conductivity, $n_{CP} = (\sqrt{c^3/\hbar G})^3$ the Planck carrier density.

5.3.8.1. Ordering 1−b−3j = 0, and 5−5b−9j = N → b =(2−N) / 2 and j = N / 6, Where N is a Fitted Number, We Gain

$\rho_R T^{(2-N)/2} n_C^{N/6} \sim \hbar^{(2-N)/2} c^{-N/2} \kappa^{-(2-N)/2}$

(1) Instructing N= 6, we obtain

$\rho_R \sim \kappa^2 T^2 / \hbar^2 c^3 n_C \sim AT^2$

where $A \sim \kappa^2 / \hbar^2 c^3 n_C$, that is the relation of the conductivity and temperature of monocrystalline $Sr_{1-x}La_xTiO_3$ [162].

(2) Ordering N= 4, we obtain

$\rho_R \sim \kappa T / \hbar c^2 n_C^{2/3}$?

(3) Instructing N= 3, we obtain

$\rho_R^2 \sim \kappa T / \hbar c^3 n_C$?

5.3.8.2. Instructing 1+b−3j = 0, and 5−5b−9j = N → b =(2−N) / 8 and j = (10−N) / 24, We Gain

$\rho_R T^{(2-N)/8} n_C^{(10-N)/24} \sim G^{-(2-N)/8} c^{-N/2} \kappa^{-(2-N)/8}$

Ordering N= 4, we obtain

$\rho_R^4 \sim G\kappa T / c^8 n_C$?

5.3.9. Supposing that Force F Has Relations with Hamiltonian Function H and Curvature k, We Find

$$FH^b k^j \sim F_P H_P^b k_P^j = \hbar^{-(b-j)/2} G^{-(2+b+j)/2} c^{(8+5b+3j)/2} \quad (53)$$

where $H_P = \sqrt{\hbar c^5 / G}$ is the Planck Hamiltonian function and $k_P = \sqrt{c^3/\hbar G}$ the Planck curvature.

5.3.9.1. Instructing b−j = 0, and 2+b+j = 0 → b =−1 and j =−1, also 8+5b+3j = 0, We Obtain Merely

$F \sim Hk$

That is the generalized CFL $dP / dt = -2Hkn$ [163], where P is the momentum, n the local unit normal vector, and $F \sim dP / dt$, hasn't $-2n$.

5.4. Power Product Relationship Between Four PQs

Here, we consider power product relations involving four PQs $n = 4$ via the GRE. This approach yields the centrifugal force formula, among other relations, demonstrating the applicability of the framework to more complex physical systems.

5.4.0. Similarly when n = 4, We Obtain

$$A_1^{a_1}A_2^{a_2}A_3^{a_3}A_4^{a_4}\sim A_{1P}^{a_1}A_{2P}^{a_2}A_{3P}^{a_3}A_{4P}^{a_4} \quad (54)$$

Instructing b_1= 1, b_2 = b, b_3 = j, b_3 = l, A_1= A, A_2= B, A_3= C and A_4= D, we gain

$$AB^bC^jD^l\sim A_PB_P^bC_P^jD_P^l \quad (55)$$

when l= 0, Eq. (44) is recovered. Therefore, we can determine the power product relationship between four PQs. For example

5.4.1. Supposing that Force F has Relations with Mass M, Speed v and Distance r, We Find

$$FM^bv^jr^l\sim \hbar^{(b+l)/2}G^{-(2+b-l)/2}c^{(8+b+2j-3l)/2} \quad (56)$$

5.4.1.1. Ordering b+l = 0, 2+b−l = 0 and 8+b+2j−3l = 0

→ b =−1, j =−2 and l = 1, we obtain

F∼Mv^2 / r

Above is the centrifugal force formula.

And so on.

6. Conclusion

In this paper, we have systematically investigated dimensional URs applying dimensional analysis. The main results are summarized as follows

(1) The standard form of URs were identified, wherein products of PQs on the left-hand side are equated to power products of fundamental constants such as the reduced Planck constant ħ, gravitational constant G, speed of light in vacuum c and Boltzmann constant κ are on right hand. These power products of physical constants which are rewritten appear.

(2) General Expression for URs was derived showing that the product of two or n non-commutative dimensional PQs is equivalent to a power product of ħ, G, c, κ and elementary charge e.

(3) Basic Relationship was demonstrated that every dimensional PQ corresponds to a Planck scale, expressible as a power product of the same fundamental constants. That is PQs and Planck scales having the supersymmetry [98-104].

(4) Planck Scales including Planck length L_P, Planck time t_P, Planck mass M_P, Planck temperature T_P, elementary charge Q_e(or Planck charge), Planck energy E_P, Planck momentum P_P,Planck curvature tensor $R_{\mu\nu P}$, Planck energy density ρ_P, Planck pressure p_P, Planck energy-momentum tensor $T_{\mu\nu P}$ etc. were rederived. Many PQs of identical dimension share the same Planck scale such as ρ_P, p_P and $T_{\mu\nu P}$.

(5) Planck scales were classified into two categories. First is the basic Planck scale such as L_P, t_P, M_P, T_P and Q_e, derived one for example E_P, P_P, ρ_P, p_P, $R_{\mu\nu P}$, $T_{\mu\nu P}$, and other scales such as Planck wave function ψ_P. The second is the Femi-Planck scale its exponent being half integer such as L_P, t_P, M_P, T_P, E_P, P_P, etc, the Bose-Planck scale whose exponents are integers such as Q_e, ρ_P, p_P, $R_{\mu\nu P}$, $T_{\mu\nu P}$, etc, and Other-Planck scale such as Planck wave function ψ_P.

(6) The Planck scale for any PQ was shown to be expressible as a power product of the basic Planck scales L_P, t_P, M_P, T_P and Q_e.

(7) The GRE was proposed and proved, which states that a power product of non-commutative PQs equals the one of their corresponding Planck scales. This GRE was used to verify the URs in Section 1, explaining the absence of G in some relations through dimensional reduction.

(8) Applying the GRE, some significant URs were derived: a Big Bang UR between temperature T_B and volume V_B was, suggesting the avoidance of the initial singularity with quantum gravity effects; a related UR between acceleration a_B and volume V_B; a SBH UR between mass M_H and volume V_H, also indicating the absence of a singularity under quantum effects; URs between the density ρ_H of a SBH and its mass M_H or volume V_H.

(9) The GRE provides a unified framework for a broad class of dimensional URs. It reproduces known URs as special cases. Note that dimensional arguments alone cannot determine numerical prefactors or fully capture dimensionless relations.

(10) Monomial scaling relations between two PQs were derived for the case $n = 2$ within the GRE framework. In particular, direct or inverse proportionality between two quantities arises when their exponents equal 1 or –1, respectively.

(11) General formulae were obtained by introducing physical assumptions relating energy to mass, energy to frequency, energy density to distance, force per unit area to distance, radiation density to temperature, energy density to temperature, acceleration to temperature, entropy density to temperature, energy density to acceleration, curvature tensor to the energy-momentum tensor, Lagrange density function to the electromagnetic field tensor, superfluid density to voltage, and so on.

(12) Numerous fundamental physical equations were recovered without prefactors, including the Einstein's mass– energy relation, event horizon temperature of a SBH [3], light quantum relation, inverse correlation between high-frequency quasi-periodic oscillation and black hole mass [139-147], relativistic gravitational energy, biquadratic relation between photon energy and energy density [139-147], event horizon radius of stationary black holes [131], A.H. Compton wavelength formula, age of a SBH [3], observed density of dark energy [135, 136], Casimir effect equation, relativistic gravitational pressure or negative pressure in HDE [137, 148-152], Planck blackbody radiation law, Stefan-Boltzmann law, relativistic gravitational energy density with square temperature, Unruh formula [134], relativistic gravitational temperature, cubic relation between entropy density and temperature [144], relativistic gravitational entropy density, relativistic gravitational acceleration, quantized acceleration, Einstein field equations [138], electromagnetic Lagrange density function under the Lorentz gauge [153], relation of quasiparticle character and superfluid density of $F_eT_{e0.55}S_{e0.45}$ [154], and so on.

(13) Several new relations were identified, including those between the square of energy and its density in SBH, the volume of event horizon of stationary black holes and its age, the energy density and the sixth power of the radius in SBH, the central pressure inside an SBH, the gravitational energy density far within the horizon, and the entropy density at the SBH center [137], the acceleration far from the horizon inside SBH.

(14) The analysis was extended to systems of three and four PQs, corresponding to $n = 3$ or 4 in the GRE, respectively.

(15) Additional general formulae were formulated by postulating relations among energy, mass and distance; energy, charge and distance; acceleration, force and mass; acceleration, mass and distance; pressure, volume and temperature; thickness, temperature and resistance; temperature, superfluid density and mass; conductivity, temperature and carrier density; force, Hamiltonian function and curvature, etc.

(16) Many well-known factor-free equations were reproduced, including Newton's law, Schrödinger equation, the temperature in Newtonian attraction, Coulomb law, Newton's second law, Newtonian gravitational acceleration, height of the nth energy level of neutrons in the Earth's gravitational field [159], acceleration of HDE, Clapeyron equation, superconducting thin film power law [155-156], Uemura's law [160], two formulas of critical temperature of LSCO [157, 158], relation of the conductivity and temperature of monocrystalline $Sr_{1-x}La_xTiO_3$ [162], generalized CFL [163], and centrifugal force formula.

(17) Certain derived relations currently lack a clear physical interpretation.

(18) Three methods are used to determine the relationships of three PQs, one is the exponential equation of G and c, ħ and c or ħ and G being equal to zero; another is the one of G being equal to zero, then consider the circumstances of ħ and c; the third is the one of G being equal to zero and one of c being equal to a fitted number, because the exponential equation of c is not necessarily equal to zero.

(19) The GRE proves to be a powerful tool for determining power product relationships among two, three, and four PQs, although it does not predict numerical prefactors. This approach offers a unified and conceptually significant method for deriving scaling laws across multiple domains of physics.

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