Article

Not peer-reviewed version

Hypercomplex Dynamics and Turbulent Flows in Sobolev and Besov Spaces: A Rigorous Analysis of the Navier-Stokes Equations

Rômulo Damasclin Chaves dos Santos * and Jorge Henrique de Oliveira Sales *

Posted Date: 7 October 2025

doi: 10.20944/preprints202510.0499.v1

Keywords: Sobolev and Besov spaces; Navier-Stokes equations; hypercomplex bifurcations; Littlewood-Paley decomposition; Millennium Prize Problem

doi:10.20944/preprints202510.0499.v1

Article

Hypercomplex Dynamics and Turbulent Flows in Sobolev and Besov Spaces: A Rigorous Analysis of the Navier-Stokes Equations

Rômulo Damasclin Chaves dos Santos * and Jorge Henrique de Oliveira Sales *

Department of Exact Sciences, Santa Cruz State University, Ilhéus, Bahia, Brazil
* Correspondence: rdcsantos@uesc.br (R.D.C.d.S.); jhosales@uesc.br (J.H.d.O.S.)

Abstract

This study presents a rigorous mathematical framework for the analysis of the Navier-Stokes equations within the context of **Sobolev and Besov functional spaces**, with a particular emphasis on the **regularity of solutions, hypercomplex bifurcations, and turbulence** in fluid dynamics. By employing advanced mathematical tools such as **interpolation theory, Littlewood-Paley decomposition, and energy cascade models**, we provide a comprehensive analysis of the intricate behaviors exhibited by fluid systems. The research establishes **higher-order Sobolev regularity** for solutions to the Navier-Stokes equations, demonstrating enhanced smoothness under appropriate conditions on initial data and external forces. Additionally, the **characterization of Besov spaces** through the Littlewood-Paley decomposition captures multifractal and irregular behaviors in turbulent flows, offering critical insights into energy dissipation mechanisms. A **quaternionic formulation** of the Navier-Stokes equations is introduced, providing a novel approach to modeling rotational symmetries and bifurcation phenomena in three-dimensional fluid dynamics. The study further confirms the **regularity and uniqueness of solutions** in Besov spaces, contributing to the ongoing exploration of the Millennium Prize Problem. Overall, this work advances the mathematical understanding of fluid dynamics and establishes a robust foundation for future research in this challenging field.

Keywords: Sobolev and Besov spaces; Navier-Stokes equations; hypercomplex bifurcations; Littlewood-Paley decomposition; Millennium Prize Problem

---

1. Introduction

The investigation of regularity, bifurcation phenomena, and turbulence in fluid dynamics has been a central theme in mathematical analysis, particularly in the context of the incompressible Navier–Stokes equations. This work employs the rigorous functional framework of Sobolev and Besov spaces to address these foundational challenges. Our approach extends classical bifurcation theory to encompass hypercomplex dynamical systems by incorporating quaternionic structures, which naturally encode the rotational symmetries intrinsic to fluid flows.

We first establish higher-order regularity theorems for solutions to the Navier–Stokes equations within Sobolev spaces, providing refined estimates that extend beyond classical results. Subsequently, we conduct a detailed analysis of Besov spaces through the Littlewood–Paley decomposition, highlighting their critical role in capturing the multiscale nature of turbulent flows. Furthermore, we explore the intricate structure of bifurcations in quaternionic dynamical systems, offering novel applications to the rotational dynamics of fluids.

Building on these foundations, we propose a comprehensive theoretical framework that unifies the analysis of regularity, bifurcation, and turbulence in fluid mechanics. This framework not only strengthens the mathematical understanding of the Navier–Stokes equations but also contributes to the broader effort toward resolving the Millennium Prize Problem, offering fresh perspectives on the deep mathematical structure governing fluid flows.

The seminal work of Constantin and Foias [1] provides a profound treatment of the long-time behavior and regularity properties of solutions, forming a cornerstone of the modern theory. The geometric underpinnings of fluid dynamics are elegantly articulated in the contributions of Marsden and Ratiu [2], whose exploration of mechanics and symmetry underscores the significance of rotational invariants. Triebel's foundational work on function spaces [3], particularly Sobolev and Besov spaces, furnishes a rigorous analytical backbone for the study of partial differential equations (PDEs) and turbulence.

Complementary perspectives are provided by Temam [4], whose comprehensive treatment encompasses both theoretical and numerical aspects, and by Doering and Gibbon [5], whose work emphasizes the critical dynamics underlying regularity and turbulence. The mathematical theory of viscous incompressible flow has been profoundly shaped by the pioneering contributions of Ladyzhenskaya [6], whose rigorous analysis laid the groundwork for contemporary developments in regularity theory. Sohr [7] offers an elegant functional analytic formulation, while Galdi [8] extends the analysis to steady-state regimes and further regularity properties.

A pivotal contribution to the harmonic analysis of PDEs is found in the work of Bahouri, Chemin, and Danchin [9], who advanced Fourier-based techniques specifically tailored to nonlinear PDEs, greatly enhancing the treatment of regularity within Sobolev and Besov frameworks. The foundational texts by Grafakos [10] and Stein [11] remain indispensable in harmonic analysis, providing essential tools that underpin modern approaches to the Navier–Stokes equations. Muscalu and Schlag [12] further developed advanced methods in both classical and multilinear harmonic analysis, while Runst and Sickel [13] presented a thorough exposition of Sobolev spaces of fractional order, critical for handling nonlinear PDEs.

The classical reference by Adams and Fournier [14] remains a fundamental resource for understanding Sobolev space theory and its applications to PDE regularity. Evans' textbook [15] continues to serve as an authoritative reference for PDE theory, balancing clarity with mathematical rigor. In parallel, the work of Bertozzi and Majda [16] provides deep insights into vorticity dynamics, incompressible flows, and the onset of turbulence, particularly in relation to bifurcation phenomena. The classical results of Fefferman and Stein [17] on H^p spaces are pivotal for the development of modern harmonic analysis in several variables, directly impacting the analysis of fluid equations. Cannone [18] significantly contributed to the application of wavelet-based techniques and paraproducts in the analysis of turbulence within the Navier–Stokes framework. Lemarié-Rieusset [19] offers an exhaustive survey of the modern developments surrounding the Navier–Stokes equations, while Chemin's influential monograph [20] on perfect incompressible fluids delves deeply into the interplay between bifurcation dynamics and turbulence formation. More recently, dos Santos and Sales [22] proposed a robust mathematical framework that advances the regularity theory of the Navier–Stokes equations. Their innovative integration of the Smagorinsky model within the Large Eddy Simulation (LES) paradigm, grounded in Banach and Sobolev space theory, led to the formulation of a novel theorem that supports the development of anisotropic viscosity models. Their contribution provides rigorous analytical tools that advance the understanding of the regularity problem from both a theoretical and applied perspective.

In summary, this work advances the mathematical analysis of the Navier–Stokes equations, with a particular focus on the interplay between Sobolev and Besov spaces and the nonlinear dynamics of fluid flows. By synthesizing classical theories with contemporary analytical tools including quaternionic dynamics, harmonic analysis, and bifurcation theory it offers a unified and rigorous framework for addressing regularity, bifurcation, and turbulence. Ultimately, this contributes to the deeper mathematical foundations required for progress toward resolving the Millennium Prize Problem associated with the Navier–Stokes equations.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

2. Sobolev Spaces and Regularity of Navier–Stokes Solutions

Definition 1 (Functional Framework and Governing Equations). *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $T > 0$. The incompressible Navier–Stokes equations read:*

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{u} = 0, & \text{in } \Omega \times (0,T), \\ \mathbf{u}|_{\partial\Omega} = 0, \quad \mathbf{u}(\cdot,0) = \mathbf{u}_0. \end{cases} \tag{1}$$

We denote by $L^2_\sigma(\Omega)$ the closure of divergence-free smooth functions in $L^2(\Omega)$ and by $H^1_{0,\sigma}(\Omega)$ its closure in $H^1_0(\Omega)$.

Definition 2 (Leray–Hopf Weak Solution). *Let $\mathbf{u}_0 \in L^2_\sigma(\Omega)$ and $\mathbf{f} \in L^2(0,T;H^{-1}(\Omega))$. A function*

$$\mathbf{u} \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;H^1_{0,\sigma}(\Omega))$$

is a Leray–Hopf weak solution *of* (1) *if it satisfies, for all divergence-free $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^n)$,*

$$-\int_0^T (\mathbf{u}, \partial_t \varphi)\, dt + \int_0^T ((\mathbf{u} \cdot \nabla)\mathbf{u}, \varphi)\, dt + \nu \int_0^T (\nabla \mathbf{u}, \nabla \varphi)\, dt = (\mathbf{u}_0, \varphi(\cdot,0)) + \int_0^T \langle \mathbf{f}, \varphi \rangle\, dt,$$

and the global energy inequality

$$\frac{1}{2}\|\mathbf{u}(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla \mathbf{u}(s)\|_{L^2}^2\, ds \le \frac{1}{2}\|\mathbf{u}_0\|_{L^2}^2 + \int_0^t \langle \mathbf{f}(s), \mathbf{u}(s) \rangle\, ds. \tag{2}$$

2.1. Technical Lemmas

Lemma 1 (Gagliardo–Nirenberg Inequality). *Let $u \in H^1_0(\Omega)$ and $\Omega \subset \mathbb{R}^3$. Then*

$$\|u\|_{L^4(\Omega)} \le C\|u\|_{L^2(\Omega)}^{1/4} \|\nabla u\|_{L^2(\Omega)}^{3/4}.$$

Proof. The inequality follows from interpolation between $H^1(\Omega) \hookrightarrow L^6(\Omega)$ and $L^2(\Omega) \hookrightarrow L^2(\Omega)$, using the Riesz–Thorin theorem with $\theta = 3/4$. Explicitly,

$$\|u\|_{L^4} \le \|u\|_{L^2}^{1-\theta} \|u\|_{L^6}^{\theta} \le C\|u\|_{L^2}^{1/4} \|\nabla u\|_{L^2}^{3/4}.$$

□

Lemma 2 (Sobolev Algebra Property). *If $s > n/2$, then $H^s(\Omega)$ is an algebra:*

$$\|uv\|_{H^s} \le C\|u\|_{H^s}\|v\|_{H^s}.$$

Proof. Using Fourier transform, $\widehat{uv}(\xi) = \int \hat{u}(\eta)\hat{v}(\xi - \eta)d\eta$, we obtain

$$(1 + |\xi|^2)^{s/2}|\widehat{uv}(\xi)| \le C \int_{\mathbb{R}^n} (1 + |\eta|^2)^{s/2}|\hat{u}(\eta)| \times (1 + |\xi - \eta|^2)^{s/2}|\hat{v}(\xi - \eta)|\, d\eta. \tag{3}$$

The estimate follows by Young's inequality and the embedding $H^s \hookrightarrow L^\infty$ for $s > n/2$. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

Lemma 3 (Kato–Ponce Commutator Estimate)**.** *Let $s > 0$ and $1 < p < \infty$. Then for smooth $f, g : \mathbb{R}^n \to \mathbb{R}$,*

$$\|\Lambda^s(fg) - f\Lambda^s g - g\Lambda^s f\|_{L^p} \le C\Big(\|\nabla f\|_{L^{p_1}}\|\Lambda^{s-1}g\|_{L^{p_2}} + \|\nabla g\|_{L^{q_1}}\|\Lambda^{s-1}f\|_{L^{q_2}}\Big), \tag{4}$$

where $\Lambda^s = (I-\Delta)^{s/2}$ and $1/p = 1/p_1 + 1/p_2 = 1/q_1 + 1/q_2$.

Proof. Use Bony's paraproduct decomposition $fg = T_f g + T_g f + R(f,g)$, apply Λ^s and estimate each dyadic block via Littlewood–Paley theory. Orthogonality and Bernstein inequalities yield the stated bound. □

2.2. Higher-Order Sobolev Regularity

Theorem 1 (Higher-Order Sobolev Regularity)**.** *Let $\mathbf{u}_0 \in H^s_\sigma(\Omega)$, $\mathbf{f} \in L^2(0,T;H^{s-1}(\Omega))$, and $s > n/2$. Then the Leray–Hopf solution $\mathbf{u}$ satisfies*

$$\mathbf{u} \in L^\infty(0,T;H^s_\sigma(\Omega)) \cap L^2(0,T;H^{s+1}_\sigma(\Omega)),$$

with the estimate

$$\|\mathbf{u}(t)\|^2_{H^s} + \nu\int_0^t \|\nabla\mathbf{u}(\tau)\|^2_{H^s}\,d\tau \le C\Big(\|\mathbf{u}_0\|^2_{H^s} + \int_0^t \|\mathbf{f}(\tau)\|^2_{H^{s-1}}\,d\tau\Big).$$

Proof. Apply Λ^s to (1) and take the L^2 inner product with $\Lambda^s\mathbf{u}$. We obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s\mathbf{u}\|^2_{L^2} + \nu\|\nabla\Lambda^s\mathbf{u}\|^2_{L^2} &= -(\Lambda^s((\mathbf{u}\cdot\nabla)\mathbf{u}),\Lambda^s\mathbf{u}) \\ &= -([\Lambda^s,\mathbf{u}]\cdot\nabla\mathbf{u},\Lambda^s\mathbf{u}) + (\Lambda^s\mathbf{f},\Lambda^s\mathbf{u}).\end{aligned}$$

Using Lemma 3 and Young's inequality,

$$\begin{aligned}|([\Lambda^s,\mathbf{u}]\cdot\nabla\mathbf{u},\Lambda^s\mathbf{u})| &\le C\|\nabla\mathbf{u}\|_{L^\infty}\|\mathbf{u}\|^2_{H^s} \\ &\le C\|\mathbf{u}\|_{H^s}\|\mathbf{u}\|^2_{H^s} \\ &\le \frac{\nu}{2}\|\nabla\mathbf{u}\|^2_{H^s} + C_\nu\|\mathbf{u}\|^4_{H^s}.\end{aligned}$$

Gronwall's inequality then yields the stated H^s estimate. □

2.3. Mild Solutions and Higher-Order Sobolev Estimates

We now consider the Navier–Stokes equations in mild form:

$$\mathbf{u}(t) = e^{\nu t\Delta}\mathbf{u}_0 - \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})(\tau)\,d\tau + \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\mathbf{f}(\tau)\,d\tau.$$

Theorem 2 (High-Order Sobolev Regularity via Mild Solutions)**.** *Let $s > n/2$, $\mathbf{u}_0 \in H^s_\sigma(\Omega)$, and $\mathbf{f} \in L^2(0,T;H^{s-1}(\Omega))$. Then the mild solution $\mathbf{u}$ satisfies*

$$\mathbf{u} \in C([0,T];H^s_\sigma(\Omega)) \cap L^2(0,T;H^{s+1}_\sigma(\Omega))$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

with the estimate

$$\|\mathbf{u}(t)\|_{H^s} \leq \|\mathbf{u}_0\|_{H^s} + C\int_0^t \|\mathbf{u}(\tau)\|_{H^s}^2 \, d\tau + \int_0^t \|\mathbf{f}(\tau)\|_{H^{s-1}} \, d\tau.$$

Proof. We apply $\Lambda^s = (I-\Delta)^{s/2}$ and estimate each term:

1. Linear term:

$$\|\Lambda^s e^{\nu t\Delta}\mathbf{u}_0\|_{L^2} \leq \|\mathbf{u}_0\|_{H^s}.$$

2. Non-linear term: Using Lemma 3 and the algebra property of H^s for $s > n/2$,

$$\left\|\Lambda^s \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})\, d\tau\right\|_{L^2} \leq \int_0^t \|\Lambda^s\nabla\cdot(\mathbf{u}\otimes\mathbf{u})(\tau)\|_{L^2}\, d\tau \leq C\int_0^t \|\mathbf{u}(\tau)\|_{H^s}^2\, d\tau.$$

3. Forcing term: The smoothing property of the semigroup gives

$$\left\|\Lambda^s \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\mathbf{f}(\tau)\, d\tau\right\|_{L^2} \leq \int_0^t \|\mathbf{f}(\tau)\|_{H^{s-1}}\, d\tau.$$

Combining the three estimates, we obtain

$$\|\mathbf{u}(t)\|_{H^s} \leq \|\mathbf{u}_0\|_{H^s} + C\int_0^t \|\mathbf{u}(\tau)\|_{H^s}^2\, d\tau + \int_0^t \|\mathbf{f}(\tau)\|_{H^{s-1}}\, d\tau.$$

Applying Gronwall's lemma in the integral form gives global-in-time bounds for $\mathbf{u}$ in H^s, and the semigroup smoothing ensures $\mathbf{u} \in L^2(0,T;H^{s+1})$.

4. Asymptotic Expansion: Iterating the mild formulation gives the Duhamel expansion

$$\mathbf{u}(t) = e^{\nu t\Delta}\mathbf{u}_0 - \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\nabla\cdot\left(e^{\nu\tau\Delta}\mathbf{u}_0 \otimes e^{\nu\tau\Delta}\mathbf{u}_0\right) d\tau + \mathcal{O}\Big(\int_0^t \|\mathbf{u}(\tau)\|_{H^s}^3\, d\tau\Big),$$

which shows explicitly how higher-order Sobolev norms evolve and how the non-linear interactions contribute in a controlled manner. □

Remark 1. *The above approach gives a rigorous framework for controlling the non-linear term in H^s via Kato–Ponce commutator estimates, semigroup smoothing, and Gronwall inequalities. This establishes both existence, uniqueness (locally), and higher-order regularity of mild solutions in Sobolev spaces.*

2.4. Voronovskaya-Type Expansion for Navier–Stokes in Sobolev Spaces

Consider a smooth solution of the Navier–Stokes equations and decompose it via the Littlewood–Paley dyadic blocks:

$$\mathbf{u} = \sum_{j\geq -1} \Delta_j \mathbf{u}, \qquad \Delta_j\mathbf{u} = \mathcal{F}^{-1}(\varphi_j \hat{\mathbf{u}}),$$

where φ_j are smooth frequency cut-offs supported on dyadic annuli.

Theorem 3 (Voronovskaya-Type Expansion [21]). *Let $s > n/2$, $\mathbf{u}_0 \in H^s_\sigma(\Omega)$, and $\mathbf{f} \in L^2(0,T;H^{s-1})$. For each dyadic block $\Delta_j\mathbf{u}$, the solution admits the decomposition*

$$\begin{aligned}\Delta_j\mathbf{u}(t) &= e^{\nu t\Delta}\Delta_j\mathbf{u}_0 \\ &\quad - \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\nabla\cdot(\Delta_j(\mathbf{u}\otimes\mathbf{u}))(\tau)\,d\tau \\ &\quad + \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\Delta_j\mathbf{f}(\tau)\,d\tau \\ &= e^{\nu t\Delta}\Delta_j\mathbf{u}_0 - t\,\mathbb{P}\nabla\cdot(\Delta_j(\mathbf{u}_0\otimes\mathbf{u}_0)) + R_j(t),\end{aligned}$$

where the remainder $R_j(t)$ satisfies, for sufficiently small $t > 0$,

$$\|R_j(t)\|_{L^2} \le Ct^2 2^{2js}\|\mathbf{u}_0\|^3_{H^s} + Ct\int_0^t \|\Delta_j\mathbf{f}(\tau)\|_{L^2}\,d\tau.$$

This expansion explicitly separates the linear evolution, the leading-order nonlinearity, and the higher-order remainder, providing precise control of the solution in Sobolev spaces.

Proof. The first equality follows directly from the mild formulation of the solution applied to each dyadic block. Expanding the non-linear term around the initial data yields the linear approximation $t\,\mathbb{P}\nabla\cdot(\Delta_j(\mathbf{u}_0\otimes\mathbf{u}_0))$, while the remainder is controlled by Taylor expansion and the smoothing properties of the semigroup. The estimate on $R_j(t)$ follows from Bernstein inequalities and the Sobolev embedding $H^s \hookrightarrow L^\infty$ for $s > n/2$, ensuring that the nonlinear interactions are bounded in L^2 for small times. □

Proof. Applying the dyadic block operator to the mild formulation, the solution can be expressed as

$$\begin{aligned}\Delta_j\mathbf{u}(t) &= e^{\nu t\Delta}\Delta_j\mathbf{u}_0 \\ &\quad - \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\Delta_j\nabla\cdot(\mathbf{u}\otimes\mathbf{u})(\tau)\,d\tau \\ &\quad + \int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\Delta_j\mathbf{f}(\tau)\,d\tau.\end{aligned}$$

Expanding the nonlinear term around the initial data yields

$$(\mathbf{u}\otimes\mathbf{u})(\tau) = \mathbf{u}_0\otimes\mathbf{u}_0 + \tau\,\partial_t(\mathbf{u}\otimes\mathbf{u})|_{\tau=0} + \mathcal{O}(\tau^2),$$

so that the linear approximation in time is given by $t\,\mathbb{P}\nabla\cdot(\Delta_j(\mathbf{u}_0\otimes\mathbf{u}_0))$, while the remainder is controlled using dyadic Sobolev estimates:

$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\Delta_j\nabla\cdot\mathcal{O}(\tau^2)\,d\tau\right\|_{L^2} \le Ct^2 2^{2js}\|\mathbf{u}_0\|^3_{H^s}.$$

The contribution of the forcing term is handled similarly, exploiting linearity and smoothing of the semigroup, giving

$$\left\|\int_0^t e^{\nu(t-\tau)\Delta}\mathbb{P}\Delta_j\mathbf{f}(\tau)\,d\tau\right\|_{L^2} \le Ct\int_0^t \|\Delta_j\mathbf{f}(\tau)\|_{L^2}\,d\tau.$$

Summation over all dyadic blocks using Littlewood–Paley equivalence then yields the expansion in H^s with explicit control of the remainder:

$$\begin{aligned}&\|\mathbf{u}(t) - e^{\nu t\Delta}\mathbf{u}_0 + t\mathbb{P}\nabla\cdot(\mathbf{u}_0\otimes\mathbf{u}_0)\|_{H^s} \\ &\quad \le Ct^2\|\mathbf{u}_0\|^3_{H^s} \\ &\qquad + Ct\int_0^t \|\mathbf{f}(\tau)\|_{H^{s-1}}\,d\tau.\end{aligned}$$

 doi:10.20944/preprints202510.0499.v1

This establishes the Voronovskaya-type expansion with explicit remainder estimates in Sobolev spaces. □

Remark 2. *This decomposition allows a precise understanding of how the quadratic non-linearity contributes to the first-order evolution of each dyadic block and ensures that higher-order Sobolev norms remain controlled for small times. It also provides a framework for constructing **iterated expansions**, useful in turbulence analysis and high-regularity perturbation theory.*

3. Advanced Sobolev Spaces, Besov Spaces, and Fractional Regularity

Fractional Sobolev and Besov spaces provide a unified framework for quantifying smoothness beyond integer-order differentiability. They interpolate between local and nonlocal regularity scales, forming the analytical backbone for the study of elliptic, parabolic, and fluid-dynamical equations, including the Navier–Stokes and fractional diffusion models.

3.1. Fractional Sobolev Spaces

For any real exponent $s \in \mathbb{R}$, the fractional Sobolev space $H^s(\Omega)$ extends the classical integer-order spaces $W^{k,p}(\Omega)$ by allowing noninteger smoothness indices, thus capturing intermediate degrees of regularity.

Definition 3 (Fractional Sobolev Space)**.** *Let $s \in \mathbb{R}$ and $n \in \mathbb{N}$. The* fractional Sobolev space $H^s(\mathbb{R}^n)$ *consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|u\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} (1+|\xi|^2)^s \, |\hat{u}(\xi)|^2 \, d\xi < \infty, \tag{5}$$

where $\hat{u}$ denotes the Fourier transform of u. Formally,

$$H^s(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{H^s(\mathbb{R}^n)} < \infty\}.$$

If $s = k \in \mathbb{N}_0$, then $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$, the classical Sobolev space of square-integrable functions with weak derivatives up to order k in L^2. Positive exponents $s > 0$ measure differentiability, negative $s < 0$ correspond to distributional regularity, and $s = 0$ reduces to L^2.

For open $\Omega \subset \mathbb{R}^n$, $H^s(\Omega)$ can be defined by restriction from $\mathbb{R}^n$, via spectral calculus, or interpolation.

Lemma 4 (Density and Duality)**.** *For $s > 0$, $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$, and the dual of $H^s(\mathbb{R}^n)$ is $H^{-s}(\mathbb{R}^n)$, under the pairing $\langle u, v\rangle = \int_{\mathbb{R}^n} \hat{u}(\xi)\overline{\hat{v}(\xi)}\, d\xi$.*

Proof. Approximate u by $\rho_\varepsilon * u$ with a mollifier ρ_ε and use the Fourier characterization (5). Plancherel's theorem then yields convergence in H^s and identifies the dual via the Riesz isomorphism $(I - \Delta)^{s/2}$. □

These fractional spaces provide the natural setting for weak formulations of PDEs with fractional diffusion or nonlocal operators.

3.2. Besov Spaces and Littlewood–Paley Decomposition

Besov spaces refine Sobolev regularity by decomposing functions into dyadic frequency bands. They allow the separation of local and global smoothness, crucial for nonlinear PDE analysis.

Definition 4 (Besov Space)**.** *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. Let $(\Delta_j)_{j \ge 0}$ be a homogeneous dyadic Littlewood–Paley decomposition defined by smooth cut-off functions φ_j supported in annuli such that $\sum_{j=0}^{\infty} \varphi_j(\xi) = 1$ for all $\xi \neq 0$. The* Besov space *$B^s_{p,q}(\mathbb{R}^n)$ consists of all $u \in \mathcal{S}'(\mathbb{R}^n)$ for which*

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq} \|\Delta_j u\|^q_{L^p(\mathbb{R}^n)} \right)^{1/q} < \infty, \tag{6}$$

with the usual modification

$$\|u\|_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{j \ge 0} 2^{js} \|\Delta_j u\|_{L^p(\mathbb{R}^n)}. \tag{7}$$

The operators Δ_j isolate frequency components in dyadic shells:

$$\begin{aligned} \widehat{\Delta_j u}(\xi) &= \varphi(2^{-j}\xi)\, \widehat{u}(\xi), \\ \varphi &\in C_c^{\infty}(\{\xi \in \mathbb{R}^n : c_1 \le |\xi| \le c_2\}), \quad 0 < c_1 < c_2. \end{aligned} \tag{8}$$

Lemma 5 (Equivalence with Sobolev Spaces)**.** *For $s \in \mathbb{R}$,*

$$B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n),$$

and the norms induced by (5) and (6) are equivalent.

Proof. By Plancherel's theorem,

$$\begin{aligned} \|u\|^2_{H^s} &\simeq \textstyle\sum_{j=0}^{\infty} \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\varphi(2^{-j}\xi)\hat{u}(\xi)|^2 \, d\xi \\ &\simeq \textstyle\sum_{j=0}^{\infty} 2^{2js} \|\Delta_j u\|^2_{L^2}. \end{aligned}$$

□

For noninteger $s > 0$ and $p = q = \infty$, we recover the Hölder–Zygmund space $C^s(\mathbb{R}^n)$.

Analytical significance. Besov spaces are stable under nonlinear operations, real interpolation, and scaling. They constitute the optimal setting for studying fine regularity, singularities, and multifractal structures in fluid equations, particularly Navier–Stokes, where critical spaces correspond to the natural scaling of the equations.

3.3. Interpolation and Embedding Theorems

Theorem 4 (Real Interpolation)**.** *Let $s_0 < s_1$ and $\theta \in (0, 1)$. For $1 \le p \le \infty$ and suitable q_0, q_1, the real interpolation identity holds:*

$$(B^{s_0}_{p,q_0}, B^{s_1}_{p,q_1})_{\theta,q} = B^s_{p,q}, \qquad s = (1-\theta)s_0 + \theta s_1. \tag{9}$$

In particular,

$$\big(H^{s_0}(\Omega), H^{s_1}(\Omega)\big)_{\theta,2} = H^s(\Omega), \qquad s = (1-\theta)s_0 + \theta s_1. \tag{10}$$

Theorem 5 (Embedding Properties)**.** *If $s > t$ or $s = t$ and $q \le r$, then*

$$B^s_{p,q}(\Omega) \hookrightarrow B^t_{p,r}(\Omega). \tag{11}$$

In particular, $H^s(\Omega) = B^s_{2,2}(\Omega)$ embeds continuously into $L^p(\Omega)$ whenever $s > n(\frac{1}{2} - \frac{1}{p})$.

Sketch. Use Bernstein inequalities for dyadic blocks:

$$\|\nabla^k \Delta_j u\|_{L^p} \le C\, 2^{jk} \|\Delta_j u\|_{L^p}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

From this and monotonicity of ℓ^q norms, one deduces (11). The Sobolev case follows from Hausdorff–Young and interpolation. □

These results provide the precise analytical machinery for establishing regularity, compactness, and stability of weak solutions to nonlinear PDEs.

3.4. Fractional Operators and Fluid Dynamics

The fractional Laplacian $(-\Delta)^s$, $s \in (0,1)$, generalizes classical diffusion and naturally acts on fractional Sobolev or Besov spaces. In Fourier variables,

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s}\,\hat{u}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{12}$$

Equivalent integral formulations involve hypersingular kernels:

$$(-\Delta)^s u(x) = C_{n,s}\,\text{p.v.}\int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

Fractional operators capture long-range interactions and anomalous diffusion effects, providing more accurate descriptions of turbulent energy cascades and nonlocal dissipation. They are crucial in modern models of turbulence, such as fractional or hyper-dissipative Navier–Stokes systems.

Remark 3 (Fractional Regularity in Fluid Dynamics)**.** *Fractional Sobolev and Besov spaces furnish the correct functional framework to quantify partial smoothness of velocity fields, vorticity, and energy spectra in turbulent flows. They allow rigorous formulation of conditional regularity criteria, intermittency analysis, and scaling laws consistent with Kolmogorov's phenomenology.*

4. Higher-Order Sobolev Regularity of Navier–Stokes Equations

In this section we upgrade weak L^2–H^1 regularity to higher Sobolev classes under suitable hypotheses. We present a clean statement and a detailed proof based on energy methods, elliptic regularity for the Laplacian (or Stokes operator) and careful estimates of the nonlinear term.

Theorem 6 (Higher-order Sobolev regularity)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^∞ boundary. Fix $T > 0$. Assume*

$$u_0 \in H^k(\Omega) \cap L^2_\sigma(\Omega), \qquad f \in L^2(0,T;H^k(\Omega)),$$

for some integer $k \geq 0$. Let u be a Leray–Hopf weak solution on $[0,T]$. If additionally the initial data and forcing satisfy a smallness condition

$$\|u_0\|^2_{H^k} + \int_0^T \|f(t)\|^2_{H^k}\,dt \leq \varepsilon \tag{13}$$

for $\varepsilon = \varepsilon(\nu,k,\Omega) > 0$ sufficiently small, then

$$u \in L^\infty(0,T;H^k(\Omega)) \cap L^2(0,T;H^{k+1}(\Omega)).$$

Moreover, if u is smooth on $(0,T_*)$ then the standard bootstrap gives $u \in L^2(0,T;H^{k+2}(\Omega))$.

Remark 4. *The smallness hypothesis (13) may be removed if one assumes global-in-time control of a Serrin-type norm (see Ladyzhenskaya–Prodi–Serrin criteria) or if one works locally in time: given $u_0 \in H^k$ there exists $T^* > 0$ (depending on $\|u_0\|_{H^k}$) such that the same conclusion holds on $[0,T^*]$.*

Strategy of proof

We outline the main steps: (i) construct Galerkin approximations and obtain uniform H^k–energy estimates; (ii) estimate the nonlinear term using Sobolev product/commutator bounds; (iii) apply Grönwall to close the estimate; (iv) pass to the limit $N \to \infty$ to get the result for the weak solution;

(v) apply elliptic regularity / Stokes resolvent estimates to raise spatial regularity by two derivatives when appropriate.

Proof. 6. Let $\{w_j\}_{j\geq 1}$ be the divergence-free eigenfunctions of the Stokes operator with Dirichlet boundary conditions (or Laplacian eigenfunctions projected by Leray projector P). For each N consider the Galerkin solution

$$u^N(t,x) = \sum_{j=1}^{N} c_j^N(t) w_j(x),$$

which satisfies the finite-dimensional ODE system obtained by projecting the Navier–Stokes equations. Testing the Galerkin equations with $A^k u^N$ (where $A = -P\Delta$ is the Stokes operator and $A^{k/2}$ defines the H^k–norm on divergence-free fields) yields, after standard manipulations,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^N\|_{H^k}^2 + \nu\|u^N\|_{H^{k+1}}^2 \\ &= \langle f, A^k u^N\rangle \\ &\quad - \langle (u^N\cdot\nabla)u^N, A^k u^N\rangle.\end{aligned}$$

Here $\langle\cdot,\cdot\rangle$ denotes the L^2 pairing (or duality when appropriate).

The forcing term is controlled by Cauchy–Schwarz and Young:

$$|\langle f, A^k u^N\rangle| \leq \|f\|_{H^k}\|u^N\|_{H^k} \leq \frac{\nu}{4}\|u^N\|_{H^{k+1}}^2 + C(\nu)\|f\|_{H^k}^2, \tag{14}$$

where we used the elliptic equivalence $\|u\|_{H^{k+1}} \simeq \|A^{(k+1)/2}u\|_{L^2}$ and interpolation $\|u\|_{H^k} \leq \delta\|u\|_{H^{k+1}} + C(\delta)\|u\|_{L^2}$ if needed.

We must estimate

$$N_k := \left|\langle (u^N\cdot\nabla)u^N,\ A^k u^N\rangle\right|.$$

There are two standard approaches:

(A) Commutator decomposition. Write

$$\begin{aligned}\langle (u\cdot\nabla)u,\ A^k u\rangle &= \langle A^{k/2}[(u\cdot\nabla)u],\ A^{k/2}u\rangle \\ &= \langle [(u\cdot\nabla), A^{k/2}]u,\ A^{k/2}u\rangle.\end{aligned}$$

since $\langle (u\cdot\nabla)A^{k/2}u,\ A^{k/2}u\rangle = 0$ by skew-symmetry when $\nabla\cdot u = 0$ and appropriate boundary conditions. Thus

$$N_k \leq \|[(u\cdot\nabla), A^{k/2}]u\|_{L^2}\,\|u\|_{H^k}.$$

By Kato–Ponce type commutator estimates (or Coifman–Meyer paraproduct calculus) one has, for integer $k \geq 0$,

$$\|[(u\cdot\nabla), A^{k/2}]u\|_{L^2} \leq C\big(\|\nabla u\|_{L^\infty}\|u\|_{H^k} + \|u\|_{H^{k+1}}\|\nabla u\|_{L^r}\big), \tag{15}$$

for some r depending on n, k, and using embeddings $H^s \hookrightarrow L^\infty$ when $s > n/2$. Consequently,

$$N_k \leq C\|\nabla u\|_{L^\infty}\|u\|_{H^k}^2 + C\|u\|_{H^{k+1}}\|\nabla u\|_{L^r}\|u\|_{H^k}.$$

Using Young's inequality with $\varepsilon > 0$,

$$N_k \leq \frac{\nu}{4}\|u\|_{H^{k+1}}^2 + C\|\nabla u\|_{L^\infty}^2\|u\|_{H^k}^2 + C\|\nabla u\|_{L^r}^2\|u\|_{H^k}^2. \tag{16}$$

(B) Direct product estimate. If $k > n/2$ then H^k is an algebra and we can use

$$\|(u\cdot\nabla)u\|_{H^k} \leq C\|u\|_{H^k}\|\nabla u\|_{H^k} \leq C\|u\|_{H^k}\|u\|_{H^{k+1}},$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

which yields

$$N_k \le C\|u\|_{H^k}^2 \|u\|_{H^{k+1}} \le \frac{\nu}{4}\|u\|_{H^{k+1}}^2 + C\|u\|_{H^k}^4. \tag{17}$$

Either route produces an estimate of the schematic form

$$N_k \le \frac{\nu}{4}\|u\|_{H^{k+1}}^2 + G(\|u\|_X)\|u\|_{H^k}^2,$$

where $G(\|u\|_X)$ is a (locally) bounded function of a lower norm of u (e.g. $\|\nabla u\|_{L^\infty}$ or $\|u\|_{H^k}$).

Insert (14) and the nonlinear bound into (**??**) to obtain

$$\frac{d}{dt}\|u^N\|_{H^k}^2 + \nu\|u^N\|_{H^{k+1}}^2 \le C\|f\|_{H^k}^2 + C\Phi(t)\|u^N\|_{H^k}^2, \tag{18}$$

where $\Phi(t)$ equals either $\|\nabla u^N(t)\|_{L^\infty}^2 + \|\nabla u^N(t)\|_{L^r}^2$ or $\|u^N(t)\|_{H^k}^2$, depending on which nonlinear estimate was used.

Applying Grönwall's inequality on $[0, T]$ yields, for some constants C_1, C_2,

$$\begin{aligned} &\|u^N(T)\|_{H^k}^2 + \nu\int_0^T \|u^N(t)\|_{H^{k+1}}^2\,dt \\ &\quad\le \Big(\|u_0^N\|_{H^k}^2 \\ &\qquad + C_1\int_0^T \|f\|_{H^k}^2\,dt\Big) \\ &\qquad \times \exp\Big(C_2\int_0^T \Phi(t)\,dt\Big). \end{aligned} \tag{19}$$

If we assume the smallness condition (13), then $\int_0^T \Phi(t)\,dt$ can be made arbitrarily small (one uses embedding $H^{k+1} \subset W^{1,\infty}$ when $k+1 > n/2$ or bootstrapping argument for short time), so the exponential factor is bounded by a universal constant and we get uniform-in-N bounds

$$\begin{aligned} &\sup_{t\in[0,T]} \|u^N(t)\|_{H^k}^2 \\ &+ \nu\int_0^T \|u^N\|_{H^{k+1}}^2\,dt \\ &\le C\big(\|u_0\|_{H^k}^2 \\ &\quad + \|f\|_{L^2(0,T;H^k)}^2\big). \end{aligned} \tag{20}$$

Hence the sequence $\{u^N\}$ is bounded in $L^\infty(0, T; H^k) \cap L^2(0, T; H^{k+1})$.

With the uniform bounds and compactness (Aubin–Lions lemma), extract a subsequence $u^{N_j} \to u$ weakly-* in $L^\infty(0, T; H^k)$ and weakly in $L^2(0, T; H^{k+1})$, and strongly in lower norms; pass to the limit in the Galerkin equations to conclude that the weak solution u satisfies the same a priori bounds. Thus

$$u \in L^\infty(0, T; H^k(\Omega)) \cap L^2(0, T; H^{k+1}(\Omega)).$$

Suppose now u enjoys the above bound and $f \in L^2(0, T; H^k)$. Apply the elliptic regularity to the stationary Stokes operator at each time slice: formally,

$$-\nu\Delta u + \nabla p = f - \partial_t u - (u\cdot\nabla)u.$$

If the right-hand side belongs to $L^2(0, T; H^k)$ (which follows from the previous step and $u \in L^\infty(0, T; H^k)$ together with product estimates), then elliptic regularity for the Stokes system implies

$u \in L^2(0,T;H^{k+2})$ (and $p \in L^2(0,T;H^{k+1})$), completing the bootstrap. Rigorous justification uses the time-differentiability in H^{k-1} or mollification in time with standard limiting arguments. □

Auxiliary estimates and comments

- *Product/commutator controls.* The commutator bound (15) can be proved using Kato–Ponce inequalities or Bony paraproduct decomposition; for integer k one also can use classical Moser-type product estimates:
$$\|uv\|_{H^k} \le C(\|u\|_{L^\infty}\|v\|_{H^k} + \|v\|_{L^\infty}\|u\|_{H^k}).$$
- *Role of critical indices.* When $k > n/2$ the space H^k is an algebra, simplifying nonlinear estimates; when $k \le n/2$ one must use mixed space estimates (Serrin conditions) or work via Besov embeddings.
- *Local vs global.* The proof above gives a *local-in-time* existence and regularity result for arbitrary H^k data and a *global* result under the smallness condition (13) (or alternative global control like Serrin norms).

This completes the higher-order regularity analysis and shows how harmonic-analytic tools, elliptic regularity and careful commutator estimates combine to lift weak solutions into higher Sobolev classes under standard hypotheses.

5. Characterization of Besov Spaces

5.1. Rigorous Equivalence Proof and Functional Properties

Besov spaces are fundamental in harmonic analysis, partial differential equations, and approximation theory. They generalize Sobolev and Hölder spaces, capturing fine regularity properties via smoothness and integrability indices. We provide a rigorous equivalence of norms characterization using the Littlewood-Paley decomposition.

Theorem 7 (Equivalence of Besov Norms via Littlewood-Paley Decomposition)**.** *Let $s \in \mathbb{R}$, $1 \le p,q \le \infty$. Then the inhomogeneous Besov space $B^s_{p,q}(\mathbb{R}^n)$ admits an equivalent norm via a smooth dyadic decomposition. Concretely, there exists a sequence $\{\varphi_j\}_{j\ge -1} \subset C_c^\infty(\mathbb{R}^n)$ satisfying*

$$\operatorname{supp}(\varphi_j) \subseteq \left\{\xi \in \mathbb{R}^n : 2^{j-1} \le |\xi| \le 2^{j+1}\right\}, \quad j \ge 0, \tag{21}$$

and a low-frequency cutoff φ_{-1} supported in $\{|\xi| \le 2\}$, forming a dyadic partition of unity:

$$\sum_{j=-1}^{\infty} \varphi_j(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n. \tag{22}$$

For any $u \in \mathscr{S}'(\mathbb{R}^n)$, define
$$\Delta_j u := \mathscr{F}^{-1}[\varphi_j \hat{u}].$$
Then, there exists a constant $C > 0$ independent of u such that

$$\begin{aligned} C^{-1}\|u\|_{B^s_{p,q}} &\le \|\Delta_{-1}u\|_{L^p} + \left(\sum_{j=0}^{\infty} 2^{jsq}\|\Delta_j u\|^q_{L^p}\right)^{1/q} \\ &\le C\|u\|_{B^s_{p,q}}. \end{aligned} \tag{23}$$

Moreover, $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space, and pointwise multiplication is continuous under suitable conditions on smoothness and integrability indices.

Proof. Define dyadic blocks as

$$\Delta_j u := \mathscr{F}^{-1}[\varphi(2^{-j}\cdot)\hat{u}], \quad j \ge 0, \quad \Delta_{-1}u := \mathscr{F}^{-1}[\varphi_{-1}\hat{u}].$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

The smoothness of φ_j ensures that $\Delta_j u \in \mathscr{S}(\mathbb{R}^n)$ for $u \in \mathscr{S}(\mathbb{R}^n)$, and the partition of unity guarantees

$$u = \sum_{j=-1}^{\infty} \Delta_j u, \quad \text{in } \mathscr{S}'(\mathbb{R}^n).$$

The inhomogeneous Besov norm is given by

$$\|u\|_{B^s_{p,q}} := \|u\|_{L^p} + \left(\int_0^\infty t^{-sq} \omega_m(u,t)_p^q \frac{dt}{t} \right)^{1/q},$$

where $\omega_m(u,t)_p$ is the m-th order modulus of smoothness in L^p. Using standard properties of Littlewood-Paley operators, this norm is equivalent to the dyadic sum in (23) [3].

Consider the Bessel potential operator $J^s := (1-\Delta)^{s/2}$. Then

$$\|u\|_{B^s_{p,q}} \sim \|J^s u\|_{L^p} + \left(\int_0^\infty t^{-sq} \|\Delta_t u\|_{L^p}^q \frac{dt}{t} \right)^{1/q},$$
$$\Delta_t u := u * \phi_t,$$
$$\phi_t(x) := t^{-n}\phi(x/t),$$

for a suitable Schwartz function ϕ. Comparing $t \sim 2^{-j}$ yields

$$\|\Delta_t u\|_{L^p} \sim \|\Delta_j u\|_{L^p}, \quad t \in [2^{-j-1}, 2^{-j+1}],$$

thus recovering the dyadic sum representation.

Let $\{u_n\}$ be Cauchy in $B^s_{p,q}$. Then, for each j, $\{\Delta_j u_n\}$ is Cauchy in L^p and converges to some $v_j \in L^p$. Define $u := \sum_j v_j$ in $\mathscr{S}'(\mathbb{R}^n)$. By Fatou's lemma:

$$\|u_n - u\|_{B^s_{p,q}}^q \le \liminf_{m\to\infty} \sum_j 2^{jsq} \|\Delta_j(u_n - u_m)\|_{L^p}^q \to 0,$$

so $B^s_{p,q}(\mathbb{R}^n)$ is complete.

Let $u \in B^{s_1}_{p_1,q_1}$, $v \in B^{s_2}_{p_2,q_2}$ with $s_1 + s_2 > 0$ and appropriate integrability indices. Using Bony's decomposition:

$$uv = T_u v + T_v u + R(u,v),$$

where $T_u v$ and $T_v u$ are paraproducts and $R(u,v)$ is the remainder. One obtains the estimate:

$$\|uv\|_{B^s_{p,q}} \lesssim \|u\|_{L^{r_1}} \|v\|_{B^s_{r_2,q}} + \|v\|_{L^{s_1}} \|u\|_{B^s_{s_2,q}} + \|u\|_{\dot{B}^{s_1}_{p_1,q_1}} \|v\|_{\dot{B}^{s_2}_{p_2,q_2}},$$

with indices chosen to satisfy Hölder-type conditions, guaranteeing boundedness and continuity of multiplication in Besov spaces [21].

This concludes the proof. □

Remark 5. *Besov spaces naturally interpolate between Sobolev spaces $W^{k,p}$ and Hölder spaces C^α, allowing fine control of both local and global regularity. The Littlewood-Paley characterization is instrumental in PDE analysis, harmonic analysis, and nonlinear approximation.*

6. Besov Spaces and Turbulence

6.1. Littlewood-Paley Decomposition and Besov Spaces

The Littlewood-Paley decomposition is a cornerstone of modern harmonic analysis. It decomposes a tempered distribution u into frequency-localized components, enabling precise control of smoothness across scales. This multiscale framework is essential for defining Besov spaces, which provide a refined regularity scale beyond classical Sobolev spaces, particularly suitable for analyzing irregular and turbulent phenomena in fluid dynamics.

6.2. Littlewood-Paley Decomposition

Let $u \in \mathscr{S}'(\mathbb{R}^n)$, the space of tempered distributions. The Littlewood-Paley decomposition writes u as

$$u = \sum_{j=-\infty}^{\infty} \Delta_j u, \tag{24}$$

where the dyadic frequency blocks $\Delta_j u$ are defined by

$$\Delta_j u := \mathscr{F}^{-1}\big(\varphi(2^{-j}\xi)\hat{u}(\xi)\big), \tag{25}$$

with $\mathscr{F}$ denoting the Fourier transform. The cutoff $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies

$$\text{supp}(\varphi) \subseteq C(1) := \{\xi \in \mathbb{R}^n : c_1 \le |\xi| \le c_2\}, \tag{26}$$

for some constants $0 < c_1 < c_2$, ensuring smooth localization in frequency. The scaled annuli $C(2^j) = \{\xi : c_1 2^j \le |\xi| \le c_2 2^j\}$ capture dyadic frequency bands, providing a multiscale decomposition essential for Besov space characterization.

6.3. Besov Spaces via Littlewood-Paley Decomposition

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the Besov space $B^s_{p,q}(\mathbb{R}^n)$ is defined through the norm

$$\|u\|_{B^s_{p,q}} := \left(\sum_{j=-\infty}^{\infty} 2^{jsq}\|\Delta_j u\|^q_{L^p}\right)^{1/q}, \tag{27}$$

with the standard modification for $q = \infty$:

$$\|u\|_{B^s_{p,\infty}} := \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j u\|_{L^p}. \tag{28}$$

6.3.1. Special Cases and Connections

Besov spaces generalize classical function spaces:

- When $p = q = 2$, $B^s_{2,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ coincides with fractional Sobolev spaces.
- When $s > 0$ and $p = q = \infty$, $B^s_{\infty,\infty}$ is equivalent to Hölder spaces C^s.

6.3.2. Interpretation of Parameters

The Besov norm $\|u\|_{B^s_{p,q}}$ captures three fundamental aspects of function regularity:

Smoothness s

The parameter s weights high-frequency contributions:

$$\sum_{j\in\mathbb{Z}} 2^{jsq}\|\Delta_j u\|^q_{L^p} < \infty.$$

Higher s requires faster decay of $\|\Delta_j u\|_{L^p}$ as $j \to \infty$.

Integrability p

The index p controls the L^p-norm of each dyadic block:

$$\|\Delta_j u\|_{L^p} \lesssim 2^{-js},$$

ensuring frequency components are appropriately measured in the underlying Lebesgue space.

Summability q

The index q dictates how contributions from different scales combine:

$$\|u\|_{B^s_{p,q}} = \left(\sum_{j\in\mathbb{Z}} (2^{js}\|\Delta_j u\|_{L^p})^q\right)^{1/q}.$$

Smaller q emphasizes uniform decay across scales, while $q = \infty$ imposes a supremum bound.

6.4. Implications for Turbulence Analysis

Besov spaces are particularly well-suited for turbulence modeling:

- They capture intermittent and multifractal structures in velocity fields.
- The Littlewood-Paley decomposition allows scale-by-scale analysis of energy spectra.
- Smoothness indices s correspond to differentiability and regularity of the velocity field, while p, q encode integrability and summability across scales.

Thus, Besov spaces provide a mathematically rigorous framework to quantify the fine-scale structure of turbulent flows, linking harmonic analysis to physical observables.

6.5. Equivalence of Besov Norms via Littlewood-Paley and Lifting Operators

Besov norms can be equivalently characterized using lifting (Bessel potential) operators and dyadic blocks. Let

$$J^s := (1-\Delta)^{s/2}, \quad s \in \mathbb{R},$$

be the Bessel potential operator. Then, for $u \in \mathscr{S}'(\mathbb{R}^n)$,

$$\|u\|_{B^s_{p,q}} \sim \|J^s u\|_{L^p} + \left(\int_0^\infty t^{-sq}\|\Delta_t u\|^q_{L^p}\frac{dt}{t}\right)^{1/q}, \tag{29}$$

$$\Delta_t u := u * \phi_t, \tag{30}$$

$$\phi_t(x) := t^{-n}\phi(x/t), \tag{31}$$

where $\phi \in \mathscr{S}(\mathbb{R}^n)$ is a Schwartz function adapted to the dyadic decomposition.

Proof. By the Littlewood-Paley decomposition,

$$u = \sum_{j=-\infty}^{\infty} \Delta_j u, \quad \Delta_j u = \mathscr{F}^{-1}[\varphi(2^{-j}\cdot)\hat{u}].$$

Frequency localization ensures that $\Delta_j u$ has Fourier support in the annulus $C(2^j)$, and the series converges in $\mathscr{S}'(\mathbb{R}^n)$.

For the inhomogeneous Besov norm,

$$\|u\|_{B^s_{p,q}} \sim \|\Delta_{-1}u\|_{L^p} + \left(\sum_{j\geq 0} 2^{jsq}\|\Delta_j u\|^q_{L^p}\right)^{1/q}.$$

The equivalence relies on the fact that the low-frequency contribution $\Delta_{-1}u$ captures all components with $|\xi| \leq 2$, while dyadic blocks $\Delta_j u$ measure higher frequencies.

Let $\Delta_t u = u * \phi_t$ with $\phi_t(x) = t^{-n}\phi(x/t)$ and ϕ adapted to the Fourier cutoff φ. Then for $t \sim 2^{-j}$,

$$\|\Delta_t u\|_{L^p} \sim \|\Delta_j u\|_{L^p},$$

so that
$$\int_0^\infty t^{-sq} \|\Delta_t u\|_{L^p}^q \frac{dt}{t} \sim \sum_{j\in\mathbb{Z}} 2^{jsq} \|\Delta_j u\|_{L^p}^q.$$

The Bessel potential operator satisfies $\|J^s u\|_{L^p} \sim \|u\|_{\dot{B}^s_{p,q}}$ for homogeneous Besov norms. Combined with Step 3, this yields

$$\|u\|_{B^s_{p,q}} \sim \|J^s u\|_{L^p} + \left(\int_0^\infty t^{-sq} \|\Delta_t u\|_{L^p}^q \frac{dt}{t}\right)^{1/q},$$

proving the equivalence.

This equivalence implies:

- $B^s_{p,q}(\mathbb{R}^n)$ is a Banach space.
- The low-frequency term $\|J^s u\|_{L^p}$ controls smooth components, while the integral over t captures the multiscale fine structure, critical in turbulence analysis.
- Multiplication and paraproducts can be rigorously estimated using this decomposition, allowing precise control of nonlinear PDE terms.

□

Remark 6. *This formulation is particularly useful in the study of turbulent flows: dyadic blocks correspond to eddies at different scales, and Besov norms quantify energy distribution across scales, providing a rigorous mathematical framework for multiscale phenomena.*

7. Plancherel Theorem

The Plancherel Theorem is a fundamental result in Fourier analysis that establishes an *isometry* between the Hilbert space $L^2(\mathbb{R}^n)$ and itself under the Fourier transform. This property ensures preservation of energy and is crucial in harmonic analysis, quantum mechanics, signal processing, and the study of partial differential equations.

7.1. Statement of the Theorem

For $u \in L^2(\mathbb{R}^n)$, define the Fourier transform as

$$\hat{u}(\xi) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-i\xi\cdot x}\, dx. \tag{32}$$

The Plancherel Theorem asserts that $\hat{u} \in L^2(\mathbb{R}^n)$ and satisfies the norm-preserving identity

$$\|u\|_{L^2(\mathbb{R}^n)} = \|\hat{u}\|_{L^2(\mathbb{R}^n)}, \tag{33}$$

or equivalently,
$$\int_{\mathbb{R}^n} |u(x)|^2\, dx = \int_{\mathbb{R}^n} |\hat{u}(\xi)|^2\, d\xi. \tag{34}$$

7.2. Proof.

For $u, v \in L^2(\mathbb{R}^n)$, the Hilbert space inner product is

$$\langle u, v\rangle := \int_{\mathbb{R}^n} u(x)\overline{v(x)}\, dx. \tag{35}$$

To prove the theorem, it suffices to show

$$\langle u, v\rangle = \langle \hat{u}, \hat{v}\rangle. \tag{36}$$

For $u \in \mathscr{S}(\mathbb{R}^n)$ (Schwartz functions), the inversion formula is

$$u(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i\xi\cdot x}\, d\xi. \tag{37}$$

By density of $\mathscr{S}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the result extends to all L^2 functions.

Substitute (37) into (35):

$$\begin{aligned}\langle u, v\rangle &= \int_{\mathbb{R}^n} \left(\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i\xi\cdot x}\, d\xi \right) \overline{v(x)}\, dx \\ &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{u}(\xi) \left(\int_{\mathbb{R}^n} e^{i\xi\cdot x} \overline{v(x)}\, dx \right) d\xi,\end{aligned}$$

where Fubini's theorem justifies interchange of integrals.

Observe that

$$\int_{\mathbb{R}^n} e^{i\xi\cdot x} \overline{v(x)}\, dx = (2\pi)^{n/2} \overline{\hat{v}(\xi)}.$$

Hence,

$$\langle u, v\rangle = \int_{\mathbb{R}^n} \hat{u}(\xi) \overline{\hat{v}(\xi)}\, d\xi = \langle \hat{u}, \hat{v}\rangle, \tag{38}$$

which proves that the Fourier transform is an isometry.

Setting $v = u$ in (38) immediately yields

$$\|u\|^2_{L^2(\mathbb{R}^n)} = \|\hat{u}\|^2_{L^2(\mathbb{R}^n)},$$

completing the proof.

7.3. Remarks

- The Plancherel theorem implies that the Fourier transform extends to a **unitary operator** on $L^2(\mathbb{R}^n)$, preserving inner products and norms.
- The normalization factor $(2\pi)^{-n/2}$ ensures unitarity.
- It provides the basis for Parseval's identity and is fundamental in spectral analysis of linear operators, signal processing, and PDE theory.
- The theorem is a starting point for L^p-Fourier analysis and the study of Sobolev and Besov spaces, linking energy preservation to function regularity across scales.

8. Extension of Plancherel's Theorem

This section extends the classical Plancherel theorem to Sobolev spaces $W^{k,2}(\Omega)$, providing a powerful tool to analyze regularity and spectral properties of functions in PDE theory and mathematical physics.

8.1. Extended Plancherel Theorem in Sobolev Spaces

Theorem 8 (Plancherel Theorem for Sobolev Spaces)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $k \in \mathbb{N}_0$. For $u \in W^{k,2}(\Omega)$, denote by $\hat{u}$ the Fourier transform extended by zero outside Ω. Then*

$$\begin{aligned}\|u\|^2_{W^{k,2}(\Omega)} &= \sum_{|\alpha|\le k} \|D^\alpha u\|^2_{L^2(\Omega)} \\ &= \sum_{|\alpha|\le k} \int_{\mathbb{R}^n} |\xi^\alpha \hat{u}(\xi)|^2\, d\xi \\ &= \|\hat{u}\|^2_{W^{k,2}(\mathbb{R}^n)}.\end{aligned} \tag{39}$$

where multi-index notation is used and

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

$$\|\hat{u}\|^2_{W^{k,2}(\mathbb{R}^n)} := \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\xi^\alpha \hat{u}(\xi)|^2 \, d\xi.$$

Proof. We outline the proof in detail:

By definition,

$$\|u\|^2_{W^{k,2}(\Omega)} = \sum_{|\alpha|\leq k} \|D^\alpha u\|^2_{L^2(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega |D^\alpha u(x)|^2 \, dx. \tag{40}$$

Using classical properties of the Fourier transform (assuming extension by zero outside Ω), we have

$$\widehat{D^\alpha u}(\xi) = (i\xi)^\alpha \hat{u}(\xi). \tag{41}$$

For each multi-index α,

$$\begin{aligned} \|D^\alpha u\|^2_{L^2(\Omega)} &= \|\widehat{D^\alpha u}\|^2_{L^2(\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^n} |(i\xi)^\alpha \hat{u}(\xi)|^2 \, d\xi \\ &= \int_{\mathbb{R}^n} |\xi^\alpha \hat{u}(\xi)|^2 \, d\xi, \end{aligned} \tag{42}$$

where the last equality holds because $|i| = 1$.

Summing over all multi-indices α with $|\alpha| \leq k$, we get

$$\|u\|^2_{W^{k,2}(\Omega)} = \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |\xi^\alpha \hat{u}(\xi)|^2 \, d\xi = \|\hat{u}\|^2_{W^{k,2}(\mathbb{R}^n)}, \tag{43}$$

which proves the theorem. □

8.2. Remarks

- The extension relies on the zero extension of u from Ω to $\mathbb{R}^n$ and the smoothness of $\partial\Omega$.
- This identity shows that Sobolev regularity in space is encoded as polynomial decay in frequency via multiplication by ξ^α in Fourier space.
- The result is fundamental in spectral methods for PDEs, as it allows to analyze PDE operators via their symbols.

8.3. Decay Properties in Besov Spaces via Littlewood-Paley Decomposition

We now connect this spectral characterization with the multiscale decomposition used in Besov spaces.

8.3.1. Littlewood-Paley Decomposition and Plancherel Theorem

Recall the Littlewood-Paley frequency projection $\Delta_j u$, defined via Fourier multiplier $\varphi(2^{-j}\cdot)$:

$$\Delta_j u = \mathscr{F}^{-1}\big(\varphi(2^{-j}\xi)\hat{u}(\xi)\big), \tag{44}$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ is supported in the annulus

$$C(2^j) := \{\xi \in \mathbb{R}^n : c_1 2^j \leq |\xi| \leq c_2 2^j\}, \quad 0 < c_1 < c_2.$$

Applying Plancherel's theorem to $\Delta_j u$, we have

$$\|\Delta_j u\|^2_{L^2} = \int_{\mathbb{R}^n} |\varphi(2^{-j}\xi)|^2 |\hat{u}(\xi)|^2 \, d\xi. \tag{45}$$

8.3.2. Support Localization and Frequency Scaling

Since $\varphi(2^{-j}\xi)$ localizes frequencies near $|\xi| \sim 2^j$, the component $\Delta_j u$ captures the energy of u at the frequency scale 2^j.

8.3.3. Bernstein's Inequality

Bernstein's inequality quantifies the control of norms in different Lebesgue spaces based on frequency localization. For $1 \leq p \leq q \leq \infty$,

$$\|\Delta_j u\|_{L^q} \leq C 2^{jn\left(\frac{1}{p}-\frac{1}{q}\right)} \|\Delta_j u\|_{L^p}, \tag{46}$$

where C depends only on n, p, q, and the support of φ.

8.3.4. Besov Space Norm and Decay

The Besov norm $\|u\|_{B^s_{p,q}}$ is given by

$$\|u\|_{B^s_{p,q}} := \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \|\Delta_j u\|^q_{L^p} \right)^{1/q}, \tag{47}$$

which weights the L^p-norms of the frequency components by 2^{js}, capturing smoothness s.

Using Bernstein's inequality (46) with $p = 2$, for $1 \leq p \leq 2$,

$$\|\Delta_j u\|_{L^p} \leq C 2^{jn\left(\frac{1}{p}-\frac{1}{2}\right)} \|\Delta_j u\|_{L^2}. \tag{48}$$

Substituting into the Besov norm, we obtain the estimate

$$\|u\|_{B^s_{p,q}} \leq C \left(\sum_{j=-\infty}^{\infty} 2^{jsq} \left(2^{jn\left(\frac{1}{p}-\frac{1}{2}\right)} \|\Delta_j u\|_{L^2} \right)^q \right)^{1/q}. \tag{49}$$

8.3.5. Interpretation

The decay of $\|\Delta_j u\|_{L^2}$ as $j \to \infty$ reflects the smoothness of u. Faster decay corresponds to higher smoothness s. The combined weights describe how integrability (p) and smoothness (s) interact in the Besov space scale.

Summary: The extension of Plancherel's theorem to Sobolev spaces rigorously links spatial derivatives with weighted Fourier norms, while the Littlewood-Paley decomposition and Bernstein's inequality provide a multiscale framework to describe function regularity in Besov spaces, fundamental for analysis in PDEs, turbulence, and harmonic analysis.

9. Regularity of Navier-Stokes Equations in Besov Spaces

Theorem 9 (Local Existence and Regularity in Besov Spaces). *Let $n \geq 2$, $s > \frac{n}{p}$, and $1 \leq p, q \leq \infty$. Consider the incompressible Navier-Stokes system in $\mathbb{R}^n$:*

$$\begin{cases} \partial_t u - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{50}$$

where $u_0 \in B^s_{p,q}(\mathbb{R}^n)$ is divergence-free and $f \in L^1(0, T; B^s_{p,q}(\mathbb{R}^n))$.

Then there exists a time $T > 0$ and a unique solution

$$u \in X_T := L^\infty(0, T; B^s_{p,q}) \cap L^2(0, T; B^{s+1}_{p,q})$$

satisfying the system (50) and the estimate

$$\|u\|_{X_T} \leq C\Big(\|u_0\|_{B^s_{p,q}} + \|f\|_{L^1(0,T;B^s_{p,q})}\Big),$$

for some constant $C = C(s,p,q,\nu)$.

Furthermore, if $\|u_0\|_{B^s_{p,q}}$ is sufficiently small, the solution can be extended globally in time.

Proof. Define the solution map $\Phi : v \mapsto u$ where u solves the linearized problem:

$$\begin{cases} \partial_t u - \nu\Delta u + \nabla p = f - (v\cdot\nabla)v, \\ \nabla\cdot u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

Using the heat semigroup $e^{t\nu\Delta}$ acting on Besov spaces, we have the uniform bound

$$\|e^{t\nu\Delta}u_0\|_{B^s_{p,q}} \leq C\|u_0\|_{B^s_{p,q}}. \tag{51}$$

For the inhomogeneous term, the following estimate holds:

$$\left\|\int_0^t e^{(t-\tau)\nu\Delta}g(\tau)\,d\tau\right\|_{L^r(0,T;B^s_{p,q})} \leq C\|g\|_{L^\rho(0,T;B^{s-2+2/\rho}_{p,q})}, \tag{52}$$

valid for $1 \leq \rho \leq r \leq \infty$.

For the nonlinear term, the bilinear estimate (cf. [21]) states

$$\begin{aligned} \|(v\cdot\nabla)w\|_{B^{s-1}_{p,q}} &\lesssim \|v\|_{B^{n/p}_{p,1}}\|\nabla w\|_{B^{s-1}_{p,q}} \\ &\lesssim \|v\|_{B^s_{p,q}}\|w\|_{B^s_{p,q}}, \qquad s > \frac{n}{p}. \end{aligned} \tag{53}$$

Define the Banach space

$$\begin{aligned} X_T := \Big\{u : \quad &\|u\|_{X_T} := \|u\|_{L^\infty(0,T;B^s_{p,q})} \\ &+ \nu^{1/2}\|u\|_{L^2(0,T;B^{s+1}_{p,q})} < \infty\Big\}. \end{aligned} \tag{54}$$

For any $v \in X_T$, the solution $u = \Phi(v)$ satisfies

$$\begin{aligned} \|\Phi(v)\|_{X_T} \leq{}& C\|u_0\|_{B^s_{p,q}} + C\|f\|_{L^1(0,T;B^s_{p,q})} \\ &+ CT^{1/2}\nu^{-1/2}\|v\|_{L^\infty(0,T;B^s_{p,q})}\|v\|_{L^2(0,T;B^{s+1}_{p,q})} \\ \leq{}& C_0 + C_1T^{1/2}\nu^{-1/2}\|v\|^2_{X_T}, \end{aligned} \tag{55}$$

where C_0, C_1 depend on s,p,q.

Similarly, for $v_1, v_2 \in X_T$, we have

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{X_T} \leq{}& CT^{1/2}\nu^{-1/2}\|v_1 - v_2\|_{X_T} \\ &\times\big(\|v_1\|_{X_T} + \|v_2\|_{X_T}\big). \end{aligned} \tag{56}$$

Choosing $R > 2C_0$ and T sufficiently small so that

$$T < \left(4C_1R\nu^{-1/2}\right)^{-2},$$

the map Φ is a contraction on the closed ball $B_{X_T}(0, R)$, guaranteeing local existence and uniqueness by Banach's fixed point theorem.

To extend the solution globally, consider the energy inequality in Besov norms:

$$\begin{aligned} \frac{d}{dt}\|u(t)\|_{B^s_{p,q}}^q \leq C\|u(t)\|_{B^s_{p,q}}^q \|\nabla u(t)\|_{L^\infty} \\ + C\|f(t)\|_{B^s_{p,q}} \|u(t)\|_{B^s_{p,q}}^{q-1}. \end{aligned} \tag{57}$$

For $s > \frac{n}{p}$, the Gagliardo-Nirenberg inequality implies

$$\|u\|_{B^{s+1}_{p,q}} \geq c\|u\|_{B^s_{p,q}}^{\frac{s+1}{s}} \|u\|_{L^p}^{-\frac{1}{s}},$$

which controls the growth of the $B^s_{p,q}$-norm.

For the critical case $q = \infty$, logarithmic inequalities of Beale-Kato-Majda type yield

$$\|\nabla u\|_{L^\infty} \leq C\|u\|_{B^{n/p}_{p,1}} \leq C\|u\|_{B^s_{p,\infty}} \log\left(e + \|u\|_{B^{s+1}_{p,\infty}}\right),$$

providing control over growth and preventing blow-up on short time intervals.

□

10. Theorems and Proofs on Besov Spaces

10.1. Characterization of Besov Spaces via Littlewood-Paley Decomposition

Besov spaces unify and generalize classical smoothness spaces such as Sobolev and Hölder spaces and are fundamental in harmonic analysis and the theory of partial differential equations. Their characterization via Littlewood-Paley theory provides a powerful frequency-localized understanding of function regularity.

Theorem 10 (Littlewood-Paley Characterization of Besov Spaces)**.** *Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $s \in \mathbb{R}$, and $1 \leq p, q \leq \infty$. Then the Besov space $B^s_{p,q}(\mathbb{R}^n)$ consists of all tempered distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq}\|\Delta_j u\|^q_{L^p(\mathbb{R}^n)}\right)^{1/q} < \infty, \tag{58}$$

with the usual modification when $q = \infty$:

$$\|u\|_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{j\geq 0} 2^{js}\|\Delta_j u\|_{L^p(\mathbb{R}^n)}. \tag{59}$$

Here, $\{\Delta_j\}_{j\geq 0}$ is a dyadic Littlewood-Paley decomposition defined via Fourier multipliers $\varphi, \psi \in C_c^\infty(\mathbb{R}^n)$ satisfying

$$\begin{aligned} &\operatorname{supp}(\varphi) \subset \{|\xi| \leq 2\}, \\ &\operatorname{supp}(\psi) \subset \{1/2 \leq |\xi| \leq 2\}, \\ &\sum_{j=0}^{\infty} \psi(2^{-j}\xi) + \varphi(\xi) = 1, \\ &\quad \forall \xi \in \mathbb{R}^n. \end{aligned} \tag{60}$$

The operator Δ_j acts as:

$$\begin{aligned} &\Delta_j u := \mathcal{F}^{-1}\left(\psi(2^{-j}\cdot)\hat{u}\right) \quad \textit{for } j \geq 1, \\ &\qquad \textit{and} \\ &\Delta_0 u := \mathcal{F}^{-1}(\varphi\hat{u}). \end{aligned}$$

 doi:10.20944/preprints202510.0499.v1

Proof. We outline the rigorous justification of this characterization, leveraging harmonic analysis and functional analysis tools.

The Littlewood-Paley decomposition is a dyadic partition of unity in the frequency domain, localized in annuli of scale 2^j. The functions φ and ψ are smooth cutoffs enabling

$$\hat{u}(\xi) = \varphi(\xi)\hat{u}(\xi) + \sum_{j=1}^{\infty} \psi(2^{-j}\xi)\hat{u}(\xi). \tag{61}$$

Taking inverse Fourier transforms yields the decomposition

$$u = \Delta_0 u + \sum_{j=1}^{\infty} \Delta_j u, \tag{62}$$

where each $\Delta_j u$ is frequency localized in the annulus $\{\xi : |\xi| \approx 2^j\}$.

The classical definition of Besov spaces uses either differences or potential spaces; here, we focus on the equivalent norm via frequency localization:

$$\|u\|_{B^s_{p,q}} := \left\| 2^{js} \|\Delta_j u\|_{L^p} \right\|_{\ell^q(\mathbb{N})}. \tag{63}$$

This norm measures the decay/growth of the dyadic components of u weighted by 2^{js}, encoding smoothness s.

Classical Besov spaces defined via moduli of continuity or finite differences satisfy

$$\|u\|_{B^s_{p,q}} \sim \|u\|_{L^p} + \left(\int_0^{\infty} (t^{-s}\omega_k(u,t)_p)^q \frac{dt}{t} \right)^{1/q}, \tag{64}$$

where $\omega_k(u,t)_p$ is the k-th order modulus of smoothness of u in L^p.

Using the Paley-Littlewood decomposition, one can establish isomorphisms between these characterizations (see Triebel's monograph [3] for details). The dyadic decomposition captures the same smoothness scales as finite differences, enabling equivalence of norms.

Since φ and ψ are smooth, compactly supported multipliers, the operators Δ_j are bounded on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.

Furthermore, Bernstein inequalities guarantee control of derivatives and embeddings at each scale:

$$\|D^\alpha \Delta_j u\|_{L^p} \lesssim 2^{j|\alpha|} \|\Delta_j u\|_{L^p}. \tag{65}$$

This allows identification of the smoothness index s via scaling.

The space $B^s_{p,q}(\mathbb{R}^n)$ with the Littlewood-Paley norm is complete and separable (for $p, q < \infty$), making it a Banach space.

Collecting these facts, the Besov norm defined through the Littlewood-Paley projections is equivalent to classical Besov norms. The dyadic decomposition thus provides an effective, scale-localized characterization of Besov spaces.

This completes the proof. □

Remark: The Littlewood-Paley characterization of Besov spaces is pivotal in many applications, including nonlinear PDEs, where scale-by-scale analysis and frequency localization are crucial.

11. Extended Plancherel Theorem for Besov Spaces

This section presents an extension of the classical Plancherel Theorem within the framework of Besov spaces, utilizing the Littlewood-Paley decomposition. This extension is essential for capturing finer regularity properties of functions that are pivotal in the study of nonlinear PDEs and fluid dynamics.

11.1. Extended Plancherel Theorem for Besov Spaces

Theorem 11. *Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $u \in B^s_{p,q}(\mathbb{R}^n)$ be a function in the Besov space with parameters $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$. Then the Fourier transform operator $\mathcal{F}$ induces an isomorphism on $B^s_{p,q}(\mathbb{R}^n)$, i.e.,*

$$\|u\|_{B^s_{p,q}(\mathbb{R}^n)} \sim \|\hat{u}\|_{B^s_{p,q}(\mathbb{R}^n)}, \tag{66}$$

where $\hat{u} = \mathcal{F}(u)$ denotes the Fourier transform of u.

Proof. We begin by recalling the Littlewood-Paley decomposition associated with u:

$$u = \sum_{j \geq 0} \Delta_j u, \tag{67}$$

where Δ_j are frequency localization operators defined by Fourier multipliers $\varphi(2^{-j}\xi)$ with $\varphi \in C_c^\infty(\mathbb{R}^n)$ supported in dyadic annuli. Precisely,

$$\widehat{\Delta_j u}(\xi) = \varphi(2^{-j}\xi)\hat{u}(\xi). \tag{68}$$

By the definition of the Besov norm,

$$\|u\|^q_{B^s_{p,q}} = \sum_{j \geq 0} 2^{sqj} \|\Delta_j u\|^q_{L^p}. \tag{69}$$

Using (68), the L^p-norm of $\Delta_j u$ satisfies

$$\|\Delta_j u\|_{L^p} = \left\| \mathcal{F}^{-1}\left(\varphi(2^{-j}\cdot)\hat{u}(\cdot)\right) \right\|_{L^p}. \tag{70}$$

Note that the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$ are isometries on $L^2(\mathbb{R}^n)$ but extend continuously to Besov spaces $B^s_{p,q}$ due to the smooth compact support of φ.

Now, applying the Fourier transform to $\Delta_j u$ yields

$$\widehat{\Delta_j u}(\xi) = \varphi(2^{-j}\xi)\hat{u}(\xi),$$

which can be interpreted as the Littlewood-Paley projection applied directly to $\hat{u}$. This means

$$\|\hat{u}\|^q_{B^s_{p,q}} = \sum_{j \geq 0} 2^{sqj} \|\varphi(2^{-j}\cdot)\hat{u}\|^q_{L^p}. \tag{71}$$

To complete the equivalence (66), we use the properties of the Fourier transform in L^p spaces and multiplier theory:

- Since $\varphi \in C_c^\infty$, $\varphi(2^{-j}\cdot)$ acts as a smooth Fourier multiplier localized on dyadic annuli, ensuring boundedness on L^p for $1 \leq p \leq \infty$.
- The Littlewood-Paley operators Δ_j and their Fourier multipliers $\varphi(2^{-j}\cdot)$ satisfy the partition of unity property with disjoint supports, allowing the norm equivalences to hold by Plancherel-type arguments extended to Besov scales.
- Utilizing standard multiplier theorems (Mihlin-Hörmander), we conclude the operator norms are uniformly bounded and the dyadic decompositions of u and $\hat{u}$ correspond in $B^s_{p,q}$.

Therefore, there exist constants $C_1, C_2 > 0$ independent of u such that

$$C_1\|u\|_{B^s_{p,q}} \leq \|\hat{u}\|_{B^s_{p,q}} \leq C_2\|u\|_{B^s_{p,q}},$$

which establishes the isomorphism (66) and completes the proof. □

Remarks: This theorem generalizes the classical Plancherel identity on L^2 to the richer framework of Besov spaces. It allows one to analyze the regularity and integrability properties of functions and their Fourier transforms in a unified way, proving especially useful in nonlinear PDE analysis where the finer structure of solutions must be captured.

The Littlewood-Paley decomposition plays a pivotal role in defining and understanding Besov spaces, providing a multiscale framework where frequency localization reflects function smoothness and oscillation.

This extended Plancherel theorem is instrumental for studying advanced fluid dynamics, dispersive PDEs, and signal processing, where control of both space and frequency behavior is essential.

12. Regularity of Navier–Stokes Equations in Besov Spaces

In this section, we rigorously investigate the regularity properties of solutions to the Navier–Stokes equations within the framework of Besov spaces. This approach provides a refined scale of regularity beyond classical Sobolev spaces and is particularly suited for analyzing the interplay between nonlinear terms and dissipation mechanisms inherent in the equations.

12.1. Theorem: Regularity in Besov Spaces

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space with $n \geq 2$ and $T > 0$. Consider the incompressible Navier–Stokes equations:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{72}$$

where $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity field, $p = p(t,x)$ is the pressure, $\nu > 0$ is the kinematic viscosity, and $\mathbf{f} = \mathbf{f}(t,x)$ is an external force.

Theorem 12 (Regularity in Besov Spaces)**.** *Let $s > \frac{n}{p}$ with $1 \leq p,q \leq \infty$, and let the initial data satisfy*

$$\mathbf{u}_0 \in B^s_{p,q}(\mathbb{R}^n), \quad \nabla \cdot \mathbf{u}_0 = 0,$$

and the external force

$$\mathbf{f} \in L^1(0,T; B^s_{p,q}(\mathbb{R}^n)).$$

Then, there exists a time $T^* \leq T$ such that the Navier–Stokes equations (130) admit a unique solution

$$\mathbf{u} \in C([0,T^*]; B^s_{p,q}(\mathbb{R}^n)) \cap L^2(0,T^*; B^{s+1}_{p,q}(\mathbb{R}^n)).$$

Moreover, if the initial norm $\|\mathbf{u}_0\|_{B^s_{p,q}}$ is sufficiently small (relative to ν), the solution exists globally in time, i.e., $T^* = T$.

12.2. Proof.

The proof proceeds via a fixed point argument on an appropriate Banach space, exploiting the linear semigroup associated with the heat operator and the bilinear structure of the nonlinearity.

12.2.1. Functional Setting

Define the solution space

$$X_T := \left\{ \mathbf{u} \in C([0,T]; B^s_{p,q}(\mathbb{R}^n)) \cap L^2(0,T; B^{s+1}_{p,q}(\mathbb{R}^n)) \right\}, \tag{73}$$

 doi:10.20944/preprints202510.0499.v1

endowed with the norm

$$\|\mathbf{u}\|_{X_T} := \|\mathbf{u}\|_{L^\infty(0,T;B^s_{p,q})} + \nu^{1/2}\|\mathbf{u}\|_{L^2(0,T;B^{s+1}_{p,q})}. \tag{74}$$

12.2.2. Mild Formulation

The mild formulation of (130) reads:

$$\begin{aligned}\mathbf{u}(t) = e^{t\nu\Delta}\mathbf{u}_0 - \int_0^t e^{(t-\tau)\nu\Delta}\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})(\tau)\,d\tau \\ + \int_0^t e^{(t-\tau)\nu\Delta}\mathbb{P}\mathbf{f}(\tau)\,d\tau.\end{aligned} \tag{75}$$

where $\mathbb{P}$ is the Leray projection onto divergence-free vector fields.

12.2.3. Linear Estimates

The heat semigroup satisfies the smoothing estimate in Besov spaces:

$$\|e^{t\nu\Delta}\mathbf{u}_0\|_{B^s_{p,q}} \le \|\mathbf{u}_0\|_{B^s_{p,q}}, \tag{76}$$

and for the inhomogeneous term:

$$\left\|\int_0^t e^{(t-\tau)\nu\Delta}g(\tau)\,d\tau\right\|_{L^\infty(0,T;B^s_{p,q})} \le C\|g\|_{L^1(0,T;B^s_{p,q})}. \tag{77}$$

12.2.4. Nonlinear Estimates

The bilinear estimate for the convection term is

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{B^{s-1}_{p,q}} \le C\|\mathbf{u}\|^2_{B^s_{p,q}}, \tag{78}$$

valid whenever $s > n/p$, due to the algebra property of Besov spaces:

$$B^s_{p,q}(\mathbb{R}^n) \text{ is an algebra if } s > \frac{n}{p}.$$

Moreover, the composition of the divergence operator with the paraproduct satisfies

$$\|\nabla\cdot(\mathbf{u}\otimes\mathbf{v})\|_{B^{s-1}_{p,q}} \le C\|\mathbf{u}\|_{B^s_{p,q}}\|\mathbf{v}\|_{B^s_{p,q}}. \tag{79}$$

12.2.5. Fixed Point Argument

Define the solution map

$$\begin{aligned}\Phi(\mathbf{v})(t) := e^{t\nu\Delta}\mathbf{u}_0 - \int_0^t e^{(t-\tau)\nu\Delta}\mathbb{P}\nabla\cdot(\mathbf{v}\otimes\mathbf{v})(\tau)\,d\tau \\ + \int_0^t e^{(t-\tau)\nu\Delta}\mathbb{P}\mathbf{f}(\tau)\,d\tau.\end{aligned} \tag{80}$$

Using (76), (77), and (79), we obtain the estimate:

$$\|\Phi(\mathbf{v})\|_{X_T} \le C_0 + C_1T^{1/2}\nu^{-1/2}\|\mathbf{v}\|^2_{X_T}, \tag{81}$$

where C_0 depends on $\|\mathbf{u}_0\|_{B^s_{p,q}}$ and $\|\mathbf{f}\|_{L^1(0,T;B^s_{p,q})}$.
Similarly, for the difference,

$$\begin{aligned}\|\Phi(\mathbf{v}_1) - \Phi(\mathbf{v}_2)\|_{X_T} \le CT^{1/2}\nu^{-1/2}\|\mathbf{v}_1 - \mathbf{v}_2\|_{X_T} \\ \times(\|\mathbf{v}_1\|_{X_T} + \|\mathbf{v}_2\|_{X_T}).\end{aligned} \tag{82}$$

doi:10.20944/preprints202510.0499.v1

Choosing $T > 0$ sufficiently small such that

$$C_1 T^{1/2} \nu^{-1/2} R < \frac{1}{2}$$

for a suitable radius R, the map Φ becomes a contraction on the ball of radius R in X_T.

12.2.6. A Priori Estimates and Global Existence

Deriving an a priori estimate from the mild formulation leads to

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{B^s_{p,q}} \leq C\|\mathbf{u}(t)\|^2_{B^s_{p,q}} + \|\mathbf{f}(t)\|_{B^s_{p,q}}. \tag{83}$$

Applying Grönwall's inequality yields the bound

$$\|\mathbf{u}(t)\|_{B^s_{p,q}} \leq \left(\|\mathbf{u}_0\|_{B^s_{p,q}} + \int_0^t \|\mathbf{f}(\tau)\|_{B^s_{p,q}} d\tau\right) e^{Ct\|\mathbf{u}\|_{B^s_{p,q}}}. \tag{84}$$

Thus, for small initial data relative to the viscosity ν, the solution persists globally.

13. Detailed Proof of Higher-Order Sobolev Regularity

Proof. Consider the Navier-Stokes equations in weak formulation:

$$\int_\Omega \partial_t u \cdot v \,\mathrm{d}x + \nu \int_\Omega \nabla u : \nabla v \,\mathrm{d}x + \int_\Omega (u \cdot \nabla) u \cdot v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x \tag{85}$$

for all divergence-free $v \in H^1_0(\Omega)$. For Galerkin approximation $u^N = \sum_{k=1}^N c_k^N(t) w_k$, we have:

$$\frac{dc_k^N}{dt} + \nu\lambda_k c_k^N + \sum_{i,j=1}^N B_{ijk} c_i^N c_j^N = f_k^N \tag{86}$$

where $B_{ijk} = \int_\Omega (w_i \cdot \nabla) w_j \cdot w_k \,\mathrm{d}x$ and $f_k^N = \int_\Omega f \cdot w_k \,\mathrm{d}x$.

To establish higher regularity, multiply by $\lambda_k c_k^N$ and sum:

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\nabla u^N\|^2_{L^2} + \nu\|\Delta u^N\|^2_{L^2} \\ &= -\int_\Omega (u^N \cdot \nabla) u^N \cdot \Delta u^N \,\mathrm{d}x \\ &\quad + \int_\Omega f \cdot \Delta u^N \,\mathrm{d}x \end{aligned} \tag{87}$$

Using Ladyzhenskaya's inequality:

$$\begin{aligned} \left|\int_\Omega (u^N \cdot \nabla) u^N \cdot \Delta u^N \,\mathrm{d}x\right| &\leq C\|u^N\|_{L^4}\|\nabla u^N\|_{L^4}\|\Delta u^N\|_{L^2} \\ &\leq C\|\nabla u^N\|^{1/2}_{L^2}\|\Delta u^N\|^{3/2}_{L^2} \\ &\quad \text{(by Sobolev embedding).} \end{aligned} \tag{88}$$

By Young's inequality with ϵ:

$$C\|\nabla u^N\|^{1/2}_{L^2}\|\Delta u^N\|^{3/2}_{L^2} \leq \frac{\nu}{4}\|\Delta u^N\|^2_{L^2} + C(\nu)\|\nabla u^N\|^2_{L^2} \tag{89}$$

Similarly for the forcing term:

$$\left|\int_\Omega f \cdot \Delta u^N \,\mathrm{d}x\right| \leq \frac{1}{\nu}\|f\|^2_{L^2} + \frac{\nu}{4}\|\Delta u^N\|^2_{L^2} \tag{90}$$

Combining estimates:

$$\begin{aligned}\frac{d}{dt}\|\nabla u^N\|_{L^2}^2 + \nu\|\Delta u^N\|_{L^2}^2 \le C(\nu)\|\nabla u^N\|_{L^2}^2 + \frac{2}{\nu}\|f\|_{L^2}^2 \\ + \frac{\nu}{2}\|\Delta u^N\|_{L^2}^2 \\ - \frac{\nu}{2}\|\Delta u^N\|_{L^2}^2\end{aligned} \tag{91}$$

Thus:

$$\frac{d}{dt}\|\nabla u^N\|_{L^2}^2 + \frac{\nu}{2}\|\Delta u^N\|_{L^2}^2 \le C(\nu)\|\nabla u^N\|_{L^2}^2 + \frac{2}{\nu}\|f\|_{L^2}^2 \tag{92}$$

Grönwall's lemma yields uniform bounds in $L^\infty(0,T;H^1) \cap L^2(0,T;H^2)$. For higher regularity H^k, use induction: assume $u \in L^2(0,T;H^m)$ for $m < k$ and consider $\partial^\alpha u$ for $|\alpha| = k$:

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|\partial^\alpha u^N\|_{L^2}^2 + \nu\|\nabla\partial^\alpha u^N\|_{L^2}^2 \\ &= -\int_\Omega \partial^\alpha\left[(u^N\cdot\nabla)u^N\right]\cdot\partial^\alpha u^N \, \mathrm{d}x \\ &\quad + \int_\Omega \partial^\alpha f\cdot\partial^\alpha u^N \, \mathrm{d}x.\end{aligned} \tag{93}$$

The critical term is decomposed using Bony's paraproduct:

$$\begin{aligned}\partial^\alpha\left[(u^N\cdot\nabla)u^N\right] &= \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}u^N\cdot\nabla\partial^\beta u^N \\ &= T_{\nabla u^N}(\partial^\alpha u^N) + T_{\partial^\alpha u^N}(\nabla u^N) \\ &\quad + R(u^N,\nabla u^N).\end{aligned} \tag{94}$$

By Coifman-Meyer estimates in Sobolev spaces:

$$\begin{aligned}\|T_{\nabla u^N}(\partial^\alpha u^N)\|_{L^2} &\le C\|\nabla u^N\|_{L^\infty}\|\partial^\alpha u^N\|_{L^2} \\ \|T_{\partial^\alpha u^N}(\nabla u^N)\|_{L^2} &\le C\|\partial^\alpha u^N\|_{L^2}\|\nabla u^N\|_{L^\infty} \\ \|R(u^N,\nabla u^N)\|_{L^2} &\le C\|u^N\|_{\dot H^k}\|\nabla u^N\|_{\dot H^{k-1}}\end{aligned} \tag{95}$$

When $k > n/2+1$, Sobolev embedding $H^k \hookrightarrow W^{1,\infty}$ gives:

$$\frac{d}{dt}\|u^N\|_{H^k}^2 \le C(\nu)\|u^N\|_{H^k}^3 + C\|f\|_{H^k}^2 \tag{96}$$

Local existence follows from Grönwall. Global existence requires small data: if $\|u_0\|_{H^2} + \|f\|_{L^2(0,T;H^2)}$ sufficiently small, the solution remains bounded. □

14. Energy Dissipation in Besov Spaces

Consider the Navier-Stokes equations in $\mathbb{R}^n$:

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \tag{97}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{98}$$

where $\mathbf{u} = \mathbf{u}(x,t)$ is the velocity field, $p = p(x,t)$ is the pressure, $\nu > 0$ is the kinematic viscosity, and $\mathbf{f}$ is an external force.

The total kinetic energy of the fluid is defined as

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^n}|\mathbf{u}(x,t)|^2\,dx. \tag{99}$$

 doi:10.20944/preprints202510.0499.v1

Differentiating $E(t)$ with respect to time gives

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^n} \mathbf{u} \cdot \frac{\partial \mathbf{u}}{\partial t}\, dx. \tag{100}$$

Substituting the momentum equation (97) into (100), we obtain

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^n} \mathbf{u} \cdot (-\nabla p + \nu \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} + \mathbf{f}) dx. \tag{101}$$

Analyzing each term:

Pressure term:

$$\int_{\mathbb{R}^n} \mathbf{u} \cdot \nabla p\, dx = 0, \tag{102}$$

since

$$\int_{\mathbb{R}^n} \nabla \cdot (p\mathbf{u})\, dx = 0, \tag{103}$$

by the divergence theorem and the incompressibility condition (98).

Viscous dissipation term:

$$\int_{\mathbb{R}^n} \mathbf{u} \cdot \nu \Delta \mathbf{u}\, dx = -\nu \int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2\, dx, \tag{104}$$

via integration by parts.

Nonlinear term:

$$\int_{\mathbb{R}^n} \mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u}\, dx = 0, \tag{105}$$

because

$$\int_{\mathbb{R}^n} \nabla \cdot \left(\frac{|\mathbf{u}|^2}{2} \mathbf{u} \right) dx = 0, \tag{106}$$

using incompressibility.

External force term:

$$\int_{\mathbb{R}^n} \mathbf{u} \cdot \mathbf{f}\, dx, \tag{107}$$

which represents the energy input from the external force.

Substituting (102)–(107) into (101) yields the classical energy balance equation:

$$\frac{d}{dt}E(t) = -\nu \int_{\mathbb{R}^n} |\nabla \mathbf{u}|^2\, dx + \int_{\mathbb{R}^n} \mathbf{u} \cdot \mathbf{f}\, dx. \tag{108}$$

Energy Dissipation at Each Scale

Applying the Littlewood-Paley decomposition

$$\mathbf{u} = \sum_{j=-\infty}^{\infty} \Delta_j \mathbf{u}, \tag{109}$$

the energy at scale j is

$$E_j(t) = \frac{1}{2} \|\Delta_j \mathbf{u}(t)\|_{L^2}^2. \tag{110}$$

Differentiating $E_j(t)$ gives

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

$$\begin{aligned} \frac{d}{dt}E_j(t) = -\nu\|\nabla\Delta_j\mathbf{u}(t)\|_{L^2}^2 \\ + \text{Nonlinear Interactions} \\ + \int \Delta_j\mathbf{u}\cdot\Delta_j\mathbf{f}\,dx. \end{aligned} \tag{111}$$

The dissipation term satisfies the scaling

$$\|\nabla\Delta_j\mathbf{u}\|_{L^2}^2 \approx 2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}^2, \tag{112}$$

so the viscous dissipation at frequency 2^j grows quadratically with 2^j.

Total Energy Dissipation Formula in Besov Norm

Summing over all scales leads to the dissipation expressed in terms of Besov norms:

$$\sum_j 2^{2j}\|\Delta_j\mathbf{u}\|_{L^2}^2 \sim \|\mathbf{u}\|_{B^1_{2,2}}^2 = \|\mathbf{u}\|_{H^1}^2, \tag{113}$$

showing that energy dissipation corresponds to the H^1 (or $B^1_{2,2}$) norm squared of the velocity field.

Interpretation

Equation (112) reveals that energy dissipation predominantly occurs at high frequencies, where j is large, consistent with the physical intuition that smaller scales (eddies) are responsible for most of the viscous dissipation.

This analysis rigorously connects the classical energy balance to the frequency-localized perspective provided by Besov spaces and Littlewood-Paley theory, offering a refined understanding of how dissipation is distributed across scales in turbulent flows.

15. Quaternionic Bifurcations in Fluid Dynamics

Quaternionic analysis provides a powerful framework for modeling rotations, symmetries, and instabilities in three-dimensional fluid dynamics. This approach allows us to encode the velocity field and its rotational structures compactly using quaternion-valued functions.

15.1. Quaternionic Formulation of Navier-Stokes Equations

Let a quaternion-valued velocity field be expressed as

$$q(x,t) = q_0(x,t) + q_1(x,t)i + q_2(x,t)j + q_3(x,t)k, \tag{114}$$

where i, j, k are the quaternionic imaginary units satisfying the relations

$$i^2 = j^2 = k^2 = ijk = -1, \tag{115}$$

and the multiplication rules

$$\begin{aligned} ij = k, \quad jk = i, \quad ki = j, \\ ji = -k, \quad kj = -i, \quad ik = -j. \end{aligned} \tag{116}$$

The quaternionic Navier–Stokes equations for an incompressible flow read

$$\frac{\partial q}{\partial t} + (q\cdot\nabla)q = -\nabla p + \nu\Delta q, \tag{117}$$

subject to the incompressibility constraint

$$\mathrm{Re}(\nabla\cdot q) = 0. \tag{118}$$

 doi:10.20944/preprints202510.0499.v1

Here, the quaternionic gradient operator acts as

$$\nabla = \partial_{x_1} i + \partial_{x_2} j + \partial_{x_3} k, \tag{119}$$

and the Laplacian is applied component-wise.

15.2. Linearization and Stability Analysis

Consider a steady-state solution q_s of (117). Introduce a perturbation of the form

$$q(x,t) = q_s(x) + \varepsilon q_1(x,t), \tag{120}$$

where $0 < \varepsilon \ll 1$ and q_1 is the perturbation field.

Substituting (120) into (117) and linearizing in ε yields the linearized equation:

$$\frac{\partial q_1}{\partial t} = L(q_1), \tag{121}$$

where the linear operator L is defined by

$$L(q_1) = -(q_s \cdot \nabla) q_1 - (q_1 \cdot \nabla) q_s + \nu \Delta q_1 - \nabla p_1, \tag{122}$$

subject to the divergence-free constraint

$$\mathrm{Re}(\nabla \cdot q_1) = 0. \tag{123}$$

15.3. Eigenvalue Problem and Bifurcation Criterion

The linear stability is determined by solving the eigenvalue problem

$$L(q_1) = \lambda q_1, \tag{124}$$

where $\lambda \in \mathbb{C}$ is the eigenvalue.

Bifurcation Criterion: A bifurcation occurs when the real part of an eigenvalue crosses zero:

$$\mathrm{Re}(\lambda) = 0. \tag{125}$$

In particular, if a pair of complex conjugate eigenvalues cross the imaginary axis, a Hopf-type bifurcation occurs, leading to oscillatory dynamics:

$$\lambda = \pm i\omega, \quad \omega \in \mathbb{R}^+. \tag{126}$$

15.4. Energy Method for Stability

Taking the L^2 inner product of (121) with q_1 and using the incompressibility condition, we obtain the energy balance equation

$$\frac{1}{2}\frac{d}{dt}\|q_1\|_{L^2}^2 = -\nu \|\nabla q_1\|_{L^2}^2 + B(q_s, q_1), \tag{127}$$

where the bilinear interaction term is

$$B(q_s, q_1) = -\langle (q_s \cdot \nabla) q_1, q_1 \rangle - \langle (q_1 \cdot \nabla) q_s, q_1 \rangle. \tag{128}$$

The viscous term always dissipates energy, while the bilinear term can either stabilize or destabilize depending on the alignment of the base flow q_s and the perturbation q_1.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 7 October 2025 doi:10.20944/preprints202510.0499.v1

15.5. Rigorous Bifurcation Theorem

Theorem 13 (Quaternionic Bifurcation Criterion)**.** *Let q_s be a steady solution of the quaternionic Navier-Stokes equations (117). Suppose the linearized operator L defined in (122) has a simple eigenvalue $\lambda(\mu)$ depending smoothly on a parameter μ (e.g., Reynolds number). If at $\mu = \mu_c$ the following hold:*

$$\mathrm{Re}(\lambda(\mu_c)) = 0 \quad \textit{and} \quad \frac{d}{d\mu}\mathrm{Re}(\lambda(\mu))|_{\mu=\mu_c} \neq 0, \tag{129}$$

then a local bifurcation occurs from the trivial solution at $\mu = \mu_c$, leading to the emergence of a branch of nontrivial time-periodic or steady-state solutions, depending on whether $\lambda(\mu_c)$ is purely imaginary or real.

This quaternionic formulation captures both the rotational symmetries and the bifurcation phenomena intrinsic to three-dimensional fluid flows. The bifurcation analysis via the spectrum of the linearized quaternionic operator L provides a rigorous tool to detect the onset of instabilities, oscillations, or transition to turbulence.

16. The Navier–Stokes Equations in $\mathbb{R}^3$

The incompressible Navier–Stokes equations describe the evolution of a velocity field $\mathbf{u}(t,x)$ and pressure $p(t,x)$ in a viscous incompressible fluid:

$$\begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla\cdot\mathbf{u} = 0, \end{cases} \tag{130}$$

where $\nu > 0$ is the kinematic viscosity, and $\mathbf{f}$ is a given external force. We consider $(t,x) \in [0,T] \times \mathbb{R}^3$.

17. Existence of Solutions in Besov Spaces

Theorem 14 (Existence in Besov Spaces)**.** *Let $\mathbf{u}_0 \in B^s_{p,q}(\mathbb{R}^3)$ and $\mathbf{f} \in L^r(0,T;B^s_{p,q}(\mathbb{R}^3))$ with $s > 3/p$ and $r \geq 1$. Then, for sufficiently small $T > 0$ (or sufficiently large $\nu > 0$), there exists a unique mild solution*

$$\mathbf{u} \in C([0,T];B^s_{p,q}(\mathbb{R}^3))$$

to the Navier–Stokes system (130).

Proof. Introduce the Stokes semigroup $e^{t\nu\Delta}$ and the Leray projector $\mathbb{P}$ onto divergence-free vector fields. Then a mild solution satisfies

$$\begin{aligned} \mathbf{u}(t) = e^{t\nu\Delta}\mathbf{u}_0 & \\ & - \int_0^t e^{(t-\tau)\nu\Delta}\mathbb{P}\nabla\cdot(\mathbf{u}\otimes\mathbf{u})(\tau)\,d\tau \\ & + \int_0^t e^{(t-\tau)\nu\Delta}\mathbb{P}\mathbf{f}(\tau)\,d\tau \end{aligned} \tag{131}$$

Using Bony's paraproduct decomposition, the nonlinear term satisfies

$$\|(\mathbf{u}\cdot\nabla)\mathbf{u}\|_{B^s_{p,q}} \leq C\|\mathbf{u}\|^2_{B^s_{p,q}}, \quad s > 3/p. \tag{132}$$

This relies on the embedding $B^s_{p,q} \hookrightarrow L^\infty$ for $s > 3/p$ and continuity of the paraproduct in Besov spaces.

 doi:10.20944/preprints202510.0499.v1

Define the mapping Φ on $X_T := C([0,T]; B^s_{p,q})$ by the right-hand side of (131). Using the bilinear estimate (132) and the contractivity of $e^{t\nu\Delta}$ in $B^s_{p,q}$, we obtain

$$\|\Phi(\mathbf{u})\|_{X_T} \leq \|\mathbf{u}_0\|_{B^s_{p,q}} + CT\|\mathbf{u}\|^2_{X_T} + \int_0^T \|\mathbf{f}(\tau)\|_{B^s_{p,q}} d\tau.$$

For small T, Φ is a contraction, and Banach's fixed-point theorem guarantees existence and uniqueness of $\mathbf{u} \in X_T$. □

18. Uniqueness of Solutions

Theorem 15 (Uniqueness)**.** *Let $\mathbf{u}_1, \mathbf{u}_2 \in C([0,T]; B^s_{p,q})$ be two solutions with the same initial data $\mathbf{u}_0$ and force $\mathbf{f}$. Then $\mathbf{u}_1 = \mathbf{u}_2$.*

Proof. Set $\mathbf{v} = \mathbf{u}_1 - \mathbf{u}_2$. Then $\mathbf{v}$ satisfies

$$\partial_t \mathbf{v} - \nu\Delta\mathbf{v} + (\mathbf{u}_1 \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u}_2 = 0, \quad \mathbf{v}(0) = 0. \tag{133}$$

Using the bilinear estimates in Besov spaces,

$$\|(\mathbf{u}_1 \cdot \nabla)\mathbf{v}\|_{B^s_{p,q}} + \|(\mathbf{v} \cdot \nabla)\mathbf{u}_2\|_{B^s_{p,q}} \leq C\big(\|\mathbf{u}_1\|_{B^s_{p,q}} + \|\mathbf{u}_2\|_{B^s_{p,q}}\big)\|\mathbf{v}\|_{B^s_{p,q}}.$$

Hence,

$$\frac{d}{dt}\|\mathbf{v}\|_{B^s_{p,q}} \leq C\big(\|\mathbf{u}_1\|_{B^s_{p,q}} + \|\mathbf{u}_2\|_{B^s_{p,q}}\big)\|\mathbf{v}\|_{B^s_{p,q}}.$$

By Grönwall's inequality and $\mathbf{v}(0) = 0$, we conclude $\mathbf{v} \equiv 0$, proving uniqueness. □

19. Regularity of Solutions

Theorem 16 (Regularity in Besov Spaces)**.** *Under the assumptions of Theorem 14, the solution satisfies*

$$\mathbf{u} \in C([0,T]; B^s_{p,q}(\mathbb{R}^3)).$$

Proof. From (130), we obtain

$$\frac{d}{dt}\|\mathbf{u}(t)\|_{B^s_{p,q}} \leq \|\mathbf{f}(t)\|_{B^s_{p,q}} + C\|\mathbf{u}(t)\|^2_{B^s_{p,q}}. \tag{134}$$

Integrating (134) in time gives

$$\begin{aligned}\|\mathbf{u}(t)\|_{B^s_{p,q}} \leq \|\mathbf{u}_0\|_{B^s_{p,q}} &\\ &+ \int_0^t \|\mathbf{f}(\tau)\|_{B^s_{p,q}}\, d\tau \\ &+ C\int_0^t \|\mathbf{u}(\tau)\|^2_{B^s_{p,q}}\, d\tau.\end{aligned}$$

Setting $A(t) = \|\mathbf{u}(t)\|_{B^s_{p,q}}$ and $F(t) = \int_0^t \|\mathbf{f}(\tau)\|_{B^s_{p,q}} d\tau$, we have

$$A(t) \leq A(0) + F(t) + C\int_0^t A(\tau)^2 d\tau.$$

By standard differential inequalities for quadratic nonlinearities, this implies

$$A(t) \leq \frac{A(0) + F(t)}{1 - C(A(0) + F(t))t},$$

provided $C(A(0) + F(t))t < 1$. This guarantees that the solution remains in $B^s_{p,q}$ for short times or sufficiently small initial data.

doi:10.20944/preprints202510.0499.v1

Since $t \mapsto A(t)$ is continuous, we conclude

$$\mathbf{u} \in C([0,T]; B^s_{p,q}(\mathbb{R}^3)).$$

□

20. Energy Estimates in Besov Spaces and Turbulence

In turbulent flows, energy cascades from large to small scales. Besov spaces provide a natural framework to quantify this multiscale behavior because they capture both local regularity and scale-dependent decay.

20.1. Energy Estimates

Let $\mathbf{u} \in C([0,T]; B^s_{p,q}(\mathbb{R}^3))$ be a solution to (130). Applying the Littlewood-Paley decomposition $\mathbf{u} = \sum_j \Delta_j \mathbf{u}$ and using the projector $\mathbb{P}$, we obtain for each dyadic block:

$$\begin{aligned} \frac{d}{dt}\|\Delta_j \mathbf{u}\|^2_{L^2} + 2\nu\|\nabla\Delta_j \mathbf{u}\|^2_{L^2} \\ \le 2\left|\langle \Delta_j((\mathbf{u}\cdot\nabla)\mathbf{u}), \Delta_j \mathbf{u}\rangle\right| \\ + 2\langle \Delta_j \mathbf{f}, \Delta_j \mathbf{u}\rangle. \end{aligned} \tag{135}$$

20.2. Nonlinear Term Estimate

Using Bony's paraproduct decomposition,

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = T_{\mathbf{u}}\nabla\mathbf{u} + T_{\nabla\mathbf{u}}\mathbf{u} + R(\mathbf{u}, \nabla\mathbf{u}),$$

and the continuity of paraproducts in Besov spaces, we have for $s > 3/p$:

$$\|\Delta_j((\mathbf{u}\cdot\nabla)\mathbf{u})\|_{L^2} \le C2^{-js}\|\mathbf{u}\|_{B^s_{p,q}}\|\mathbf{u}\|_{B^s_{p,q}}. \tag{136}$$

20.3. Global Energy Bound in Besov Norm

Summing over $j \ge -1$ and taking the l^q-norm in j, we obtain

$$\frac{d}{dt}\|\mathbf{u}\|_{B^s_{p,q}} \le C\|\mathbf{u}\|^2_{B^s_{p,q}} + \|\mathbf{f}\|_{B^s_{p,q}}, \tag{137}$$

which is consistent with the a priori estimate in Theorem 16. This inequality captures the *energy transfer across scales*, a key feature in turbulence modeling.

20.4. Relation with Turbulence

Besov norms $\|\Delta_j \mathbf{u}\|_{L^p}$ quantify the energy content at scale 2^{-j}. For turbulent flows:

- Large j correspond to small scales (high frequencies) where dissipation dominates.
- Small j correspond to large scales where energy is injected.
- The decay of $2^{js}\|\Delta_j \mathbf{u}\|_{L^p}$ with j characterizes the energy cascade, consistent with Kolmogorov-type scaling in isotropic turbulence.

Thus, Besov spaces provide a rigorous framework to **analyze multiscale energy distributions**, offering a bridge between PDE theory and physical turbulence phenomena.

Results

Higher-Order Sobolev Regularity

We established **higher-order Sobolev regularity** for solutions to the Navier-Stokes equations using **Galerkin approximations** and detailed **energy estimates**. Under suitable conditions on the initial data $u_0 \in H^k(\Omega)$ and external forces $f \in L^2(0,T; H^k(\Omega))$, the solution u satisfies:

doi:10.20944/preprints202510.0499.v1

$$u \in L^2(0,T;H^{k+2}(\Omega)).$$

This result is pivotal for understanding the **mathematical structure of fluid flows** and contributes to the exploration of the **Millennium Prize Problem**.

Characterization of Besov Spaces

We provided a **rigorous characterization of Besov spaces** through the **Littlewood-Paley decomposition**, enabling the analysis of **multifractal and irregular behaviors** in turbulent flows. The **extended Plancherel theorem** for Besov spaces connects **spatial derivatives** with **frequency-localized norms**, offering a refined framework for studying **nonlinear PDEs** and **energy dissipation** in turbulence.

Quaternionic Bifurcations

The **quaternionic formulation** of the Navier-Stokes equations introduces a novel approach to modeling **rotational symmetries and bifurcation phenomena** in three-dimensional fluid dynamics. The **bifurcation criterion**, derived from the spectrum of the linearized quaternionic operator, provides a rigorous tool for detecting **instabilities and transitions to turbulence**.

Energy Dissipation in Besov Spaces

Our analysis of **energy dissipation** in Besov spaces reveals that dissipation predominantly occurs at **high frequencies**, where smaller scales (eddies) dominate viscous dissipation. This aligns with physical intuition and provides a **mathematical framework** for understanding the **energy distribution** across scales in turbulent flows.

Regularity and Uniqueness of Solutions

We established the **regularity and uniqueness** of solutions to the Navier-Stokes equations in Besov spaces. For initial data $\mathbf{u}_0 \in B^s_{p,q}(\mathbb{R}^3)$ and external forces $\mathbf{f} \in L^r(0,T;B^s_{p,q}(\mathbb{R}^3))$, there exists a unique solution:

$$\mathbf{u} \in C([0,T];B^s_{p,q}(\mathbb{R}^3)).$$

This result is fundamental for addressing the **Millennium Prize Problem** and provides a **solid foundation** for future research.

Conclusions

This study advances the **mathematical understanding** of the Navier-Stokes equations within the framework of **Sobolev and Besov functional spaces**, offering new insights into **regularity, bifurcations, and turbulence** in fluid dynamics. By integrating **interpolation theory, Littlewood-Paley decomposition, and energy cascade models**, we developed a **unified framework** for analyzing these complex phenomena.

A key contribution of this work is the establishment of **higher-order Sobolev regularity** for solutions to the Navier-Stokes equations, achieved through **Galerkin approximations** and **energy estimates**. This not only deepens our understanding of the **mathematical structure of fluid flows** but also contributes to the exploration of the **Millennium Prize Problem**.

The **characterization of Besov spaces** via the **Littlewood-Paley decomposition** is pivotal for capturing **multifractal and irregular behaviors** in turbulent flows. The **extended Plancherel theorem** for Besov spaces strengthens the link between **spatial derivatives** and **frequency-localized norms**, which is essential for studying **nonlinear PDEs** and **energy dissipation mechanisms**.

The **quaternionic formulation** of the Navier-Stokes equations provides a novel approach to modeling **rotational symmetries and bifurcation phenomena** in three-dimensional fluid dynamics. The **bifurcation criterion**, derived from the spectrum of the linearized quaternionic operator, offers a rigorous tool for detecting the **onset of instabilities and transitions to turbulence**.

 doi:10.20944/preprints202510.0499.v1

In summary, this research advances the **mathematical theory of fluid dynamics** and lays the groundwork for future studies aimed at resolving the **Millennium Prize Problem**. The findings open new avenues for research in **mathematical fluid dynamics** and provide a robust foundation for addressing the **complex behaviors** exhibited by fluid systems.

Acknowledgments: Santos gratefully acknowledges the support of the PPGMC Program for the Postdoctoral Scholarship PROBOL/UESC nr. 218/2025. Sales would like to express his gratitude to CNPq for the financial support under grant 308816/2025-0.

References

1. Constantin, P., & Foiaş, C. (1988). *Navier-stokes equations*. University of Chicago press.
2. Marsden, J. E., & Ratiu, T. S. (2013). *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems* (Vol. 17). Springer Science & Business Media.
3. Triebel, H. (1983). Theory of function spaces, Birkhauser, Basel.
4. Temam, R. (2024). *Navier–Stokes equations: theory and numerical analysis* (Vol. 343). American Mathematical Society.
5. Doering, C. R., & Gibbon, J. D. (1995). *Applied analysis of the Navier-Stokes equations* (No. 12). Cambridge university press.
6. Ladyzhenskaya, O. A. (1969). *The Mathematical Theory of Viscous Incompressible Flow*, Gordon Breach Pub. Co., New York.
7. Sohr, H. (2012). *The Navier-Stokes equations: An elementary functional analytic approach*. Springer Science & Business Media.
8. Galdi, G. (2011). *An introduction to the mathematical theory of the Navier-Stokes equations: Steady-state problems.* Springer Science & Business Media.
9. Bahouri, H. (2011). *Fourier analysis and nonlinear partial differential equations.* Springer.
10. Grafakos, L. (2008). Grafakos, L. (2008). *Classical fourier analysis* (Vol. 2). New York: Springer.
11. Stein, E. M., & Murphy, T. S. (1993). *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals* (Vol. 3). Princeton University Press.
12. Muscalu, C., & Schlag, W. (2013). *Classical and Multilinear Harmonic Analysis: Volume 1* (Vol. 137). Cambridge University Press.
13. Runst, T., & Sickel, W. (2011). *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations* (Vol. 3). Walter de Gruyter.
14. Adams, R. A., & Fournier, J. J. (2003). *Sobolev spaces*, volume 140 of Pure and Applied Mathematics.
15. Evans, L. C. (2022). *Partial differential equations* (Vol. 19). American mathematical society.
16. Majda, A. J., Bertozzi, A. L., & Ogawa, A. (2002). Vorticity and incompressible flow. Cambridge texts in applied mathematics. *Appl. Mech. Rev.*, 55(4), B77-B78.
17. Fefferman, C., & Stein, E. M. (1972). H^p spaces of several variables.
18. Cannone, M. (1994). *Ondelettes, paraproduits et Navier-Stokes* (Doctoral dissertation, Paris 9).
19. Lemarié-Rieusset, P. G. (2002). Recent developments in the Navier-Stokes problem.
20. Chemin, J. Y. (1998). *Perfect incompressible fluids* (No. 14). Oxford University Press.
21. Bahouri, H. (2011). *Fourier analysis and nonlinear partial differential equations*. Springer.
22. dos Santos, R. D. C., & de Oliveira Sales, J. H. (2023). Treatment for regularity of the Navier-Stokes equations based on Banach and Sobolev functional spaces coupled to anisotropic viscosity for analysis of vorticity transport. *The Journal of Engineering and Exact Sciences*, 9(8), 16656-01e.