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Takaaki Fujita *

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HyperGeography and SuperHyperGeography

Takaaki Fujita

Independent Researcher, Tokyo, Japan; takaaki.fujita060@gmail.com

Abstract

A *Hyperstructure* is built on the concept of the powerset, providing a framework to model interactions among elements of a set. Extending this concept, a *Superhyperstructure* employs the n-th powerset to represent hierarchical systems with multiple layers, enabling richer abstractions and more complex relationships. Geography is a mathematical framework on a 2D Riemannian manifold encoding regions, features, attributes, networks, and projections to analyze spatial relations. In this paper, we examine whether Hyperstructures and SuperHyperstructures can be employed to define *HyperGeography* and *SuperHyperGeography*, and we provide a concise discussion including potential applications.

Keywords: HyperStructure; SuperHyperStructure; HyperGeography; SuperHyperGeography

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1. Preliminaries

This section gathers the basic notions and notation used throughout the paper. Unless explicitly stated otherwise, we work in the finite setting. By convention, the empty set is treated as an element of every set.

1.1. Hyperstructure and Superhyperstructure

A *Hyperstructure* is organized around the powerset and serves as a vehicle for modeling relations among elements of a set [1–6]. Owing to its flexibility, the hyperstructure framework has been investigated across several areas, including mathematics and chemistry [7–10]. A *Superhyperstructure* advances this idea by utilizing the n-th powerset to encode multi-layered hierarchical interactions, thereby enabling deeper abstraction and greater structural complexity [11–13]. Related concepts such as *SuperHyperGraph* are also known [14–18]. We next record the n-th powerset, which underpins these structures.

Definition 1 (Base Set)**.** *A base set S is the underlying collection from which higher-level constructions—powersets and (super)hyperstructures—are built. Formally,*

$$S = \{\, x \mid x \text{ is an element of a specified domain} \,\}.$$

All elements appearing in $\mathcal{P}(S)$ or in the iterated powersets $\mathcal{P}_n(S)$ ultimately arise from members of S.

Definition 2 (Powerset)**.** *[19] The* powerset *of a set S, denoted $\mathcal{P}(S)$, is the family of all subsets of S, including $\emptyset$ and S itself:*

$$\mathcal{P}(S) = \{\, A \mid A \subseteq S \,\}.$$

Example 1 (Powerset in a geographic partition)**.** *Let $S = \{N, S, C\}$ denote three disjoint administrative regions of a city:* North (N), South (S), *and* Coast (C). *The powerset*

$$\mathcal{P}(S) = \{\emptyset, \{N\}, \{S\}, \{C\}, \{N, S\}, \{N, C\}, \{S, C\}, \{N, S, C\}\}$$

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contains $2^{|S|} = 2^3 = 8$ *subsets. Suppose their areas are*

$$\mu(N) = 30 \text{ km}^2, \quad \mu(S) = 25 \text{ km}^2, \quad \mu(C) = 20 \text{ km}^2.$$

For any $A \subseteq S$*, the union* $\bigcup A$ *represents the geographic footprint of the chosen regions (with* $\bigcup \emptyset = \emptyset$ *and* $\mu(\emptyset) = 0$*). For instance,*

$$A = \{N, C\} \implies \bigcup A = N \cup C, \qquad \mu(N \cup C) = \mu(N) + \mu(C) = 30 + 20 = 50 \text{ km}^2,$$

since the regions are disjoint. This concretely interprets $\mathcal{P}(S)$ *as all possible selections of regions for a planning scenario.*

Definition 3 (n-th Powerset)**.** *(cf.[20–23]) For a set* H*, the* n*-th powerset* $\mathcal{P}_n(H)$ *is defined recursively by*

$$\mathcal{P}_1(H) = \mathcal{P}(H), \qquad \mathcal{P}_{n+1}(H) = \mathcal{P}(\mathcal{P}_n(H)), \quad n \geq 1.$$

The nonempty version $\mathcal{P}_n^*(H)$ *is given by*

$$\mathcal{P}_1^*(H) = \mathcal{P}^*(H), \qquad \mathcal{P}_{n+1}^*(H) = \mathcal{P}^*(\mathcal{P}_n^*(H)),$$

where $\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$*.*

Example 2 (n-th powerset (here $n = 2$) for district portfolios)**.** *Let* $H = \{D_1, D_2, D_3\}$ *be three disjoint districts with areas*

$$\mu(D_1) = 12 \text{ km}^2, \qquad \mu(D_2) = 9 \text{ km}^2, \qquad \mu(D_3) = 5 \text{ km}^2.$$

Then

$$\mathcal{P}_1(H) = \mathcal{P}(H) \text{ has cardinality } 2^3 = 8, \qquad \mathcal{P}_2(H) = \mathcal{P}(\mathcal{P}_1(H)) \text{ has cardinality } 2^8 = 256.$$

For the nonempty versions, $\mathcal{P}_1^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}$ *has* $2^3 - 1 = 7$ *elements, hence*

$$|\mathcal{P}_2^*(H)| = |\mathcal{P}^*(\mathcal{P}_1^*(H))| = 2^7 - 1 = 127.$$

A concrete element of $\mathcal{P}_2^*(H)$ *(a nonempty set of nonempty district-sets) is*

$$X = \{\{D_1, D_2\}, \{D_2\}, \{D_3\}\} \in \mathcal{P}_2^*(H).$$

Its geographic coverage is the union of all districts that appear in X*:*

$$\bigcup(\bigcup X) = D_1 \cup D_2 \cup D_3, \qquad \mu(D_1 \cup D_2 \cup D_3) = 12 + 9 + 5 = 26 \text{ km}^2.$$

Thus, $\mathcal{P}_2(H)$ *parameterizes* portfolios of district groupings *(e.g., policy or service bundles), while* $\mathcal{P}_2^*(H)$ *restricts to portfolios with no empty members.*

To provide a self-contained foundation for hyperstructures and superhyperstructures, we recall the following standard notions.

Definition 4 (Classical Structure)**.** *(cf. [20,21,24]) A* Classical Structure *consists of a nonempty set* H *together with one or more* classical operations *satisfying specified axioms. A classical* m*-ary operation has the form*

$$\#_0 : H^m \to H,$$

with $m \geq 1$*. Familiar examples include the operations defining groups, rings, and fields.*

Definition 5 (Hyperoperation)**.** *(cf. [11,25–27]) A* hyperoperation *on a set S is a map*

$$\circ : S \times S \longrightarrow \mathcal{P}(S),$$

so that combining two inputs returns a set *of outcomes (not necessarily a singleton).*

Definition 6 (Hyperstructure)**.** *(cf. [20,21,28,29]) A* Hyperstructure *augments a base set S by operating on its powerset. Formally,*

$$\mathcal{H} = (\mathcal{P}(S), \circ),$$

where $\circ$ acts on subsets of S.

Example 3 (HyperStructure — capacity–constrained parcel consolidation)**.** *Let the base set be parcels $S = \{a, b, c, d\}$ with weights (kg)*

$$w(a) = 3, \quad w(b) = 4, \quad w(c) = 5, \quad w(d) = 6,$$

and let the vehicle capacity be $C = 10$ kg. Consider the HyperStructure

$$\mathcal{H} = (\mathcal{P}(S), \circ_{cap}),$$

where the hyperoperation $\circ_{cap} : \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S)) \setminus \{\emptyset\}$ maps two order sets $X, Y \subseteq S$ to the set of all feasible trip partitions *Z such that: (i) $\bigcup Z = X \cup Y$, (ii) for every $T \in Z$ one has $\sum_{x \in T} w(x) \leq C$.*

Take $X = \{a, b\}$ and $Y = \{c\}$. Then $X \cup Y = \{a, b, c\}$ with total weight $3 + 4 + 5 = 12 > 10$, so at least two trips are needed. Enumerating all set partitions of $\{a, b, c\}$ and keeping only capacity–feasible ones gives

$$\Big\{\{a,b\},\{c\}\Big\} : \ 3+4 = 7 \leq 10,\ 5 \leq 10; \qquad \Big\{\{a,c\},\{b\}\Big\} : \ 3+5 = 8 \leq 10,\ 4 \leq 10;$$

$$\Big\{\{b,c\},\{a\}\Big\} : \ 4+5 = 9 \leq 10,\ 3 \leq 10; \qquad \Big\{\{a\},\{b\},\{c\}\Big\} : \ 3,4,5 \leq 10.$$

The single–trip partition $\{\{a, b, c\}\}$ is infeasible *since $12 > 10$. Therefore*

$$X \circ_{cap} Y = \Big\{ \ \{\{a,b\},\{c\}\},\ \{\{a,c\},\{b\}\},\ \{\{b,c\},\{a\}\},\ \{\{a\},\{b\},\{c\}\} \ \Big\},$$

illustrating a concrete hyperoperation on $\mathcal{P}(S)$ returning multiple feasible consolidation outcomes under a real capacity constraint.

Definition 7 (SuperHyperOperation)**.** *[20] Let H be nonempty. Define recursively, for $k \geq 0$,*

$$\mathcal{P}^0(H) = H, \qquad \mathcal{P}^{k+1}(H) = \mathcal{P}\big(\mathcal{P}^k(H)\big).$$

*For fixed $m, n \geq 0$ and arity $s \geq 1$, an (m, n)-*SuperHyperOperation *is a map*

$$\odot^{(m,n)} : \ \big(\mathcal{P}^m(H)\big)^s \longrightarrow \mathcal{P}^n(H).$$

If the codomain may include $\emptyset$, we obtain the neutrosophic variant; otherwise we are in the classical case.

Definition 8 (n-Superhyperstructure)**.** *(cf. [20,23,30,31]) An n-*Superhyperstructure *generalizes hyperstructures by acting on the n-th powerset:*

$$\mathcal{SH}_n = (\mathcal{P}_n(S), \circ),$$

with $\circ$ defined on $\mathcal{P}_n(S)$.

Example 4 (n-Superhyperstructure ($n = 2$) — team reorganization with size constraints). *Let $S = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be employees. A level–2 object $X \in \mathcal{P}_2(S) = \mathcal{P}(\mathcal{P}(S))$ is a set of* teams *(each team is a nonempty subset of S). Define an operation*

$$\circ_{team} : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \longrightarrow \mathcal{P}(\mathcal{P}_2(S)) \setminus \{\emptyset\}$$

that maps programs X, Y to the set of all reorganizations *Z whose ground union is preserved and whose teams have size in $\{2, 3\}$ (no singletons):*

$$\bigcup Z = \bigcup(X \cup Y) \subseteq S, \qquad \forall T \in Z : \ 2 \leq |T| \leq 3.$$

Take

$$X = \{\{e_1, e_2\}, \{e_3\}\}, \qquad Y = \{\{e_4, e_5\}, \{e_6\}\}.$$

Then $\bigcup(X \cup Y) = \{e_1, \ldots, e_6\}$. Two canonical families in $X \circ_{team} Y$ are:

$$Z_{\text{triads}} = \{\{e_1, e_2, e_3\}, \{e_4, e_5, e_6\}\}, \qquad Z_{\text{pairs}} = \{\{e_1, e_4\}, \{e_2, e_5\}, \{e_3, e_6\}\}.$$

Counting shows the operation returns many *outcomes: the number of partitions of six labeled employees into two unlabeled triads is*

$$\frac{1}{2}\binom{6}{3} = \frac{1}{2} \cdot 20 = 10,$$

and the number of partitions into three pairs (perfect matchings) is

$$(6-1)!! = 5!! = 15.$$

Hence $X \circ_{team} Y$ contains at least $10 + 15 = 25$ reorganizations satisfying the size constraint, which demonstrates a concrete n-Superhyperstructure on $\mathcal{P}_2(S)$ producing multiple valid teamings while preserving the same ground set.

Definition 9 (SuperHyperStructure of order (m, n)). *(cf. [11,32,33]) Let S be nonempty and $m, n \geq 0$. A* (m, n)-SuperHyperStructure *of arity s is any choice of*

$$\odot^{(m,n)} : \left(\mathcal{P}^m(S)\right)^s \longrightarrow \mathcal{P}^n(S).$$

The special cases recover standard settings: $m = n = 0$ gives ordinary s-ary operations; $m = 0, n = 1$ yields hyperoperations; and $s = 1$ corresponds to superhyperfunctions.

Example 5 ((m, n)-SuperHyperStructure with $(m, n) = (2, 1)$ — two–layer service bundles to single–layer deployment). *Let $S = \{A_1, A_2, A_3, A_4\}$ be neighborhoods with areas (km^2)*

$$\mu(A_1) = 8, \quad \mu(A_2) = 5, \quad \mu(A_3) = 6, \quad \mu(A_4) = 7,$$

and population densities (persons/km^2)

$$\rho(A_1) = 7{,}000, \ \rho(A_2) = 6{,}000, \ \rho(A_3) = 3{,}500, \ \rho(A_4) = 2{,}000.$$

Thus atomic populations are

$$P(A_1) = 8 \times 7{,}000 = 56{,}000, \quad P(A_2) = 5 \times 6{,}000 = 30{,}000,$$

$$P(A_3) = 6 \times 3{,}500 = 21{,}000, \quad P(A_4) = 7 \times 2{,}000 = 14{,}000,$$

and over S one has

$$P(S) = 56{,}000 + 30{,}000 + 21{,}000 + 14{,}000 = 121{,}000$$

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and total area $8+5+6+7=26$ *km*^{2}*, hence the average density on the union is*

$$\bar{\rho} = \frac{121{,}000}{26} \approx 4{,}653.846 \text{ persons/km}^2.$$

A level–2 bundle *is an element of* $\mathcal{P}_2(S)$*. Consider two agencies' proposals*

$$X = \{\{A_1, A_2\}, \{A_3\}\}, \qquad Y = \{\{A_4\}\}.$$

Define the $(2,1)$*-superhyperoperation*

$$\odot^{(2,1)} : \mathcal{P}_2(S) \times \mathcal{P}_2(S) \longrightarrow \mathcal{P}(\mathcal{P}(S)) \setminus \{\emptyset\}$$

by

$$X \odot^{(2,1)} Y := \Big\{ Z \in \mathcal{P}(S) \setminus \{\emptyset\} : \bigcup Z = \bigcup \big(\bigcup X \cup \bigcup Y\big) \Big\}.$$

A canonical output is the flattened *selection*

$$Z_0 = \{A_1, A_2, A_3, A_4\} \in X \odot^{(2,1)} Y,$$

and any other Z *in the output has the same ground union. If each person needs* $r = 0.05$ *kits/month, the single–layer monthly demand is*

$$D = r \cdot P(S) = 0.05 \times 121{,}000 = 6{,}050 \text{ kits/month},$$

which is invariant *across all* $Z \in X \odot^{(2,1)} Y$ *because it depends only on the flattened union. This realizes a concrete* (m,n)*-SuperHyperStructure with* $(m,n) = (2,1)$*: inputs at level 2 (bundles) and outputs at level 1 (deployable region selections) while preserving measurable aggregates.*

2. Main Results

In this section, we present the main contributions of this paper.

2.1. Geography

Geography is a mathematical framework on a 2D Riemannian manifold encoding regions, features, attributes, networks, and projections to analyze spatial relations [34–39].

Definition 10 (Geography as a mathematical structure)**.** *(cf.[40,41]) Fix a nonempty, connected, oriented 2-dimensional* C^1 *manifold* M *(the* ground*) endowed with a Riemannian metric* g*. Let*

$$d := d_g \text{ be the induced geodesic distance}, \qquad \mu := \mathcal{H}^2_g \text{ the induced Borel area measure},$$

and let Σ *be the Borel* σ*-algebra on* M*. A* geography *is a tuple*

$$\mathbf{Geo} = (M, g, d, \mu, \Sigma, \mathcal{P}, \mathcal{F}, \mathcal{A}, \mathrm{att}, \mathcal{N}, \mathrm{proj})$$

satisfying the axioms (G1)–(G6) below.

(G1) **Regions (administrative/physical partition).** $\mathcal{P} = \{R_i\}_{i \in I} \subseteq \Sigma$ *is a countable family of pairwise* μ*-almost disjoint measurable sets with*

$$\mu\left(M \setminus \bigcup_{i \in I} R_i\right) = 0, \qquad \mathcal{H}^1_g(\partial R_i) < \infty \text{ for all } i \in I.$$

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For $R, S \in \mathcal{P}$ with $R \neq S$, define adjacency

$$R \sim S \quad :\Longleftrightarrow \quad \mathcal{H}^1_g(\partial R \cap \partial S) > 0.$$

(G2) ***Features (typed geometric objects).*** *$\mathcal{F}$ is a set of triples $f = (S_f, \dim f, \alpha_f)$ where $S_f \subseteq M$ is a countably $\mathcal{H}^{\dim f}_g$-rectifiable set of finite $\mathcal{H}^{\dim f}_g$-measure, $\dim f \in \{0,1,2\}$, and $\alpha_f : S_f \to V_f$ is a measurable* attribute map *into a measurable space $(V_f, \mathcal{B}_f)$. Conventionally, $\dim f = 0$ (points, e.g. cities), 1 (curves, e.g. rivers/roads), or 2 (areas, e.g. lakes/parks).*

(G3) ***Attribute fields (spatial variables).*** *$\mathcal{A}$ is a (finite or countable) index set. For each $a \in \mathcal{A}$ there is a measurable value space $(V_a, \mathcal{B}_a)$ and a measurable field*

$$A_a : (M, \Sigma) \longrightarrow (V_a, \mathcal{B}_a), \qquad \textit{written } \operatorname{att}(a) = A_a.$$

Numerical fields satisfy $A_a \in L^1_{\mathrm{loc}}(M, \mu)$; for such a and any $E \in \Sigma$ of finite μ-measure, define the aggregate

$$\mathsf{Agg}_a(E) := \int_E A_a \, \mathrm{d}\mu.$$

(G4) ***Embedded networks (transport/flow).*** *$\mathcal{N} = (G, \varphi)$ where $G = (V, E)$ is a (finite or countable) simple graph and φ embeds G into M via*

$$\varphi_V : V \to M, \qquad \varphi_E : E \to \{\textit{rectifiable curves in } M\},$$

so that each $e = \{u, v\} \in E$ is mapped to a rectifiable curve $\varphi_E(e)$ whose endpoints are $\varphi_V(u), \varphi_V(v)$. The network length measure ℓ is the 1-dimensional Hausdorff measure restricted to $\bigcup_{e \in E} \varphi_E(e)$.

(G5) ***Spatial relations.*** *From (G1)–(G4) derive measurable binary relations on regions/features, e.g.*

$$\begin{aligned} R \text{ touches } S &:\Longleftrightarrow \mathcal{H}^1_g(\partial R \cap \partial S) > 0, \\ R \text{ contains } S &:\Longleftrightarrow \mu(S \setminus R) = 0, \\ \operatorname{dist}(E, F) &:= \inf\{\, d(x, y) : x \in E,\ y \in F \,\}, \end{aligned}$$

for measurable $E, F \subseteq M$. These are well-defined because d is a metric and $\mu, \mathcal{H}^k_g$ are Borel regular.

(G6) ***Coordinate realization / projection (cartography).*** proj *is an optional specification of coordinates: either a chart $\pi : M \to S^2$ (reference ellipsoid/sphere) or a locally C^1 map $P : M \to \mathbb{R}^2$ (a cartographic projection) which is a local diffeomorphism off a set of μ-measure 0.*

Remark 1 (Minimal consequences). *For any finite subpartition $\{R_i\}_{i=1}^n \subset \mathcal{P}$ with $\mu(M \setminus \bigcup_{i=1}^n R_i) = 0$, σ-additivity yields the exact decomposition*

$$\mu(M) = \sum_{i=1}^n \mu(R_i).$$

For any integrable numerical attribute $A_a \in L^1(M, \mu)$,

$$\int_M A_a \, \mathrm{d}\mu \;=\; \sum_{i=1}^n \int_{R_i} A_a \, \mathrm{d}\mu,$$

and for regions $R \neq S$ one has

$$\operatorname{dist}(R, S) = 0 \;\Longleftrightarrow\; \mathcal{H}^1_g(\partial R \cap \partial S) > 0,$$

so adjacency $R \sim S$ implies $\operatorname{dist}(R, S) = 0$, while $\operatorname{dist}(R, S) > 0$ implies non-adjacency.

Example 6 (Urban water supply and mobility planning)**.** *We instantiate* **Geo** *on a metropolitan area.*

(G1) Ground and regions. Let $M \subset \mathbb{R}^2$ be a bounded open set with smooth boundary, endowed with the Euclidean metric $g = \mathrm{Id}$. Let Σ be the Borel σ-algebra and μ the Lebesgue area measure (so μ is the 2-dimensional Hausdorff measure). Partition M into three measurable regions (almost disjoint, finite perimeter)

$$\mathcal{P} = \{R_1, R_2, R_3\}, \qquad \mu(R_1) = 40\ \mathrm{km}^2,\ \ \mu(R_2) = 25\ \mathrm{km}^2,\ \ \mu(R_3) = 15\ \mathrm{km}^2,$$

with adjacencies $R_1 \sim R_2$, $R_1 \sim R_3$, and $\mathrm{dist}(R_2, R_3) > 0$ (separated by a river corridor).

(G2) Features. Let $\mathcal{F}$ contain: (i) a curve feature $f_{\text{river}} = (S_{\text{riv}}, 1, \alpha_{\text{riv}})$ where S_{riv} is a rectifiable river centerline crossing R_1 and separating R_2 from R_3; (ii) point features $f_{\text{res},k} = (\{p_k\}, 0, \alpha_{\text{res},k})$ for reservoirs p_k with storage attribute $\alpha_{\text{res},k}(p_k) \in \mathbb{R}_+$; (iii) area features $f_{\text{park},j} = (S_{\text{park},j}, 2, \alpha_{\text{park},j})$ for urban parks.

(G3) Attribute fields. Define population density and monthly rainfall fields

$$A_{\text{pop}}(x) = \begin{cases} 10{,}000 & x \in R_1, \\ 7{,}000 & x \in R_2, \\ 3{,}000 & x \in R_3, \end{cases} \quad A_{\text{rain}}(x) = \begin{cases} 180 & x \in R_1, \\ 160 & x \in R_2, \\ 150 & x \in R_3, \end{cases}$$

measured in $\mathrm{persons}/\mathrm{km}^2$ *and* $\mathrm{mm}/\mathrm{month}$*, respectively. Then the regional populations are*

$$\mathsf{Agg}_{\text{pop}}(R_i) = \int_{R_i} A_{\text{pop}}\, \mathrm{d}\mu = (\mathit{density}_i) \cdot \mu(R_i),$$

so numerically

$$\begin{aligned} \mathsf{Pop}(R_1) &= 10{,}000 \times 40 = 400{,}000, \\ \mathsf{Pop}(R_2) &= 7{,}000 \times 25 = 175{,}000, \\ \mathsf{Pop}(R_3) &= 3{,}000 \times 15 = 45{,}000, \end{aligned} \qquad \Rightarrow \qquad \mathsf{Pop}(M) = 620{,}000.$$

Monthly rainfall volume over R_i (assuming spatially constant per region) is

$$V_{\text{rain}}(R_i) = \left(\frac{A_{\text{rain}}(R_i)}{1000}\right) \cdot \left(\mu(R_i) \times 10^6\right)\ \mathrm{m}^3,$$

since $1\ \mathrm{km}^2 = 10^6\ \mathrm{m}^2$. *Hence*

$$\begin{aligned} V_{\text{rain}}(R_1) &= 0.18 \times 40 \times 10^6 = 7.2 \times 10^6\ \mathrm{m}^3, \\ V_{\text{rain}}(R_2) &= 0.16 \times 25 \times 10^6 = 4.0 \times 10^6\ \mathrm{m}^3, \\ V_{\text{rain}}(R_3) &= 0.15 \times 15 \times 10^6 = 2.25 \times 10^6\ \mathrm{m}^3. \end{aligned}$$

(G4) Embedded networks. Let $G = (V, E)$ be the primary road graph with major hubs $V = \{v_1, v_2, v_3\}$ (one per region) and edges $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}\}$. Embed each edge as a rectifiable curve $\varphi_E(e) \subset M$ with lengths

$$\ell(\varphi_E(\{v_1, v_2\})) = 12\ \mathrm{km}, \quad \ell(\varphi_E(\{v_1, v_3\})) = 10\ \mathrm{km}, \quad \ell(\varphi_E(\{v_2, v_3\})) = 8\ \mathrm{km}.$$

The total embedded edge length is $\ell(\cup_{e \in E} \varphi_E(e)) = 30$ km (edges disjoint except at endpoints).

(G5) Spatial relations. By construction, $R_1 \sim R_2$ and $R_1 \sim R_3$ since $\mathcal{H}^1_g(\partial R_1 \cap \partial R_i) > 0$ for $i = 2, 3$, while $\mathrm{dist}(R_2, R_3) > 0$ (the river corridor enforces a positive gap).

(G6) Projection. Take $P : M \to \mathbb{R}^2$ to be the local UTM projection (chart on the chosen zone); P is C^1 and a local diffeomorphism off a null set, enabling planar cartography.

This instantiation supports concrete planning queries such as allocating stormwater storage to match $\{V_{\text{rain}}(R_i)\}_{i=1}^3$ and sizing road capacities along φ_E to serve $\{\mathsf{Pop}(R_i)\}$.

Example 7 (Coastal evacuation and shelter capacity)**.** *We instantiate* **Geo** *on a coastal city with tsunami hazard zoning.*

(G1) Ground and regions. Let $M \subset \mathbb{R}^2$ be a simply connected coastal municipality with Euclidean metric g, Borel σ-algebra Σ, and Lebesgue area measure μ. Partition M into hazard/land-use regions

$$\mathcal{P} = \{H_1, H_2, H_3\}, \qquad \mu(H_1) = 12\ \mathrm{km}^2,\ \mu(H_2) = 28\ \mathrm{km}^2,\ \mu(H_3) = 10\ \mathrm{km}^2,$$

where H_1 is the tsunami inundation zone (low elevation), H_2 is inland residential, and H_3 is an industrial port. Assume $H_1 \sim H_2$, $H_1 \sim H_3$, and $\mathrm{dist}(H_2, H_3) > 0$.

(G2) Features. Include: (i) shoreline curve $f_{\mathrm{shore}} = (S_{\mathrm{shore}}, 1, \alpha_{\mathrm{shore}})$ with α_{shore} carrying local berm height; (ii) evacuation shelters as point features $f_{\mathrm{sh},k} = (\{q_k\}, 0, \alpha_{\mathrm{cap},k})$ with capacities $\alpha_{\mathrm{cap},1}(q_1) = 60{,}000$, $\alpha_{\mathrm{cap},2}(q_2) = 50{,}000$, $\alpha_{\mathrm{cap},3}(q_3) = 30{,}000$; (iii) critical bridges as curve features with width attributes.

(G3) Attribute fields. Define elevation (meters above sea level) and population density

$$A_{\mathrm{elev}}(x) = \begin{cases} 2 & x \in H_1, \\ 12 & x \in H_2, \\ 4 & x \in H_3, \end{cases} \qquad A_{\mathrm{pop}}(x) = \begin{cases} 8{,}000 & x \in H_1, \\ 6{,}000 & x \in H_2, \\ 1{,}500 & x \in H_3, \end{cases}$$

in m *and* persons/km^2. *Then regional populations are*

$$\begin{aligned} \mathsf{Pop}(H_1) &= \int_{H_1} A_{\mathrm{pop}}\, \mathrm{d}\mu = 8{,}000 \times 12 = 96{,}000, \\ \mathsf{Pop}(H_2) &= 6{,}000 \times 28 = 168{,}000, \\ \mathsf{Pop}(H_3) &= 1{,}500 \times 10 = 15{,}000, \end{aligned} \qquad \Rightarrow \qquad \mathsf{Pop}(M) = 279{,}000.$$

Suppose the evacuation target is all residents in H_1. Total designated shelter capacity is

$$\mathsf{Cap}_{\mathrm{tot}} = \sum_{k=1}^{3} \alpha_{\mathrm{cap},k}(q_k) = 140{,}000 \quad \Longrightarrow \quad \mathsf{Cap}_{\mathrm{tot}} - \mathsf{Pop}(H_1) = 44{,}000\ \ (\textit{sufficient by } 44{,}000).$$

(G4) Embedded evacuation network. Let $G = (V, E)$ with $V = \{u_1, u_2, u_3\}$ representing the centroids of H_1, H_2, H_3 and with edges along designated evacuation corridors:

$$\ell(\varphi_E(\{u_1, u_2\})) = 5.0\ \mathrm{km}, \quad \ell(\varphi_E(\{u_1, u_3\})) = 4.2\ \mathrm{km}, \quad \ell(\varphi_E(\{u_2, u_3\})) = 3.0\ \mathrm{km}.$$

The total corridor length is 12.2 km. *If a bridge feature on $\{u_1, u_2\}$ has width $w = 8$ m and safe pedestrian flux $J_{\max} = 1.5$ persons/(m · s), then the link's peak throughput is*

$$Q_{\max} = w \cdot J_{\max} = 12\ \text{persons/s} \approx 43{,}200\ \text{persons/h},$$

so in principle one such link can clear H_1 in about $96{,}000/43{,}200 \approx 2.22$ h under steady flow (ignoring transients and routing constraints).

(G5) Spatial relations. Because $\Upsilon(H_1)$ meets both $\Upsilon(H_2)$ and $\Upsilon(H_3)$ along positive-length arcs, $H_1 \sim H_2$ and $H_1 \sim H_3$. Moreover, $\mathrm{dist}(H_2, H_3) > 0$ implies non-adjacency between H_2 and H_3.

(G6) Projection. Take P to be a conformal coastal chart (e.g. local Mercator/UTM); since P is C^1 and locally invertible away from a null set, lengths/areas are well-defined up to the standard Jacobian factors used in map production.

This instantiation quantifies (i) required shelter capacity relative to $\mathsf{Pop}(H_1)$ and (ii) corridor throughput via embedded network geometry, enabling data-driven evacuation timing analyses.

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2.2. HyperGeography

HyperGeography extends Geography by treating sets of atomic regions as hyperregions and using a hyperoperation to generate partitions of their union.

Definition 11 (HyperGeography). *Fix a countable family* $S = \{A_j\}_{j\in J} \subseteq \Sigma$ *of* atomic regions *such that*

$$\mu\left(M \setminus \bigcup_{j\in J} A_j\right) = 0, \qquad \mu(A_j) > 0, \qquad \mathcal{H}^1_g(\partial A_j) < \infty,$$

and $A_i \cap A_j$ *are* μ*-almost disjoint for* $i \neq j$*. Set the* hyperregion universe *to be the nonempty powerset*

$$\mathcal{R} \;:=\; \mathcal{P}^*(S) = \mathcal{P}(S) \setminus \{\emptyset\}.$$

Define the region hyperoperation $\circ_{\mathrm{reg}} : \mathcal{R} \times \mathcal{R} \to \mathcal{P}(\mathcal{R}) \setminus \{\emptyset\}$ *by*

$$X \circ_{\mathrm{reg}} Y \;:=\; \left\{ Z \in \mathcal{R} \;:\; \bigcup Z \;=\; \bigcup (X \cup Y) \;\text{ and } Z \text{ is a finite subfamily of } S \right\}.$$

Equivalently,

$$X \circ_{\mathrm{reg}} Y \;\supseteq\; \{\, X \cup Y \,\} \qquad \text{and} \qquad X \circ_{\mathrm{reg}} Y \;\subseteq\; \mathcal{P}\Big(\{A \in S : A \subseteq \bigcup (X \cup Y)\}\Big) \setminus \{\emptyset\}.$$

(Thus $X \circ_{\mathrm{reg}} Y$ *returns all finite admissible partitions of the geometric union* $\cup(X \cup Y)$ *by atoms from* S*, including the coarse partition* $\{\cup(X \cup Y)\}$ *whenever* $\cup(X \cup Y) \in S$*; if not, the set* $\{X \cup Y\}$ *is still an element of* $\mathcal{R}$ *because* $X \cup Y \subseteq S$*.)*

Let $\mathcal{F}$ *denote the set of typed geometric features as in Definition 10(G2); we allow* hyperattachment *to hyperregions via measurable maps*

$$\alpha_f^\sharp : \mathcal{R} \longrightarrow \mathcal{P}(V_f) \setminus \{\emptyset\}, \qquad \alpha_f^\sharp(X) := \{\alpha_f(x) : x \in \cup X \cap S_f\}^*.$$

For spatial attributes, let $\mathcal{A}$ *be an index set and for each* $a \in \mathcal{A}$ *fix a measurable field* $A_a : (M, \Sigma) \to (V_a, \mathcal{B}_a)$ *as in (G3). For numeric attributes take* $V_a = \mathrm{R}$ *and assume* $A_a \in L^1_{\mathrm{loc}}(M, \mu)$*. Define the* attribute hypervaluation *on hyperregions:*

$$\mathrm{Agg}_a^\sharp : \mathcal{R} \longrightarrow \mathcal{P}(\mathrm{R}) \setminus \{\emptyset\}, \qquad \mathrm{Agg}_a^\sharp(X) := \left\{ \tfrac{1}{\mu(E)} \int_E A_a \, \mathrm{d}\mu \;:\; E \in \Sigma,\, \emptyset \neq E \subseteq \cup X,\, \mu(E) < \infty \right\}.$$

(Other admissible weightings can be inserted; this simple choice suffices for our theorems.)

Define the hypernetwork $\mathcal{N}_h = (G_h, \varphi_h)$ *by taking* $G_h = (V_h, E_h)$ *with* $V_h = \mathcal{R}$ *and*

$$\{X, Y\} \in E_h \quad \Longleftrightarrow \quad \mathcal{H}^1_g\big(\partial(\cup X) \cap \partial(\cup Y)\big) > 0,$$

and embed G_h *in* M *by* $\varphi_h(X) \in M$ *chosen, e.g., as the* μ*-barycenter of* $\cup X$ *and edges mapped to rectifiable curves contained in a small tubular neighborhood of* $\partial(\cup X) \cap \partial(\cup Y)$*.*

A HyperGeography *is the tuple*

$$\mathbf{HGeo} \;:=\; \big(M, g, d, \mu, \Sigma,\; S,\; \mathcal{R},\; \circ_{\mathrm{reg}},\; \mathcal{F},\; \alpha^\sharp,\; \mathcal{A},\; \mathrm{Agg}^\sharp,\; \mathcal{N}_h,\; \mathrm{proj}\big).$$

Remark 2 (Well-definedness). *(i)* $X \circ_{\mathrm{reg}} Y \neq \emptyset$ *because* $X \cup Y \in \mathcal{R}$ *and* $X \cup Y \in X \circ_{\mathrm{reg}} Y$*. (ii)* $\mathrm{Agg}_a^\sharp(X)$ *is nonempty since* $\mu(\cup X) > 0$ *and we can choose any measurable* $E \subseteq \cup X$ *with* $0 < \mu(E) < \infty$*. (iii)* E_h *is well-defined because* $\partial(\cup X)$ *and* $\partial(\cup Y)$ *have finite* $\mathcal{H}^1_g$*-measure whenever* $\cup X$ *and* $\cup Y$ *are finite unions of finite-perimeter atoms.*

Example 8 (Multi-district hospital catchment as a hyperregion)**.** *We build a concrete* HyperGeography *on a city partitioned into atomic districts.*

Atoms and measure. *Let* $M \subset \mathbb{R}^2$ *be a bounded open set with Euclidean metric* g*, Borel* σ*-algebra* Σ*, and area measure* μ*. Fix the atomic family* $S = \{A_1, \dots, A_5\} \subset \Sigma$ *with*

$$\mu(A_1) = 12,\ \mu(A_2) = 9,\ \mu(A_3) = 5,\ \mu(A_4) = 7,\ \mu(A_5) = 10 \quad (\textit{in } km^2),$$

finite-perimeter boundaries $\mathcal{H}^1_g(\partial A_j) < \infty$*, and pairwise* μ*-almost disjoint interiors. Assume adjacencies*

$$A_1 \sim A_2, \quad A_2 \sim A_3, \quad A_3 \sim A_4, \quad A_4 \sim A_5, \quad A_1 \sim A_5,$$

and all other pairs non-adjacent.

Hyperregions and hyperoperation. *The hyperregion universe is* $\mathcal{R} = \mathcal{P}^*(S)$*. Consider two (possibly disconnected) service catchments:*

$$X = \{A_1, A_3\}, \qquad Y = \{A_2\}.$$

Their union in the ground space is $\cup(X \cup Y) = A_1 \cup A_2 \cup A_3$ *(area* $= 12 + 9 + 5 = 26$ *km*^{2}*). By Definition 11,*

$$X \circ_{\text{reg}} Y = \Big\{ Z \in \mathcal{R} : \ \cup Z = \cup(X \cup Y),\ Z \subseteq S \textit{ finite} \Big\} \ \supseteq \ \{\{A_1, A_2, A_3\}\},$$

so the fine *partition* $\{A_1, A_2, A_3\}$ *is a valid hyperselection; any other finite atom-cover of* $A_1 \cup A_2 \cup A_3$ *is also admissible.*

Attributes and hypervaluation. *Let population density be piecewise constant (persons/km*^{2}*):*

$$A_{\text{pop}}(x) = \begin{cases} 8{,}000 & x \in A_1, \\ 6{,}000 & x \in A_2, \\ 4{,}500 & x \in A_3, \\ 3{,}000 & x \in A_4, \\ 2{,}000 & x \in A_5. \end{cases}$$

Then regional populations on the atoms are

$$\begin{aligned} \mathsf{Pop}(A_1) &= 8{,}000 \times 12 = 96{,}000, \\ \mathsf{Pop}(A_2) &= 6{,}000 \times 9 = 54{,}000, \\ \mathsf{Pop}(A_3) &= 4{,}500 \times 5 = 22{,}500. \end{aligned}$$

For the hyperregion $X \cup Y$*, the* classical *average density is*

$$\bar{\rho}_{\text{pop}}(X \cup Y) = \frac{\int_{\cup(X \cup Y)} A_{\text{pop}}\, d\mu}{\mu(\cup(X \cup Y))} = \frac{96{,}000 + 54{,}000 + 22{,}500}{26} = \frac{172{,}500}{26} \approx 6{,}634.615 \textit{ persons/km}^2.$$

By Definition of $\mathsf{Agg}^\sharp$*,*

$$\mathsf{Agg}^\sharp_{\text{pop}}(X \cup Y) = \left\{ \frac{1}{\mu(E)} \int_E A_{\text{pop}}\, d\mu : \ \emptyset \neq E \subseteq A_1 \cup A_2 \cup A_3 \right\} \ \supseteq \ [4{,}500,\ 8{,}000],$$

since choosing $E \subseteq A_3$ *(resp.* $E \subseteq A_1$*) realizes the lower (resp. upper) endpoint.*

If the expected monthly visit rate is $r = 0.05$ *visits/person/month, then the hyperregion's* total *baseline demand is*

$$D_{\text{month}} = r \cdot \big(\mathsf{Pop}(A_1) + \mathsf{Pop}(A_2) + \mathsf{Pop}(A_3)\big) = 0.05 \times 172{,}500 = 8{,}625.$$

This number is independent of the chosen hyperselection $Z \in X \circ_{\text{reg}} Y$ *because it depends only on the ground-space union* $\cup(X \cup Y)$*.*

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Hypernetwork. *Vertices are all hyperregions* $V_h = \mathcal{R}$. *In particular,*

$$\{\{A_1\},\{A_2\}\} \in E_h \quad \text{and} \quad \{\{A_2\},\{A_3\}\} \in E_h$$

since $\mathcal{H}^1_g(\partial A_1 \cap \partial A_2) > 0$ *and* $\mathcal{H}^1_g(\partial A_2 \cap \partial A_3) > 0$. *Thus a path* $\{A_1\} - \{A_2\} - \{A_3\}$ *exists within the hypernetwork, even though the service hyperregion* $X = \{A_1, A_3\}$ *is disconnected in the ground space.*

Example 9 (Renewable-energy siting across disjoint ridges). *We model a wind-farm siting problem where a developer considers two separated ridges and a transmission corridor.*

Atoms and adjacencies. *Let* $S = \{A_2, A_3, A_4, A_5\}$ *with areas*

$$\mu(A_2) = 14, \ \mu(A_3) = 8, \ \mu(A_4) = 9, \ \mu(A_5) = 11 \quad (km^2),$$

and adjacencies $A_2 \sim A_3$, $A_5 \sim A_4$, *all other pairs non-adjacent.*

Hyperregions. *Let the candidate generation sites be the two ridges*

$$X = \{A_2, A_5\},$$

and the supporting transmission/buffer corridor be

$$Y = \{A_3, A_4\}.$$

Then $\cup(X \cup Y) = A_2 \cup A_3 \cup A_4 \cup A_5$ *(area* $= 14 + 8 + 9 + 11 = 42$ *km*^{2}*), and by Definition 11*

$$X \circ_{\text{reg}} Y \supseteq \{\{A_2, A_3, A_4, A_5\}\},$$

with all other finite atom-covers of the same union also admissible hyperselections.

Attributes and hypervaluation. *Define wind-energy density (MWh/km*^{2}*/day) as*

$$A_{\text{wind}}(x) = \begin{cases} 5.5 & x \in A_2, \\ 0.0 & x \in A_3, \\ 0.0 & x \in A_4, \\ 4.8 & x \in A_5. \end{cases}$$

The average *wind density over the generation hyperregion X is*

$$\bar{\omega}(X) = \frac{5.5 \cdot 14 + 4.8 \cdot 11}{14 + 11} = \frac{77.0 + 52.8}{25} = \frac{129.8}{25} = 5.192.$$

The corresponding daily *potential on X is* $\bar{\omega}(X) \cdot \mu(\cup X) = 5.192 \times 25 = 129.8$ *MWh/day, and over a 30-day month this is* $129.8 \times 30 = 3{,}894$ *MWh.*

By hypervaluation,

$$\text{Agg}^{\sharp}_{\text{wind}}(X) = \left\{ \frac{1}{\mu(E)} \int_E A_{\text{wind}} \, d\mu : \emptyset \neq E \subseteq A_2 \cup A_5 \right\} \supseteq [4.8, \ 5.5],$$

with endpoints realized by $E \subseteq A_5$ *and* $E \subseteq A_2$, *respectively. For the combined siting-and-corridor hyperregion* $X \cup Y$, *the hypervaluation enlarges to*

$$\text{Agg}^{\sharp}_{\text{wind}}(X \cup Y) \supseteq [0.0, \ 5.5],$$

since one may choose $E \subseteq A_3$ *(or* A_4*) yielding zero average, showing how corridor inclusion affects averaged density though not the* total *potential restricted to X.*

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Hypernetwork and routing. *In the hypernetwork* (G_h, φ_h)*, the vertices* $\{A_2\}$ *and* $\{A_3\}$ *are adjacent, as are* $\{A_5\}$ *and* $\{A_4\}$*; thus any hyperselection* $Z \in X \circ_{\mathrm{reg}} Y$ *contains an embedded path from the generation atoms to the corridor atoms along shared positive-length boundaries, which represents feasible interconnections without leaving* $\bigcup(X \cup Y)$*.*

Operation invariants. *For all* $Z \in X \circ_{\mathrm{reg}} Y$*, the ground-space union is fixed:* $\bigcup Z = \bigcup(X \cup Y)$*. Hence any quantity expressible as an integral over* $\bigcup(X \cup Y)$ *(e.g. land take, environmental footprint governed by a density field) is invariant under the choice of hyperselection* Z*. Only* how *the union is partitioned (e.g. for permitting or ownership) varies.*

Theorem 1 (Region layer of HyperGeography is a HyperStructure)**.** *Let* **HGeo** *be as in Definition 11. Then*

$$\mathcal{H}_{\mathrm{reg}} := (\mathcal{R}, \circ_{\mathrm{reg}})$$

is a HyperStructure in the sense of the Definition.

Proof. By construction, $\mathcal{R} = \mathcal{P}^*(S)$ is nonempty. For any $X, Y \in \mathcal{R}$, the set $X \circ_{\mathrm{reg}} Y$ is nonempty because $X \cup Y \in \mathcal{R}$ and $X \cup Y \in X \circ_{\mathrm{reg}} Y$ by definition. Moreover $X \circ_{\mathrm{reg}} Y \subseteq \mathcal{R}$ since every $Z \in X \circ_{\mathrm{reg}} Y$ is a nonempty finite subfamily of S. Hence $\circ_{\mathrm{reg}}$ is a hyperoperation on $\mathcal{R}$, proving that $(\mathcal{R}, \circ_{\mathrm{reg}})$ is a HyperStructure. □

Definition 12 (Canonical embedding of regions)**.** *Given a Geography* **Geo** *as in Definition 10, set* $S := \mathcal{P} = \{R_i\}_{i \in I}$ *(the measurable partition of* M*). Let* $\mathcal{R} = \mathcal{P}^*(S)$ *and* $\circ_{\mathrm{reg}}$ *as in Definition 11. Define the* singleton embedding

$$\iota : \mathcal{P} \longrightarrow \mathcal{R}, \qquad \iota(R) := \{R\}.$$

Lemma 1 (Compatibility of union and hyperunion)**.** *For any* $R, S \in \mathcal{P}$ *one has*

$$\iota(R) \circ_{\mathrm{reg}} \iota(S) \supseteq \{\, \iota(R) \cup \iota(S) \,\} = \{\, \{R, S\} \,\},$$

and

$$\bigcup\big(\iota(R) \cup \iota(S)\big) = R \cup S.$$

Proof. By Definition 11, $X \circ_{\mathrm{reg}} Y$ always contains $X \cup Y$. Taking $X = \iota(R)$ and $Y = \iota(S)$ gives the first claim. The second identity holds because $\bigcup\{R, S\} = R \cup S$. □

Theorem 2 (HyperGeography generalizes Geography)**.** *Let* $\mathbf{Geo} = (M, g, d, \mu, \Sigma, \mathcal{P}, \mathcal{F}, \mathcal{A}, \mathrm{att}, \mathcal{N}, \mathrm{proj})$ *be a Geography. Construct* **HGeo** *from* **Geo** *by taking* $S := \mathcal{P}$ *and defining* $\mathcal{R}$*,* $\circ_{\mathrm{reg}}$*,* $\alpha^\sharp$*,* $\mathrm{Agg}^\sharp$ *and* $\mathcal{N}_h$ *as in Definition 11. Then:*

(a) ***Region recovery.*** *The map* $\iota : \mathcal{P} \to \mathcal{R}$ *in Definition 12 is injective and*

$$\bigcup \iota(R) = R, \qquad \forall R \in \mathcal{P}.$$

(b) ***Adjacency recovery.*** *For* $R \neq S$*,*

$$R \sim S \quad \Longleftrightarrow \quad \mathcal{H}^1_g\big(\partial(\bigcup\iota(R)) \cap \partial(\bigcup\iota(S))\big) > 0,$$

so adjacency in **Geo** *is exactly the edge relation between* $\iota(R)$ *and* $\iota(S)$ *in* $\mathcal{N}_h$*.*

(c) ***Attribute recovery.*** *For any numeric attribute* $A_a \in L^1_{\mathrm{loc}}(M, \mu)$ *and any* $R \in \mathcal{P}$ *with* $0 < \mu(R) < \infty$*,*

$$\frac{1}{\mu(R)} \int_R A_a \, \mathrm{d}\mu \;\in\; \mathrm{Agg}^\sharp_a\big(\,\iota(R)\,\big),$$

so the classical region-average is an element of the hypervaluation set.

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(d) ***Union is a hyperselection.*** *For $R, S \in \mathcal{P}$ there exists a canonical* selection *map*

$$\mathrm{Sel} : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}, \qquad \mathrm{Sel}(X, Y) := X \cup Y,$$

such that

$$\mathrm{Sel}\big(\iota(R), \iota(S)\big) \in \iota(R) \circ_{\mathrm{reg}} \iota(S) \quad \text{and} \quad \bigcup \mathrm{Sel}\big(\iota(R), \iota(S)\big) = R \cup S.$$

Hence the ordinary union of regions is recovered as a deterministic choice inside the hyperunion.

Consequently, the composite

$$\mathcal{P} \xrightarrow{\iota} \mathcal{R} \xrightarrow{\cup} \Sigma$$

identifies **Geo** *with the* singleton slice *of* **HGeo**, *and* **HGeo** *strictly generalizes* **Geo** *by allowing operations and evaluations on arbitrary hyperregions $X \in \mathcal{R}$ (finite unions of atoms) rather than only single regions $R \in \mathcal{P}$.*

Proof. (a) Injectivity of ι is immediate: if $\iota(R) = \iota(S)$ then $\{R\} = \{S\}$, so $R = S$. Also $\bigcup \iota(R) = \cup\{R\} = R$.

(b) Using (a) and Definition 10(G1), adjacency $R \sim S$ means $\mathcal{H}^1_g(\partial R \cap \partial S) > 0$. Since $\cup\iota(R) = R$ and $\cup\iota(S) = S$, the stated equivalence follows.

(c) By Definition 11, for $X = \iota(R)$ we may choose the admissible measurable set $E = R$; then

$$\frac{1}{\mu(R)} \int_R A_a \, \mathrm{d}\mu \in \mathrm{Agg}^{\sharp}_a\big(\iota(R)\big).$$

(d) By definition, $\mathrm{Sel}(X, Y) = X \cup Y \in X \circ_{\mathrm{reg}} Y$, so for $X = \iota(R)$ and $Y = \iota(S)$, $\mathrm{Sel}(\iota(R), \iota(S)) = \{R, S\} \in \iota(R) \circ_{\mathrm{reg}} \iota(S)$ and $\cup \mathrm{Sel}(\iota(R), \iota(S)) = R \cup S$. This shows that ordinary union is realized as a distinguished element of the hyperunion, proving that classical region algebra sits inside the hyperalgebra by a canonical choice. □

2.3. *(m,n)-SuperHyperGeography*

We formalize an *(m,n)-SuperHyperGeography*, show that its region layer is an (m, n)-SuperHyperStructure, and prove that it strictly generalizes the *HyperGeography* introduced earlier.

Let **Geo** be as in Definition 10 with measurable region partition $\mathcal{P} = \{R_i\}_{i \in I} \subseteq \Sigma$ and let $S \subseteq \Sigma$ be a fixed countable family of *atomic regions* such that

$$\mu\Big(M \setminus \bigcup_{A \in S} A\Big) = 0, \qquad 0 < \mu(A) < \infty, \qquad \mathcal{H}^1_g(\partial A) < \infty \quad (\forall A \in S),$$

and distinct atoms are μ-almost disjoint. To control regularity, we also track the *atomic support* of a k-level object.

Definition 13 (Atomic support and flattening)**.** *Define recursively the* atom map $\mathrm{At}_k : \mathcal{P}^*_k(S) \to \mathcal{P}^*(S)$ *and the* flattening $\mathrm{Y}_k : \mathcal{P}^*_k(S) \to \Sigma$ *by*

$$\mathrm{At}_1(X) = X, \qquad \mathrm{At}_{k+1}(X) = \bigcup_{Y \in X} \mathrm{At}_k(Y),$$

$$\mathrm{Y}_k(X) = \bigcup_{A \in \mathrm{At}_k(X)} A \quad (\in \Sigma).$$

We say $X \in \mathcal{P}^*_k(S)$ is support-finite *if $\mathrm{At}_k(X)$ is finite; write $\mathcal{P}^*_{k,\mathrm{fin}}(S)$ for the set of support-finite k-level objects.*

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Example 10 (Atomic support and flattening at level $k = 2$). *Let $M \subset \mathbb{R}^2$ be equipped with the Borel σ-algebra Σ and the Lebesgue area measure μ. Fix three pairwise disjoint measurable* atoms $S = \{A, B, C\} \subseteq \Sigma$ *with*

$$\mu(A) = 2\ \mathrm{km}^2, \qquad \mu(B) = 3\ \mathrm{km}^2, \qquad \mu(C) = 5\ \mathrm{km}^2.$$

Consider the level-2 object

$$X = \{\{A, B\}, \{B, C\}\} \ \in\ \mathcal{P}_2^*(S) = \mathcal{P}\big(\mathcal{P}_1^*(S)\big) \setminus \{\varnothing\}.$$

Compute the atomic support and flattening:

1) Atomic support:

$$\mathrm{At}_1(\{A, B\}) = \{A, B\}, \qquad \mathrm{At}_1(\{B, C\}) = \{B, C\},$$

$$\mathrm{At}_2(X) = \bigcup_{Y \in X} \mathrm{At}_1(Y) = \{A, B, C\}.$$

Thus X is support-finite and $|\mathrm{At}_2(X)| = 3$.

2) Flattening:

$$\Upsilon_2(X) = \bigcup_{T \in \mathrm{At}_2(X)} T = A \cup B \cup C.$$

Since A, B, C are pairwise disjoint,

$$\mu\big(\Upsilon_2(X)\big) = \mu(A) + \mu(B) + \mu(C) = 2 + 3 + 5 = 10\ \mathrm{km}^2.$$

This explicitly shows how a level-2 object X induces its atom set $\mathrm{At}_2(X)$ and the corresponding ground-set region $\Upsilon_2(X)$.

Definition 14 (Canonical nesting of atoms). *For any nonempty finite $T \subseteq S$ define the* nesting $\mathrm{Nest}_k(T) \in \mathcal{P}_{k,\mathrm{fin}}^*(S)$ *by*

$$\mathrm{Nest}_1(T) := T, \qquad \mathrm{Nest}_{k+1}(T) := \{\mathrm{Nest}_k(T)\}.$$

Then, by immediate induction,

$$\mathrm{At}_k\big(\mathrm{Nest}_k(T)\big) = T, \qquad \Upsilon_k\big(\mathrm{Nest}_k(T)\big) = \bigcup_{A \in T} A. \tag{1}$$

Example 11 (Canonical nesting of atoms and its identities). *Retain the measurable atoms $S = \{A, B, C\} \subseteq \Sigma$ from above, with the same areas, and let*

$$T = \{A, C\} \ \subseteq\ S \qquad (\textit{finite and nonempty}).$$

For $k = 3$ the canonical nesting is

$$\mathrm{Nest}_1(T) = T, \qquad \mathrm{Nest}_2(T) = \{T\}, \qquad \mathrm{Nest}_3(T) = \{\{T\}\}.$$

We verify the identities in (1):

1) Atomic support:

$$\mathrm{At}_3\big(\mathrm{Nest}_3(T)\big) = \bigcup_{Y \in \{\{T\}\}} \mathrm{At}_2(Y) = \mathrm{At}_2(\{T\}) = \bigcup_{Z \in \{T\}} \mathrm{At}_1(Z) = \mathrm{At}_1(T) = T = \{A, C\}.$$

2) Flattening:

$$\Upsilon_3\big(\mathrm{Nest}_3(T)\big) = \bigcup_{U \in \mathrm{At}_3(\mathrm{Nest}_3(T))} U = \bigcup_{U \in \{A, C\}} U = A \cup C.$$

If A and C are disjoint, then

$$\mu(\mathrm{Y}_3(\mathrm{Nest}_3(T))) = \mu(A) + \mu(C) = 2 + 5 = 7 \text{ km}^2.$$

Hence, concretely, $\mathrm{At}_k(\mathrm{Nest}_k(T)) = T$ *and* $\mathrm{Y}_k(\mathrm{Nest}_k(T)) = \bigcup_{U \in T} U$ *hold for* $k = 3$*, illustrating the general formula.*

Definition 15 ((m,n)-SuperHyperGeography)**.** *Fix integers* $m, n \geq 1$*. The* (m,n)-SuperHyperGeography *associated with* (M, g, d, μ, Σ) *and atom set* S *is the tuple*

$$\mathbf{SHGeo}^{(m,n)} = \left(M, g, d, \mu, \Sigma;\ S;\ \mathcal{U}_m, \mathcal{U}_n;\ \odot_{\mathrm{reg}}^{(m,n)};\ \mathcal{F}, \alpha^{\sharp};\ \mathcal{A}, \mathsf{Agg}^{(m)};\ \mathcal{N}^{(m)}, \mathrm{proj}\right),$$

where:

(a) $\mathcal{U}_m := \mathcal{P}^*_{m,\mathrm{fin}}(S)$ *and* $\mathcal{U}_n := \mathcal{P}^*_{n,\mathrm{fin}}(S)$ *are the domains/codomains of m- and n-level superobjects.*

(b) *The* region superhyperoperation *(binary, for definiteness)*

$$\odot_{\mathrm{reg}}^{(m,n)} : \mathcal{U}_m \times \mathcal{U}_m \longrightarrow \mathcal{P}(\mathcal{U}_n) \setminus \{\varnothing\} \tag{2}$$

is defined by

$$X \odot_{\mathrm{reg}}^{(m,n)} Y := \left\{ Z \in \mathcal{U}_n :\ \mathrm{Y}_n(Z) = \mathrm{Y}_m(X) \cup \mathrm{Y}_m(Y) \ \text{ and } \ \mathrm{At}_n(Z) \subseteq \mathrm{At}_m(X) \cup \mathrm{At}_m(Y) \right\}. \tag{3}$$

Equivalently, $X \odot Y$ *returns all n-level finite packings by atoms from* $\mathrm{At}_m(X) \cup \mathrm{At}_m(Y)$ *whose geometric union equals* $\mathrm{Y}_m(X) \cup \mathrm{Y}_m(Y)$*.*

(c) *For each geometric feature* $f = (S_f, \dim f, \alpha_f)$ *as in (G2) define the* hyperattachment

$$\alpha_f^{\sharp} : \mathcal{U}_m \longrightarrow \mathcal{P}(V_f) \setminus \{\varnothing\}, \qquad \alpha_f^{\sharp}(X) := \{\alpha_f(x) : x \in S_f \cap \mathrm{Y}_m(X)\}^*.$$

(d) *For a numeric attribute* $A_a \in L^1_{\mathrm{loc}}(M, \mu)$*, define the* level-$m$ hypervaluation

$$\mathsf{Agg}_a^{(m)}(X) := \left\{ \frac{1}{\mu(E)} \int_E A_a \, d\mu :\ \varnothing \neq E \in \Sigma,\ E \subseteq \mathrm{Y}_m(X),\ 0 < \mu(E) < \infty \right\} \subseteq \mathbb{R}. \tag{4}$$

(e) *The* level-m hypernetwork $\mathcal{N}^{(m)} = (G^{(m)}, \varphi^{(m)})$ *has*

$$V^{(m)} = \mathcal{U}_m, \quad \{X, Y\} \in E^{(m)} \iff \mathcal{H}^1_g(\partial \mathrm{Y}_m(X) \cap \partial \mathrm{Y}_m(Y)) > 0,$$

embedded in M by barycenters/rectifiable arcs along shared boundaries, as in HyperGeography.

Remark 3 (Nonemptiness and locality)**.** *For any* $X, Y \in \mathcal{U}_m$*, the finite set* $T := \mathrm{At}_m(X) \cup \mathrm{At}_m(Y)$ *yields*

$$Z_0 := \mathrm{Nest}_n(T) \in \mathcal{U}_n, \quad \mathrm{Y}_n(Z_0) = \bigcup_{A \in T} A = \mathrm{Y}_m(X) \cup \mathrm{Y}_m(Y),$$

so $Z_0 \in X \odot_{\mathrm{reg}}^{(m,n)} Y$*; hence the operation is well-defined and nonempty. All constructions depend only on atoms intersecting* $\mathrm{Y}_m(X) \cup \mathrm{Y}_m(Y)$ *(*locality*).*

Example 12 ((2,1)-SuperHyperGeography: multi-agency relief bundles)**.** *We instantiate* $\mathbf{SHGeo}^{(m,n)}$ *with* $(m, n) = (2, 1)$ *to model* bundles of service areas *proposed by two agencies and their induced single-layer deployment domain.*

Ground and atoms. *Let $M \subset \mathbb{R}^2$ be a bounded urban region with Euclidean metric g, Borel σ-algebra Σ, and area measure μ. Fix finite atoms $S = \{A_1, A_2, A_3, A_4\} \subset \Sigma$ with finite-perimeter boundaries and areas (km^2)*

$$\mu(A_1) = 10, \qquad \mu(A_2) = 8, \qquad \mu(A_3) = 6, \qquad \mu(A_4) = 12,$$

pairwise μ-almost disjoint. Define a population-density field (persons/km^2), piecewise constant on atoms:

$$A_{\text{pop}}(x) = \begin{cases} 8{,}000 & x \in A_1, \\ 6{,}000 & x \in A_2, \\ 3{,}000 & x \in A_3, \\ 2{,}000 & x \in A_4. \end{cases}$$

Level-2 superregions (bundles). *Work in $\mathcal{U}_2 = \mathcal{P}^*_{2,\text{fin}}(S)$. Consider two agencies' activation bundles*

$$X = \{\{A_1, A_2\}, \{A_3\}\}, \qquad Y = \{\{A_4\}\}.$$

By Definition 13, $\text{At}_2(X) = \{A_1, A_2, A_3\}$, $\text{At}_2(Y) = \{A_4\}$ and

$$\Upsilon_2(X) = A_1 \cup A_2 \cup A_3, \qquad \Upsilon_2(Y) = A_4.$$

Hence $\mu\big(\Upsilon_2(X)\big) = 10 + 8 + 6 = 24$ and $\mu\big(\Upsilon_2(X) \cup \Upsilon_2(Y)\big) = 36$ (km^2).

(2,1)-superhyperoperation. *By (2)–(3), $X \odot^{(2,1)}_{\text{reg}} Y$ is the nonempty set of $Z \in \mathcal{U}_1 = \mathcal{P}^*_{1,\text{fin}}(S)$ such that*

$$\Upsilon_1(Z) = \Upsilon_2(X) \cup \Upsilon_2(Y) = A_1 \cup A_2 \cup A_3 \cup A_4, \quad \text{At}_1(Z) \subseteq \{A_1, A_2, A_3, A_4\}.$$

A canonical witness is

$$Z_0 = \text{Nest}_1\big(\text{At}_2(X) \cup \text{At}_2(Y)\big) = \{A_1, A_2, A_3, A_4\} \in X \odot^{(2,1)}_{\text{reg}} Y,$$

and by (1) one has $\Upsilon_1(Z_0) = \Upsilon_2(X) \cup \Upsilon_2(Y)$.

Concrete calculations (population and demand). *Atomic populations are*

$$\text{Pop}(A_1) = 8{,}000 \cdot 10 = 80{,}000, \quad \text{Pop}(A_2) = 6{,}000 \cdot 8 = 48{,}000,$$

$$\text{Pop}(A_3) = 3{,}000 \cdot 6 = 18{,}000, \quad \text{Pop}(A_4) = 2{,}000 \cdot 12 = 24{,}000.$$

Thus

$$\text{Pop}\big(\Upsilon_2(X)\big) = 146{,}000, \qquad \text{Pop}\big(\Upsilon_2(X) \cup \Upsilon_2(Y)\big) = 170{,}000.$$

The average density on the combined deployment domain is

$$\bar{\rho} = \frac{\int_{\Upsilon_2(X) \cup \Upsilon_2(Y)} A_{\text{pop}} \, d\mu}{\mu\big(\Upsilon_2(X) \cup \Upsilon_2(Y)\big)} = \frac{170{,}000}{36} \approx 4{,}722.222 \text{ persons/km}^2.$$

If each person requires $r = 0.04$ relief kits per month, the baseline monthly demand is

$$D = r \cdot \text{Pop}\big(\Upsilon_2(X) \cup \Upsilon_2(Y)\big) = 0.04 \times 170{,}000 = 6{,}800 \text{ kits}.$$

By (4), the level-2 hypervaluation over X yields the envelope

$$\text{Agg}^{(2)}_{\text{pop}}(X) \;\supseteq\; [3{,}000,\; 8{,}000],$$

realized by $E \subseteq A_3$ *and* $E \subseteq A_1$*, respectively; for* $X \cup Y$ *the envelope expands to* $[2{,}000, 8{,}000]$*. Importantly, any* $Z \in X \odot_{\mathrm{reg}}^{(2,1)} Y$ *induces the same ground union and hence the same D; only the* partitioning *differs (e.g., for contracting or governance).*

Example 13 ((1,2)-SuperHyperGeography: logistics hub governance scenarios)**.** *We instantiate* $\mathbf{SHGeo}^{(m,n)}$ *with* $(m, n) = (1, 2)$ *to represent* site alternatives *at level 1 and the induced* portfolio of governance partitions *at level 2.*

Ground, atoms, and capacity field. *Let* $S = \{B_1, B_2, B_3\} \subset \Sigma$ *with areas (km*^{2}*)*

$$\mu(B_1) = 5, \qquad \mu(B_2) = 7, \qquad \mu(B_3) = 4.$$

Define a throughput-capacity density (trucks/day/km^2*),*

$$A_{\mathrm{cap}}(x) = \begin{cases} 120 & x \in B_1, \\ 80 & x \in B_2, \\ 150 & x \in B_3. \end{cases}$$

Level-*1 *hyperregions (site selections). *Work in* $\mathcal{U}_1 = \mathcal{P}^*_{1,\mathrm{fin}}(S)$ *and take*

$$X = \{B_1, B_2\}, \qquad Y = \{B_3\}.$$

Then $\mathrm{At}_1(X) = \{B_1, B_2\}$, $\mathrm{At}_1(Y) = \{B_3\}$ *and, by Definition 13,*

$$\Upsilon_1(X) = B_1 \cup B_2, \qquad \Upsilon_1(Y) = B_3, \qquad \Upsilon_1(X) \cup \Upsilon_1(Y) = B_1 \cup B_2 \cup B_3.$$

(1,2)-superhyperoperation and admissible outputs. *By (3),* $X \odot_{\mathrm{reg}}^{(1,2)} Y$ *consists of level-2 objects* $Z \in \mathcal{U}_2$ *with*

$$\Upsilon_2(Z) = B_1 \cup B_2 \cup B_3, \qquad \mathrm{At}_2(Z) \subseteq \{B_1, B_2, B_3\}.$$

Examples (all valid):

$$\begin{aligned} Z_{\mathrm{coarse}} &= \mathrm{Nest}_2(\{B_1, B_2, B_3\}) = \{\{B_1, B_2, B_3\}\}, \\ Z_{\mathrm{split}} &= \{\{B_1, B_3\}, \{B_2\}\}, \\ Z_{\mathrm{tri}} &= \{\{B_1\}, \{B_2\}, \{B_3\}\}. \end{aligned}$$

By (1), each has $\Upsilon_2(\cdot) = B_1 \cup B_2 \cup B_3$ *while encoding different governance/ownership partitions at level 2.*

Concrete calculations (throughput). *Total daily throughput (as an integral of* A_{cap}*) over the ground union is*

$$\int_{B_1 \cup B_2 \cup B_3} A_{\mathrm{cap}}\, \mathrm{d}\mu = 120 \cdot 5 + 80 \cdot 7 + 150 \cdot 4 = 600 + 560 + 600 = 1{,}760 \ \textit{trucks/day}.$$

Average capacity density over the union is

$$\bar{\kappa} = \frac{1{,}760}{\mu(B_1) + \mu(B_2) + \mu(B_3)} = \frac{1{,}760}{16} = 110 \ \textit{trucks/(day·km}^2\textit{)}.$$

By (4) with $m = 1$*,*

$$\mathrm{Agg}^{(1)}_{\mathrm{cap}}(X \cup Y) = \left\{ \tfrac{1}{\mu(E)} \int_E A_{\mathrm{cap}}\, \mathrm{d}\mu : \varnothing \neq E \subseteq B_1 \cup B_2 \cup B_3 \right\} \supseteq [80, 150],$$

with the endpoints realized on B_2 *and* B_3*, respectively. Note that* $\bar{\kappa}$ *and the total* 1,760 *are* invariant *across all* $Z \in X \odot_{\mathrm{reg}}^{(1,2)} Y$*, since they depend only on the flattened ground union; what changes across* $\{Z_{\mathrm{coarse}}, Z_{\mathrm{split}}, Z_{\mathrm{tri}}\}$ *is the second-order organization (single operator vs. two-tier vs. fully disaggregated management).*

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Theorem 3 ((m,n)-SuperHyperStructure representation)**.** *For fixed $m, n \geq 1$, the pair*

$$\mathcal{H}_{\mathrm{reg}}^{(m,n)} := \left(\mathcal{U}_m, \odot_{\mathrm{reg}}^{(m,n)} \right)$$

is an (m,n)-SuperHyperStructure *(with arity $s = 2$) on the base set S.*

Proof. By definition, $\mathcal{U}_m = \mathcal{P}^*_{m,\mathrm{fin}}(S) \subseteq \mathcal{P}^m(S)$ and the codomain in (2) is a nonempty subset of $\mathcal{P}^n(S)$. For any $X, Y \in \mathcal{U}_m$, the witness $Z_0 = \mathrm{Nest}_n(\mathrm{At}_m(X) \cup \mathrm{At}_m(Y)) \in \mathcal{U}_n$ (see (1)) satisfies the two constraints in (3), so $X \odot_{\mathrm{reg}}^{(m,n)} Y \neq \emptyset$ and $X \odot_{\mathrm{reg}}^{(m,n)} Y \subseteq \mathcal{U}_n \subseteq \mathcal{P}^n(S)$. Hence $\odot_{\mathrm{reg}}^{(m,n)}$ is a well-defined (m,n)-SuperHyperOperation on S. Therefore $(\mathcal{U}_m, \odot_{\mathrm{reg}}^{(m,n)})$ is an (m,n)-SuperHyperStructure. □

Lemma 2 (Level-1 identification)**.** *For all $X, Y \in \mathcal{P}^*_{1,\mathrm{fin}}(S)$ one has*

$$X \odot_{\mathrm{reg}}^{(1,1)} Y = X \circ_{\mathrm{reg}} Y.$$

Proof. When $m = n = 1$, $\mathrm{At}_1(\cdot)$ is the identity on $\mathcal{P}^*_{1,\mathrm{fin}}(S)$ and Y_1 is geometric union of atoms. Thus both definitions coincide verbatim. □

Definition 16 (Canonical embedding into level m)**.** *Define $\iota_m : \mathcal{P}^*_{1,\mathrm{fin}}(S) \to \mathcal{P}^*_{m,\mathrm{fin}}(S)$ by*

$$\iota_m(X) := \mathrm{Nest}_m\big(\mathrm{At}_1(X)\big) = \mathrm{Nest}_m(X).$$

By (1), $\mathrm{Y}_m(\iota_m(X)) = \mathrm{Y}_1(X)$ *and* $\mathrm{At}_m(\iota_m(X)) = \mathrm{At}_1(X)$.

Theorem 4 (SuperHyperGeography generalizes HyperGeography)**.** *Let* **HGeo** *be the HyperGeography built on S, and let* $\mathbf{SHGeo}^{(m,n)}$ *be as in Definition 15. Then:*

(a) ***Exact recovery at level (1,1).*** *By Lemma 2,* $\mathbf{SHGeo}^{(1,1)}$ *and* **HGeo** *have the same region universe and the same hyperoperation.*

(b) ***Embedding of HyperGeography into level m.*** *The map ι_m is injective and satisfies*

$$\mathrm{Y}_m\big(\iota_m(X)\big) = \mathrm{Y}_1(X), \qquad \mathrm{At}_m\big(\iota_m(X)\big) = \mathrm{At}_1(X),$$

hence adjacency and attribute evaluations are preserved:

$$\mathcal{H}^1_g(\partial \mathrm{Y}_1(X) \cap \partial \mathrm{Y}_1(Y)) > 0 \iff \mathcal{H}^1_g(\partial \mathrm{Y}_m(\iota_m(X)) \cap \partial \mathrm{Y}_m(\iota_m(Y))) > 0,$$

$$\frac{1}{\mu(R)} \int_R A_a \, \mathrm{d}\mu \in \mathrm{Agg}_a^{(m)}\big(\iota_m(\{R\})\big) \quad (\textit{choose } E = R).$$

(c) ***Hyperoperation compatibility.*** *For all $X, Y \in \mathcal{P}^*_{1,\mathrm{fin}}(S)$,*

$$\mathrm{Nest}_n\big(\mathrm{At}_1(X) \cup \mathrm{At}_1(Y)\big) \in \iota_m(X) \odot_{\mathrm{reg}}^{(m,n)} \iota_m(Y), \tag{5}$$

and its flattening satisfies

$$\mathrm{Y}_n\big(\mathrm{Nest}_n\big(\mathrm{At}_1(X) \cup \mathrm{At}_1(Y)\big)\big) = \mathrm{Y}_1(X) \cup \mathrm{Y}_1(Y),$$

so the level-m superhyperoperation projects to the same geometric union as in HyperGeography.

Therefore, $\mathbf{SHGeo}^{(m,n)}$ strictly *generalizes HyperGeography by allowing operations/evaluations on m-level superregions with n-level outputs, while recovering the classical case at $(m,n) = (1,1)$.*

Proof. (a) is Lemma 2. For (b), injectivity of ι_m is immediate from Definition 14, and the stated equalities follow from (1). The adjacency equivalence uses those equalities and Definition 10(G1). For the attribute statement, take $X = \{R\}$ and choose $E = R$ in (4) to obtain the classical region average as an element of $\mathrm{Agg}_a^{(m)}(\iota_m(X))$.

For (c), using Definition 15 and (1),

$$\mathrm{At}_m(\iota_m(X)) = \mathrm{At}_1(X), \quad \mathrm{At}_m(\iota_m(Y)) = \mathrm{At}_1(Y),$$

thus the candidate $Z := \mathrm{Nest}_n(\mathrm{At}_1(X) \cup \mathrm{At}_1(Y))$ satisfies $\mathrm{At}_n(Z) \subseteq \mathrm{At}_m(\iota_m(X)) \cup \mathrm{At}_m(\iota_m(Y))$ and

$$\Upsilon_n(Z) = \bigcup_{A \in \mathrm{At}_1(X) \cup \mathrm{At}_1(Y)} A = \Upsilon_1(X) \cup \Upsilon_1(Y) = \Upsilon_m(\iota_m(X)) \cup \Upsilon_m(\iota_m(Y)).$$

Hence $Z \in \iota_m(X) \odot_{\mathrm{reg}}^{(m,n)} \iota_m(Y)$, proving (5). $\square$

3. Conclusions

In this paper, we examined the viability of employing Hyperstructures and SuperHyperstructures to define *HyperGeography* and *SuperHyperGeography*, and we offered a concise discussion that included potential applications. Looking ahead, we hope to see broader set-theoretic extensions of the concepts introduced here, including Fuzzy Sets [42,43], Intuitionistic Fuzzy Sets [44], HyperFuzzy Sets [45,46], Soft Sets [47,48] and HyperSoft Sets [49,50], Rough Sets [51,52] and HyperRough Sets [53], Neutrosophic Sets [54,55], and Plithogenic Sets [56,57]. For example, I would like to explore whether concepts such as Fuzzy Geography (cf.[58–60]), Neutrosophic Geography[61,62], and Rough Geography[63,64] can be integrated with the ideas of *HyperGeography* and *SuperHyperGeography*, in order to identify new characteristics or potential applications.

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