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Posted Date: 2 October 2025

doi: 10.20944/preprints202510.0157.v1

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Foundations of the Preisach Operator in Real Options Problems with Subscription Cost and Heterogeneous Population of Consumers

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Abstract

This paper considers the pricing of a subscription service in a heterogeneous market with consumers having different discount rates. We show that in the case of non-zero enrollment/cancellation cost, solutions of the HJB equation naturally contain an equivalent of the well-known Preisach operator, a fundamental model of hysteresis in engineering applications. Singular perturbation expansions are used to approximate the optimal solution, assuming that enrollment/cancellation costs are small relative to the total subscription cost. As a case study, we consider and compare markets with one and two consumers.

Keywords: stochastic optimal control; preisach model; perturbation method

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1. Introduction

We consider a stochastic control problem in which the Hamilton-Jacobi-Bellman (HJB) admits a two-threshold solution. As a specific setting, we refer to the subscription economy. Households (consumers) make decisions to subscribe to (enroll), or unsubscribe from (cancel), a given subscription service as the flow benefit from the service fluctuates over time. There are costs from transitioning from state “subscribed" to state “unsubscribed" (cancellation cost) and vice versa (enrollment cost), resulting in a two-threshold optimal strategy, i.e. a household subscribes to the service when the flow benefit reaches a threshold value X_S, and unsubscribes when the flow benefit value drops below a threshold value X_U with $X_U < X_S$.

This paper emphasizes the heterogeneity of the market assuming a distribution of thresholds (X_S, X_U). We show that the heterogeneous setting leads to an operator, which is well-known in the engineering literature as the Preisach operator, a fundamental model of hysteresis effects, see [7] and Section 2.5 for references and a brief review of applications.

The Preisach operator was previously used as a phenomenological model of hysteresis in unemployment [8] and macroeconomic flows [9]. In the phenomenological approach, the Preisach operator is postulated as a model of the relationship between economic variables. In particular, the model parameters are not linked to parameters of the market but rather are determined from a black box identification procedure. On the other hand, in this paper, the parameters of the Preisach operator derive from the solution of the optimization problem and, as such, are functions (either explicit or amenable to approximation) of the subscription cost, enrollment/cancellation costs, discount rates and parameters of dynamics of the utility flow.

The subscription economy has grown substantially over the past years. As of 2023, U.S. households spent an average of $230 per month on subscription services [25]. Such costs span mobile phone, internet, TV streaming, Amazon Prime, and music streaming services, among others. Importantly, many of these services are easy to subscribe to but are rather cumbersome to unsubscribe from. In fact,

 doi:10.20944/preprints202510.0157.v1

in October 2024, the Federal Trade Commission (FTC) announced a final "click-to-cancel" rule that will require sellers to make it as easy for consumers to cancel their enrollment as it was to sign up [26].

Motivated by the growth of this market and its unique regulatory features, this paper develops a mathematical framework to help us understand the main economic forces driving the subscription services market. We model this market as a Stackelberg game. On the demand side, we characterize the optimal enrollment and cancellation decisions of households (i.e., the followers) as a sequence of compound real options (i.e., the decision to unsubscribe when currently subscribed, and vice versa). We assume that the household takes as given the pricing structure of the service and model the utility flow derived by the household as a diffusion process. Next, we study the firm's (i.e., the leader's) problem, in which the firm, anticipating the household's optimal response, chooses the optimal cost structure for its services. Finally, relying on singular perturbation examples, we provide a detailed characterization of the game's equilibrium strategies when enrollment/cancellation costs are small relative to the total subscription costs. The justification for small optimal enrollment/cancellation costs is application dependent. However, consider for instance, the case of Amazon Prime. As of 2025, the monthly cost of an Amazon Prime subscription stands at $14.99 a month. Assuming a real discount rate of 3% per annum and a 5-year subscription period, yields a net present cost of $846. By contrast, plausible enrollment/cancellation costs are likely to be one to two orders of magnitude smaller. That is, of the order of $10, and therefore "small" relative to the total subscription costs.

The paper is organized as follows. In Section 2, we discuss the model, the Bellman equation (in the form of a variational inequality) for the household's value function, the two-threshold optimal solution, the firm's problem in a heterogeneous market and its relationship to the Preisach operator. Section 3 presents main result. First, the household's problem is solved in the case of cost-free enrollment/cancellation. The solution provides the benchmark for further results. Next, the existence of a two-threshold solution is shown in the case of non-zero enrollment/cancellation costs. This solution is not explicit. The singular perturbation method is used to approximate the thresholds of the two-threshold solution assuming that the enrollment/cancellation costs are small compared to the total subscription cost. The perturbation expansions for the thresholds are then used to approximate the solution of the firm's problem. We present a case study of the market with one, two and N customers. Sections 4, 5 summerize a few implications of the results in the context of subscription economy.

2. Model

2.1. Dynamics

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space that supports a standard Brownian motion $(W_t)_{t\geq 0}$ with its natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Denote the state variable by X and suppose that its dynamics are given by

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \quad X_0 = x > 0. \tag{1}$$

We interpret X_t as the flow certainty equivalent utility obtained by the consumer if currently subscribed to the service. The utility from the service can vary over time, depending on the effort put by the firm to provide a higher quality product, but it is also subject to taste shocks faced by the consumer and by the quality of other products.

The second state variable is $Z \in \{0, 1\}$, where 1 means the consumer is currently subscribed to the service, and 0 means the consumer is currently not subscribed to (unsubscribed from) the service. There are costs from transitioning across these states. Denote by ξ_S the enrollment cost (i.e., of transitioning from 0 to 1). Similarly, denote by ξ_U the cancellation cost (i.e., of transitioning from 1 to 0). These costs can be the monetary cost of effort from taking each action, but also some monetary transfer offered by the firm to the consumer. We abstract away from this distinction for now.

Using an increasing sequence of times

$$\Xi = (t_1, t_2, \ldots), \qquad 0 = t_0 \leq t_1 < t_2 < \cdots,$$

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denote by $Z_t : \mathbb{R}_+ \to \{0,1\}$ the corresponding enrollment/cancellation policy

$$Z_t = \begin{cases} z, & t_{2n} \le t < t_{2n+1}, \\ 1-z, & t_{2n+1} \le t < t_{2n+2}, \end{cases} \qquad n = 0,1,2,\dots, \tag{2}$$

where

$$Z_0 = z \in \{0,1\} \tag{3}$$

is the initial state, i.e. Ξ is the sequence of transitioning times for Z_t. We denote by Ξ_S the set of times at which the consumer transitions from 0 to 1, respectively Ξ_U are the times when the consumer transitions from 1 to 0. Hence, Ξ_S is the the odd-indexed subsequence and Ξ_U is the even-indexed subsequence of Ξ if $z = 0$; conversely, Ξ_S is the the even-indexed subsequence and Ξ_U is the odd-indexed subsequence of Ξ if $z = 1$.

Denote by $p > 0$ the flow cost of the subscription. The expected pay-off for an exponential discounter with discount rate r from an enrollment/cancellation policy $Z_t : \mathbb{R}_+ \to \{0,1\}$ is given by

$$J_r(Z_t; x, z) = \mathbb{E}\left[\int_0^\infty e^{-rt}(\varkappa X_t^\alpha - p) Z_t \, dt - \left(\xi_S \sum_{t \in \Xi_S} e^{-rt} + \xi_U \sum_{t \in \Xi_U} e^{-rt}\right)\right] \tag{4}$$

with $\varkappa > 0$. The consumer's objective is to maximize the expected pay-off using an admissible enrollment/cancellation policy as a control. Hence, the value function of this optimization problem is

$$V(x,z) = \sup_{Z_t \in \mathcal{Z}} J_r(Z_t; x, z),$$

where $\mathcal{Z}$ is the set of enrollment/cancellation policies Z_t induced by transitioning time sequences Ξ.

Since X_t is a stationary Markov process, one expects optimal transitioning times to have the form of recursively as the stopping times

$$t_{n+1} = \inf\{t \ge t_n : X_t \in \mathcal{S}_z \text{ with } z = Z_{t_n}\}, \quad n = 0,1,2,\dots, \tag{5}$$

where the closed set $\mathcal{S}_z \subset \mathbb{R}_+$ and its complement $\mathcal{C}_z = \mathbb{R}_+ \setminus \mathcal{S}_z$ are the so-called stopping and continuation regions, respectively, associated with the state $z \in \{0,1\}$ [6]. In other words, the consumer transitions from state z to state $1-z$ at the nearest moment when the process X_t reaches the stopping region $\mathcal{S}_z$. In the simplest case, $\mathcal{S}_z$ is an interval with its end point(s) serving as threshold(s), i.e. a transition occurs when X_t attains a threshold.

In what follows, the enrollment/cancellation costs are assumed to satisfy

$$\xi_S \le 0 < \xi_U, \qquad \xi_S + \xi_U > 0. \tag{6}$$

Hence, there is a fine for cancellation and an incentive to enroll. The fine exceeds the incentive eliminaing the arbitrage opportunity when the flow benefit $\varkappa X_t^\alpha - p$ oscillates around the zero value, as well as eliminating excessive transitioning by spreading the thresholds.

2.2. *Variational Inequality*

In order to solve for the optimal strategy and pay-off, we represent the value function $V(Z,X) : \{0,1\} \times \mathbb{R}_+ \to \mathbb{R}$ of a consumer as a pair of value functions

$$V_U(X) = V(0,X), \qquad V_S(X) = V(1,X)$$

corresponding to the states 0 and 1, respectively. Defining the differential operator

$$\mathcal{L}_r V(X) = rV(X) - \mu X V'(X) - \frac{\sigma^2}{2} X^2 V''(X) \tag{7}$$

doi:10.20944/preprints202510.0157.v1

associated in the usual way with the process (1) and the exponential discount rate r, and following [6], the value functions $V_S, V_C \in C^1(\mathbb{R}_+)$ satisfy the following coupled variational ineqaulities (which are a form of the Bellman equation):

$$\mathcal{L}_r V_S(X) \geq \varkappa X^\alpha - p, \qquad a.e.\ X \in \mathbb{R}_+, \tag{8}$$
$$V_S(X) \geq V_U(X) - \xi_U, \qquad X \in \mathbb{R}_+, \tag{9}$$
$$\big(\mathcal{L}_r V_S(X) - \varkappa X^\alpha + p\big)\big(V_S(X) - V_U(X) + \xi_U\big) = 0, \qquad a.e.\ X \in \mathbb{R}_+, \tag{10}$$

and

$$\mathcal{L}_r V_U(X) \geq 0, \qquad a.e.\ X \in \mathbb{R}_+, \tag{11}$$
$$V_U(X) \geq V_S(X) - \xi_S, \qquad X \in \mathbb{R}_+, \tag{12}$$
$$\mathcal{L}_r V_U(X)\big(V_U(X) - V_S(X) + \xi_S\big) = 0, \qquad a.e.\ X \in \mathbb{R}_+. \tag{13}$$

2.3. Two-Threshold Solution

It is natural to expect [1,5] that the solution of the optimal control problem involves two thresholds $X_U < X_S$ at which the consumer unsubcribes if her flow benefit falls below X_U (when currently subscribed) and subscribes if her flow benefit reaches X_S (when currently unsubscribed). In other words, the stopping and continuation regions for the state $z = 1$ are

$$\mathcal{S}_1 = [0, X_U], \qquad \mathcal{C}_1 = (X_U, \infty), \tag{14}$$

and for the state $z = 0$ they are

$$\mathcal{S}_0 = [X_S, \infty), \qquad \mathcal{C}_0 = (0, X_S). \tag{15}$$

For this solution in the continuation region, the HJB variational inequality (8)–(13) leads to the equations

$$\mathcal{L}_r V_S(X) = \varkappa X^\alpha - p, \qquad X > X_U, \tag{16}$$
$$\mathcal{L}_r V_U(X) = 0, \qquad X < X_S, \tag{17}$$

subject to the set of boundary conditions, which include the value matching conditions

$$V_S(X_S) - V_U(X_S) = \xi_S, \qquad V_S(X_U) - V_U(X_U) = -\xi_U \tag{18}$$

and smooth pasting conditions

$$\frac{d}{dX}\big(V_S(X_S) - V_U(X_S)\big) = \frac{d}{dX}\big(V_S(X_U) - V_U(X_U)\big) = 0 \tag{19}$$

at the thresholds X_S, X_U; and, the conditions

$$V_U(0) = 0, \qquad \limsup_{X\to\infty} \frac{V_S(X)}{X^\alpha} < \infty \tag{20}$$

at zero and infinity. On the other hand, in the stopping region the HJB variational inequality yields

$$V_S(X) = V_U(X) - \xi_U, \qquad X \leq X_U, \tag{21}$$
$$V_U(X) = V_S(X) - \xi_S, \qquad X \geq X_S. \tag{22}$$

Moreover, the variational inequality requires that

$$\mathcal{L}_r V_S(X) \geq \varkappa X^\alpha - p, \qquad X < X_U, \tag{23}$$
$$V_S(X) \geq V_U(X) - \xi_U, \qquad X > X_U, \tag{24}$$
$$\mathcal{L}_r V_U(X) \geq 0, \qquad X > X_S, \tag{25}$$
$$V_U(X) \geq V_S(X) - \xi_S, \qquad X < X_S. \tag{26}$$

The corresponding transitioning policy is the sequence of stopping times (5) defined by the simple threshold-based rule

$$t_{n+1} = \begin{cases} \inf\{t \geq t_n : X_t \geq X_S\} & \text{if} \quad Z_{t_n} = 0, \\ \inf\{t \geq t_n : X_t \leq X_U\} & \text{if} \quad Z_{t_n} = 1, \end{cases} \qquad n = 0, 1, 2, \ldots, \tag{27}$$

i.e. a transition across the states occurs when X_t hits either the threshold X_S or X_U (depending on the current state) as well as at the initial moment if the initial conditions satisfy either $x = X_0 \geq X_S$ and $z = Z_0 = 0$ or $x = X_0 \leq X_U$ and $z = Z_0 = 1$.

Proposition 1. *Let $0 < X_U < X_S$ and let $V_S : [X_U, \infty) \to \mathbb{R}$, $V_U : [0, X_S] \to \mathbb{R}$ be a solution of problem (16)–(20) extended to $\mathbb{R}_+ \ni X$ by equations (21)–(22). Then relations (23)–(26) hold iff*

$$\varkappa X_U^\alpha - p + r\xi_U \leq 0, \tag{28}$$
$$\varkappa X_S^\alpha - p - r\xi_S \geq 0. \tag{29}$$

Proof. The proof included here for completeness follows in a standard way from the Maximum Principle.

For $X < X_U$, equation (21) combined with (17) implies

$$\mathcal{L}_r V_S(X) = \mathcal{L}_r\big(V_U(X) - \xi_U\big) = -r\xi_U,$$

hence (23) is equivalent to (28). Similarly, for $X > X_S$, from (22) and (16) it follows that

$$\mathcal{L}_r V_U(X) = \mathcal{L}_r\big(V_S(X) - \xi_S\big) = \varkappa X^\alpha - p - r\xi_S,$$

therefore (25) is equivalent to (29).

Next, define

$$u(X) := V_S(X) - V_U(X) + \xi_U.$$

Then, (24) is equivalent to

$$u(X) \geq 0, \quad X > X_U. \tag{30}$$

From equations (16), (17) and the boundary conditions (18), (19) at X_U, it follows that

$$u(X_U) = 0; \quad u'(X_U) = 0; \quad \mathcal{L}_r u(X_U) = -\frac{\sigma^2}{2} X_U^2 u''(X_U) = \varkappa X_U^\alpha - p + r\xi_U. \tag{31}$$

Therefore, (24) implies (28). Similarly, using the function

$$w(X) := V_S(X) - V_U(X) - \xi_S,$$

relation (26) is equivalent to $w(X) \leq 0$ for $X < X_S$, while equations (16), (17) and the boundary conditions (18), (19) at X_S imply

$$w(X_S) = 0, \quad w'(X_S) = 0, \quad \mathcal{L}_r w(X_S) = -\frac{\sigma^2}{2} X_S^2 w''(X_S) = \varkappa X_S^\alpha - p - r\xi_S. \tag{32}$$

Hence, (26) implies (29).

Conversely, to show that (28) implies (30), which is equivalent to (24), assume that (28) holds with the strict inequality. Then, (31) implies the existence of a sufficiently small interval $X_U \leq X < X_1$ where

$$u(X_U) = 0; \quad u(X) > 0, \; X_U < X < X_1. \tag{33}$$

From (22) and (6) it follows that

$$u(X) > 0, \quad X \geq X_S. \tag{34}$$

Assume for contradiction with (30) that

$$\min_{X > X_U} u(X) < 0.$$

Then, by (33), (34), u has a local maximum point X_2 and a local minimum point X_3 such that

$$X_U < X_2 < X_3 < X_S, \quad u(X_2) > 0, \quad u(X_3) < 0.$$

As such,

$$\mathcal{L}_r u(X_2) > 0, \quad \mathcal{L}_r u(X_3) < 0, \quad X_U < X_2 < X_3 < X_S. \tag{35}$$

But due to (16), (17),

$$\mathcal{L}_r u = \varkappa X^\alpha - p + r\xi_U, \quad X_U < X < X_S,$$

hence (35) contradicts the monotonicity of $\mathcal{L}_r u$. The contradiction proves (24) when the inequality in (28) is strict and implies (24) in the case of the equality in (28) by the continuity argument.

Similarly, relations (21) and (6) imply

$$w(X) < 0, \; X \leq X_U. \tag{36}$$

If (29) holds with the strict inequality, then due to (32),

$$w(X_S) = 0; \quad w(X) < 0, \; X_1' < X < X_S, \tag{37}$$

on a sufficiently small interval $X_1' < X \leq X_S$. If

$$\max_{X < X_S} w(X) > 0,$$

then relations (36), (37) imply that w has has a local maximum point X_2' and a local minimum point X_3' such that

$$X_U < X_2' < X_3' < X_S, \quad w(X_2') > 0, \quad w(X_3') < 0,$$

which leads to the contradiction between the relations

$$\mathcal{L}_r w(X_2') > 0, \quad \mathcal{L}_r w(X_3') < 0, \quad X_U < X_2' < X_3' < X_S,$$

and the monotonicity of the function $\mathcal{L}_r w = \varkappa X^\alpha - p - r\xi_S$ on the interval $X_U < X < X_S$. Hence, $w(x) \leq 0$ for all $X < X_S$ proving (26). The continuity argument completes the proof of the implication (29) $\Rightarrow$ (26) in the case of the equality in (29). □

The solution to problem (16)–(20) will be addressed in Section 3 under the assumption that the characteristic polynomial

$$L(m) := -\frac{1}{2}\sigma^2 \, m(m-1) - \mu \, m + r \tag{38}$$

 doi:10.20944/preprints202510.0157.v1

associated with the differential operator $\mathcal{L}_r$ satisfies

$$L(\alpha) > 0. \tag{39}$$

Denoting the roots of L by δ, γ with $\gamma < 0 < \delta$, this assumption is equivalent to

$$L(\gamma) = L(\delta) = 0, \qquad \gamma < 0 < \alpha < \delta. \tag{40}$$

As a case in point, if $\alpha = 1$, i.e. the per-period pay-off is linear, and $\mu = 0$, then $L(\alpha) = r > 0$.

In the next two subsections, we consider an extension of problem (16)–(20).

2.4. Heterogeneous Market and Firm Problem

We consider a heterogeneous market where there are N consumers with different discount rates r_i, $i = 1, \ldots, N$, each maximizing her expected pay-off (4). With each consumer we associate a weight $\beta_i > 0$, where

$$\sum_{i=1}^{N} \beta_i = 1, \tag{41}$$

and denote by Z_t^i the subscription policy of the i-th consumer. Next, we extend the model by introducing the service providing firm, thus closing the feedback loop. We assume that the flow benefit of the firm from each subscription is $\beta_i(p - c)$. Hence, the aggregated flow benefit from all the subscriptions at time t equals

$$y_t = \sum_{i=1}^{N} \beta_i Z_t^i. \tag{42}$$

Moreover, there are costs associated with enrollment/cancellation of consumers, which are assumed to be proportional to ξ_S and ξ_U. As such, the expected pay-off of the firm is

$$\hat{J}(Z_t^1, \ldots, Z_t^N; x) = \mathbb{E}\left[\int_0^\infty e^{-\rho t}(p - c)\sum_{i=1}^{N} \beta_i Z_t^i \, dt\right] + \kappa_S \xi_S - \kappa_U \xi_U \tag{43}$$

with $\kappa_S, \kappa_U > 0$, where ρ is the firm's discount rate. The firm sets the flow rate p and the enrollment/cancellation costs ξ_S, ξ_U seeking to maximize its expected pay-off (43), while each consumer sets her enrollment/cancellation thresholds X_S^i, X_U^i based on the values of p, ξ_S, ξ_U seeking to maximize her pay-off (cf. (4)). The thresholds X_S^i, X_U^i determine the enrollment/cancellation policy Z_t^i, which feeds back to (43).

The counterparts of relations (40) are assumed to hold for the roots γ_i, δ_i of each polynomial (38) each $r = r_i$, $i = 1, \ldots, N$.

2.5. Relation to Preisach Model

Relations (2), (3), (5) define a map from any space of continuous inputs $X_t : \mathbb{R}_+ \to \mathbb{R}_+$ (not necessarily realizations of the Geometric Brownian Motion) to the space of binary functions $Z_t : \mathbb{R}_+ \to \{0, 1\}$. In the case when the stopping and continuation regions have the form (14), (15) with $X_U < X_S$, i.e. (5) has the form (27), this map is well-known in engineering applications as the two-threshold two-state *non-ideal relay* (also known as *bi-stable switch*, *'lazy' switch*, *elementary rectangular hysteresis loop*, or *Schmitt trigger* depending on a particular application).

Stacking N non-ideal relays, which all have the same input $X_t : \mathbb{R}_+ \to \mathbb{R}$, and defining the output $y_t : \mathbb{R}_+ \to \mathbb{R}$ of the system as the aggregated quantity (42), has a long history in physics and engineering, starting from the fundamental phenomenological model of magnetic hysteresis proposed by F. Preisach [21]. The states of the system are naturally N-tuples $(Z^1, \ldots, Z^N) \in \{0, 1\}^N$, hence the total number of plausible states is at most 2^N (but can be smaller depending on the thresholds, see Section 3.5). The Preisach operator, which extends this hysteresis model to the continuous setting [16,19,23], has counterparts in many disciplines, most notably the Prandtl-Ishlinskii model in plasticity

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[17,18], Maxwell-slip friction model in tribology [4] and Parlange model in hydrology [11]. Further applications of the Preisach model include control based on smart materials [3,12–14,22], economics [2,8,24], neuroscience [20] and epidemiology [10,15]. The above discussion shows how the same model arises in the setting of real options and pricing.

Remark 1. *A more general market model can include a population of customers with different κ_i, α_i in their corresponding pay-offs (4). These variations still lead to the Preisach operator in the firm's pay-off. However, solutions presented below are specific to the particular case where the consumers differ by the discount rate only, see for example Remark 2. The more general case will be considered elsewhere.*

3. Results

3.1. Cost-Free Enrollment/Cancellation Benchmark

In what follows, the polynomial L of each consumer and of the firm are all assumed to satisfy condition (39), which is equivalent to (40). These polynomials can have different r and hence different roots δ, γ.

As a benchmark case, in this section we consider zero enrollment/cancellation costs, i.e. $\xi_S = \xi_U = 0$. In this case, relations (21), (22), (24), (26) imply $V_S(X) = V_U(X)$ on the whole $\mathbb{R}_+$, hence we simply write $V := V_S = V_U$. Hence, from (16), (17) it follows that $X_S = X_U$ and

$$\mathcal{L}_r V(X) = \varkappa X^\alpha - p, \qquad X > X_*, \tag{44}$$

$$\mathcal{L}_r V(X) = 0, \qquad X < X_*, \tag{45}$$

where $X_* := X_S = X_U$. Combining these equations with (23), (25) results in

$$\begin{aligned} \varkappa X^\alpha - p \le \mathcal{L}_r V(X) = 0, \qquad & X < X_*, \\ 0 \le \mathcal{L}_r V(X) = \varkappa X^\alpha - p, \qquad & X > X_*, \end{aligned}$$

therefore

$$X_* = \left(\frac{p}{\varkappa}\right)^{1/\alpha}. \tag{46}$$

Using solutions (60) of equations (44), (45) and applying the value matching and smooth pasting conditions at the threshold (46), one obtains the $C^2(\mathbb{R}_+)$ smooth value function

$$V(X) = p \cdot \begin{cases} A\left(\dfrac{X}{X_*}\right)^\delta, & X \le X_*, \\[2ex] \dfrac{1}{L(\alpha)}\left(\dfrac{X}{X_*}\right)^\alpha - \dfrac{1}{r} + B\left(\dfrac{X}{X_*}\right)^\gamma, & X > X_*, \end{cases}$$

with

$$A = \frac{r(\alpha-\gamma)+\gamma L(\alpha)}{r(\delta-\gamma)L(\alpha)}, \qquad B = \frac{r(\alpha-\delta)+\delta L(\alpha)}{r(\delta-\gamma)L(\alpha)}.$$

The corresponding transitioning policy is the simple one-threshold rule

$$Z_t = H_{X_*}(X_t), \tag{47}$$

where H_{X_*} is the Heaviside step function

$$H_{X_*}(X) = \begin{cases} 0, & X < X_*, \\ 1, & X > X_*. \end{cases} \tag{48}$$

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Accordingly, the consumer is subscribed whenever $X_t > X_*$, unsubscribed whenever $X_t < X_*$, and transitions from state 0 to 1 and vice versa at the same threshold $X_t = X_*$ specified by (46). We note that the threshold X_* doesn't depend on r.

Next, we consider the firm problem stated in the previous section. When all the consumers implement the same transitioning policy (47), the normalization condition (41) implies that the expected firm's pay-off (43) equals

$$W^0(x;p) := \mathbb{E}\left[\int_0^\infty e^{-\rho t}(p-c)H_{X_*}(X_t)dt\right].$$

Effectively, the population of consumers acts as one consumer. Therefore, the function $W^0(\cdot;p) \in C^1(\mathbb{R}_+)$ can be obtained as a solution of the differential equation

$$\mathcal{L}_\rho W^0(x;p) = (p-c)H_{X_*}(x) \tag{49}$$

(cf. (7)) subject to the boundary conditions

$$W^0(0;p) = 0, \qquad \sup_{x\in\mathbb{R}_+} W^0(x;p) < \infty. \tag{50}$$

Using the value matching and smooth pasting conditions at the discontinuity point (46) of the step function gives

$$W^0(x;p) = \frac{1}{\rho}\cdot\begin{cases} W^0_S(\chi;p) := (p-c)\left(1 - \dfrac{\delta'}{\delta'-\gamma'}\left(\dfrac{\chi}{p}\right)^{\gamma'/\alpha}\right), & p < \chi, \\ W^0_U(\chi;p) := -\dfrac{\gamma'(p-c)}{\delta'-\gamma'}\left(\dfrac{\chi}{p}\right)^{\delta'/\alpha}, & p > \chi, \end{cases} \tag{51}$$

where γ', δ' are the roots of the chracteristic polynomial

$$L'(m) := -\frac{1}{2}\sigma^2 m(m-1) - \mu m + \rho$$

satisfying

$$\gamma' < 0 < \alpha < \delta'$$

(cf. (38), (40)), and we use the variable

$$\chi = \varkappa x^\alpha.$$

Proposition 2. *The pay-off (51) of the firm achieves its maximum with respect to p at the point*

$$p^*(\chi) := \arg\max_{p>0} W^0(\cdot;\chi) = \begin{cases} p^*_U, & \chi < p^*_U, \\ p^*_S(\chi), & \chi > p^*_U, \end{cases} \tag{52}$$

where

$$p^*_U = \frac{c\delta'}{\delta'-\alpha'}$$

and $p^*_S(\chi)$ is a unique positive root p of the equation

$$\eta(\chi;p) := 1 - \frac{\delta'(\alpha-\gamma')}{\alpha(\delta'-\gamma')}\left(\frac{\chi}{p}\right)^{\gamma'/\alpha} - \frac{c\delta'\gamma'}{\alpha(\delta'-\gamma')p}\left(\frac{\chi}{p}\right)^{\gamma'/\alpha} = 0. \tag{53}$$

The function (52) satisfies

$$p^*(\chi) > c \ \text{ for all } \ \chi > 0; \tag{54}$$

$$p^*(\chi) < \chi \quad \text{for} \quad \chi > p^*_U; \qquad p^*(\chi) > \chi \quad \text{for} \quad \chi < p^*_U. \tag{55}$$

Proof. By inspection, for a fixed χ,

$$W_S^0(\chi;0) = -c; \qquad W_S^0(\chi;p) \to -\infty \quad \text{as} \quad p \to \infty;$$

$$W_U^0(\chi;p) \to -\infty \quad \text{as} \quad p \to 0; \qquad W_U^0(\chi;p) \to 0 \quad \text{as} \quad p \to \infty,$$

and each of the functions W_S^0, W_U^0 (as functions of $p > 0$) has a unique maximum. Namely,

$$\arg\max_{p>0} W_U^0(\chi;\cdot) = p_U^*$$

doesn't depend on χ; on the other hand,

$$p_S^*(\chi) := \arg\max_{p>0} W_S^0(\chi;\cdot)$$

is a unique positive root p of the equation (53). Therefore, the pay-off achieves its maximum with respect to p at the point (52). By inspection, $\chi > p_U^*$ is equivalent to $\eta(\chi;\chi) < 0$, hence $\chi > p_U^*$ iff $\chi > p_S^*(\chi)$. Also, $p_U^* > c$, hence if $\chi > p_U^*$, then

$$\eta(\chi;c) = 1 - \frac{\delta'}{\delta' - \gamma'}\left(\frac{\chi}{c}\right)^{\gamma'/\alpha} > 0$$

and consequently $c < p_S^*(\chi)$. Therefore, (54) holds and

$$c < p_S^*(\chi) < \chi \quad \text{if} \quad \chi > p_U^*,$$

which combined with $p^*(\chi) = p_U^*$ for $\chi < p_U^*$ implies (55). □

The firm maximizes its pay-off by setting $p = p^*(\chi_0)$ depending on the initial value of the state variable, $X_0 = x$, and the corresponding

$$\chi_0 := \varkappa X_0^\alpha. \tag{56}$$

In particular, if the initial value satisfies

$$\varkappa X_0^\alpha = \chi_0 < p_U^* = \frac{c\delta'}{\delta' - \alpha'}, \tag{57}$$

then the firm sets $p = p_U^*$, i.e. the firm's expected pay-off is (using the variable $\chi = \varkappa X^\alpha$)

$$W^0(\chi;p_U^*) = \frac{1}{\rho}\begin{cases} \dfrac{c\alpha}{\delta' - \alpha}\left(1 - \dfrac{\delta'}{\delta' - \gamma'}\left(\dfrac{\chi(\delta' - \alpha)}{c\delta'}\right)^{\gamma'/\alpha}\right), & \dfrac{c\delta'}{\delta' - \alpha} < \chi, \\ -\dfrac{c\alpha\gamma'}{(\delta' - \alpha)(\delta' - \gamma')}\left(\dfrac{\chi(\delta' - \alpha)}{c\delta'}\right)^{\delta'/\alpha}, & \dfrac{c\delta'}{\delta' - \alpha} > \chi. \end{cases} \tag{58}$$

Since $\chi_0 < p = p_U^*$, all the consumers with $Z_0^i = 1$ unsubscribe at the initial moment in accordance with transitioning policy (47). On the other hand, if

$$\chi_0 > p_U^* = \frac{c\delta'}{\delta' - \alpha},$$

then the firm sets $p = p_S^*(\chi_0)$, and the expected pay-off is

$$W^0(\chi;p_S^*(\chi_0)) = \frac{1}{\rho}\begin{cases} (p_S^*(\chi_0) - c)\left(1 - \dfrac{\delta'}{\delta' - \gamma'}\left(\dfrac{\chi}{p_S^*(\chi_0)}\right)^{\gamma'/\alpha}\right), & p_S^*(\chi_0) < \chi, \\ -\dfrac{\gamma'(p_S^*(\chi_0) - c)}{\delta' - \gamma'}\left(\dfrac{\chi}{p_S^*(\chi_0)}\right)^{\delta'/\alpha}, & p_S^*(\chi_0) > \chi. \end{cases} \tag{59}$$

Since due to (55) from $\chi_0 > p^*_U$ it follows that $\chi_0 > p^*(\chi_0)$, all the consumers with $Z^i_0 = 0$ subscribe at the initial moment.

Remark 2. *If consumers have different flow benefits (i.e. different κ_i or α_i), then they have different thresholds of their transitioning policies according to (47). Hence, the population is no longer represented by one consumer as above, and the solution is more complicated. This more general model will be considered elsewhere.*

3.2. Existence of a Two-Threshold Solution

In this section, we prove the existence of a two-threshold solution to the HJB variational inequality (8)–(13) assuming non-zero enrollment/cancellation rates. Recall that this solution is represented by a solution of system (16)–(26).

Proposition 3. *Let relations (6) and (39) hold. Then, system (16)–(26) has a solution.*

Proof. Applying boundary conditions (18)–(20) to the general solution of Euler's equations (16), (17) and using (40) results in

$$V_S(X) = \frac{\varkappa X^\alpha}{L(\alpha)} - \frac{p}{r} + B_S\, X^\gamma, \quad X > X_U; \qquad V_U(X) = A_U\, X^\delta, \quad X < X_S, \tag{60}$$

with

$$B_S = -\frac{\varkappa\alpha(X_S^\delta X_U^\alpha - X_S^\alpha X_U^\delta)}{\gamma L(\alpha)(X_S^\delta X_U^\gamma - X_S^\gamma X_U^\delta)}, \qquad A_U = -\frac{\varkappa\alpha(X_S^\gamma X_U^\alpha - X_S^\alpha X_U^\gamma)}{\delta L(\alpha)(X_S^\delta X_U^\gamma - X_S^\gamma X_U^\delta)}, \tag{61}$$

where the thresholds satisfy the system

$$\varkappa X_S^{\alpha-\gamma} + \left(\frac{\alpha}{\gamma} - 1\right)(p + r\xi_S)X_S^{-\gamma} = \varkappa X_U^{\alpha-\gamma} + \left(\frac{\alpha}{\gamma} - 1\right)(p - r\xi_U)X_U^{-\gamma}, \tag{62}$$

$$\varkappa X_S^{\alpha-\delta} + \left(\frac{\alpha}{\delta} - 1\right)(p + r\xi_S)X_S^{-\delta} = \varkappa X_U^{\alpha-\delta} + \left(\frac{\alpha}{\delta} - 1\right)(p - r\xi_U)X_U^{-\delta}. \tag{63}$$

According to Proposition 1, it suffices to prove that this system has a solution X_S, X_U satisfying (28), (29). To show this, we define the functions

$$\varphi(x;\xi) = \varkappa x^{\alpha-\gamma} + \left(\frac{\alpha}{\gamma} - 1\right)(p + r\xi)x^{-\gamma},$$

$$\psi(x;\xi) = \varkappa x^{\alpha-\delta} + \left(\frac{\alpha}{\delta} - 1\right)(p + r\xi)x^{-\delta}$$

with $x \geq 0$, and rewrite system (62), (63) equivalently as

$$\varphi(X_S;\xi_S) = \varphi(X_U;-\xi_U), \quad \psi(X_S;\xi_S) = \psi(X_U;-\xi_U).$$

By inspection, the functions $\varphi(\cdot,\xi)$, $\psi(\cdot,\xi)$ satisfy

$$\varphi(0;\xi) = 0; \qquad \varphi(x,\xi) \to \infty \text{ as } x \to \infty; \qquad \frac{\partial\varphi}{\partial\xi}(x,\xi) < 0 \text{ for } x > 0,$$

$$\psi(x;\xi) \to -\infty \text{ as } x \to 0; \qquad \psi(x,\xi) \to 0 \text{ as } x \to \infty; \qquad \frac{\partial\psi}{\partial\xi}(x,\xi) < 0 \text{ for } x > 0.$$

Moreover, $\varphi(\cdot,\xi)$ has a unique local minimum on the positive semiaxis at the point

$$x = x_*(\xi) := \left((p + r\xi)/\varkappa\right)^{1/\alpha};$$

and, $\psi(\cdot,\xi)$ has a unique local maximum on the positive semiaxis at the same point $x = x_*(\xi)$. Therefore, the equation $\varphi(X_S;\xi_S) = \varphi(X_U;-\xi_U)$ defines a unique implicit function

$$X_S = \Phi(X_U) : \left[0, (p - r\xi_U)/\varkappa)^{1/\alpha}\right] \to \left((p + r\xi_S)/\varkappa)^{1/\alpha}, (1 - \alpha/\gamma)(p + r\xi_S)/\varkappa)^{1/\alpha}\right].$$

Substituting this function into the equation $\psi(X_S;\xi_S) = \psi(X_U;-\xi_U)$, one obtains the scalar equation

$$\Psi(X_U) := \psi(\Phi(X_U);\xi_S) - \psi(X_U;-\xi_U) = 0.$$

From

$$\Phi(X_U) \to \Phi(0) = (1 - \alpha/\gamma)(p + r\xi_S)/\varkappa)^{1/\alpha}, \quad \psi(X_U;-\xi_U) \to -\infty \text{ as } X_U \to 0,$$

it follows that

$$\Psi(X_U) \to \infty \text{ as } X_U \to 0.$$

On the other hand, since $x_*(-\xi_U)$ is the maximum point of the function $\psi(\cdot,-\xi_U)$ and ψ decreases in the second variable,

$$\psi(x_*(-\xi_U);-\xi_U) > \psi(\Phi(x_*(-\xi_U));-\xi_U) > \psi\big(\Phi(x_*(-\xi_U));\xi_S\big),$$

hence

$$\Psi\big(x_*(-\xi_U)\big) < 0,$$

and by the intermediate value theorem Ψ has a zero X_U, which proves the existence of the thresholds. □

3.3. Asymptotic Approximation of the Two-Threshold Solution

We now develop an asymptotic expansion of the two-threshold solution assuming that the enrollment/cancellation costs are small compared to the cumulative subscription cost. More precisely, let us assume that

$$\frac{\sigma^2(\xi_U - \xi_S)}{2\alpha p} \ll 1,$$

hence

$$\varepsilon := \frac{\sigma^2(\xi_U + \xi_S)}{2\alpha p} \le \frac{\sigma^2(\xi_U - \xi_S)}{2\alpha p} \ll 1.$$

This dimensionless quantity is used as the small parameter of the asymptotic expansion. The one-threshold solution $X_S = X_U = (p/\varkappa)^{1/\alpha}$ of the cost-free enrollment/cancellation case presented in Section 3.1 serves as a reference corresponding to $\varepsilon = 0$. The following statement uses the dimensionless parameters

$$\lambda := 1 - \frac{2\mu}{\sigma^2} = \delta + \gamma, \qquad \tau = -\frac{2r}{\sigma^2} = \delta\gamma \tag{64}$$

and

$$\theta_S = \frac{\xi_S}{\xi_U + \xi_S}, \qquad \theta_U = \frac{\xi_U}{\xi_U + \xi_S}, \qquad \theta = \theta_U - \theta_S. \tag{65}$$

Proposition 4. *The singular asymptotic expansion of the solution to system (62)–(63) satisfying (28)–(29) has the form*

$$X_S = \left(\frac{p}{\varkappa}\right)^{1/\alpha}\left(1 + \sum_{n=1}^{N} a_n^S \varepsilon^{n/3}\right) + O(\varepsilon^{\frac{N+1}{3}}), \quad X_U = \left(\frac{p}{\varkappa}\right)^{1/\alpha}\left(1 + \sum_{n=1}^{N} a_n^U \varepsilon^{n/3}\right) + O(\varepsilon^{\frac{N+1}{3}}), \tag{66}$$

where

$$a_1^S = -a_1^U = \sqrt[3]{\frac{3}{2}}, \tag{67}$$
$$a_2^S = a_2^U = \frac{3-\alpha+2\lambda}{2\sqrt[3]{12}}, \tag{68}$$
$$a_3^S = \frac{1}{60}\Big(15 - 15\alpha + 2\alpha^2 + (30-8\alpha)\lambda + 2\lambda^2 - (12-30\theta)\tau\Big), \tag{69}$$
$$a_3^U = -a_3^S + \theta\tau, \tag{70}$$
$$a_4^S = \frac{1}{240\sqrt[3]{18}}\Big(45 - 90\alpha + 39\alpha^2 - 2\alpha^3 + (180 - 156\alpha + 24\alpha^2)\lambda + (84-36\alpha)\lambda^2 - 8\lambda^3 - 24(6 - 3\alpha - 3\lambda - (15-5\alpha)\theta)\tau\Big), \tag{71}$$
$$a_4^U = a_4^S - \frac{(3-\alpha)\theta\tau}{\sqrt[3]{18}}, \tag{72}$$
$$a_5^S = \frac{1}{25200\sqrt[3]{12}}\Big(945 - 3150\alpha + 2835\alpha^2 - 630\alpha^3 - 16\alpha^4 + (6300 - 11340\alpha + 5040\alpha^2 - 352\alpha^3)\lambda + 24(315 - 315\alpha + 53\alpha^2)\lambda^2 - 352\alpha\lambda^3 - 16\lambda^4 - 12\Big(630 - 840\alpha + 204\alpha^2 - (1575 - 1575\alpha + 350\alpha^2)\theta - (210 + 96\alpha + (1050-700\alpha)\theta)\lambda + 24\lambda^2\Big)\tau + 1944\tau^2\Big), \tag{73}$$
$$a_5^U = -a_5^S + \frac{(3-2\alpha)(3-\alpha+2\lambda)\theta\tau}{6\sqrt[3]{12}}. \tag{74}$$

Proof. The above asymptotic expansions of $X_{S,U}$ are obtained in a standard way by substituting equations (66) into system (62)–(63), expanding the resulting equations with respect to ε as

$$\sum_{i=1}^{N} b_n^k \varepsilon^{n/3} + O(\varepsilon^{\frac{N+1}{3}}) = 0, \quad k = 1, 2,$$

then equating each coefficient $b_n^k = b_n^k(a_1^S, a_U^1, \ldots, a_N^S, a_N^U)$ of these expansion to zero and solving the resulting linear system for $a_n^{S,U}$ for $n = 1, \ldots, N$ with $N = 5$. □

Remark 3. *According to Proposition 4, the second order approximations of $X_{S,U}$ doesn't depend on the discount rate r, the third and fourth order approximations are linear in r, and the higher order approximations starting with order five are non-linear with respect to r. The role of this nonlinearity is discussed in Section 4.*

3.4. Firm Problem: One Consumer

Next, we revisit the firm problem posed in Section 2.4 assuming large coefficients $\kappa_{U,S}$ in the firm pay-off (43). The assumption $\kappa_{U,S} \gg 1$ warrants $\varepsilon \ll 1$, i.e. small optimal enrollment/cancellation costs, letting us use the asymptotic expansions developed in Proposition 4 with the cost-free benchmark (see Section 3.1).

We begin with the firm problem with one consumer who implements the two-threshold strategy. The expected pay-off of the firm satisfies the equations

$$\mathcal{L}_\rho W_S(X) = p - c + \rho(\kappa_S \xi_S - \kappa_U \xi_U), \qquad X > X_U, \tag{75}$$
$$\mathcal{L}_\rho W_U(X) = \rho(\kappa_S \xi_S - \kappa_U \xi_U), \qquad X < X_S, \tag{76}$$

where the subscripts S, U correspond to the state of the consumer. These equations are combined with the boundary conditions

$$W_U(0) = \kappa_S \xi_S - \kappa_U \xi_U, \qquad \sup_{X \in \mathbb{R}_+} W_S(X) < \infty \tag{77}$$

 doi:10.20944/preprints202510.0157.v1

at zero and infinity, resulting in the relations

$$W_S(X) = \frac{p-c}{\rho} + \hat{B}_S X^\gamma + \kappa_S \xi_S - \kappa_U \xi_U, \tag{78}$$

$$W_U(X) = \hat{A}_U X^\delta + \kappa_S \xi_S - \kappa_U \xi_U \tag{79}$$

(cf. (49)–(50)), where the coefficients are determined by the value matching relations at the thresholds:

$$\frac{p-c}{\rho} + \hat{B}_S X_S^\gamma = \hat{A}_U X_S^\delta, \qquad \frac{p-c}{\rho} + \hat{B}_S X_U^\gamma = \hat{A}_U X_U^\delta. \tag{80}$$

Recall that the consumer sets the thresholds $X_{S,U}$ to maximize her pay-off depending on flow rate p and enrollment/cancellation costs $\xi_{S,U}$, hence the coefficients $\hat{A}_U$, $\hat{B}_S$ and the firm's pay-off $W_{S,U}$ are functions of p, ξ_S, ξ_U. As such, the firm's value function $\hat{W}_{S,U}$ is defined by

$$\hat{W}_{S,U}(X_0) = \sup_{p>0,\ \xi_S \le 0 < \xi_U,\ \xi_U + \xi_S > 0} W_{S,U}(X_0; p, \xi_S, \xi_U).$$

Using the expansions for the thresholds $X_{S,U}$ developed in Proposition 4, one obtains

$$W_S(\chi) = \frac{p-c}{\rho} + \frac{p-c}{\rho(\delta'-\gamma')}\left(\frac{\chi}{p}\right)^{\gamma'/\alpha} \sum_{n=0}^{3} c_n^S \varepsilon^{n/3} + \frac{2p\alpha(\kappa_S\theta_S - \kappa_U\theta_U)\varepsilon}{\sigma^2} + O(\varepsilon^{4/3}), \tag{81}$$

$$W_U(\chi) = \frac{p-c}{\rho(\delta'-\gamma')}\left(\frac{\chi}{p}\right)^{\delta'/\alpha} \sum_{n=0}^{3} c_n^S \varepsilon^{n/3} + \frac{2p\alpha(\kappa_S\theta_S - \kappa_U\theta_U)\varepsilon}{\sigma^2} + O(\varepsilon^{4/3}) \tag{82}$$

with

$$c_0^S = -\delta', \ c_0^U = -\gamma', \ c_1^S = c_1^U = 0, \ c_2^S = \frac{\rho(\alpha - 3\gamma')}{\sigma^2\sqrt[3]{12}}, \quad c_2^U = \frac{\rho(\alpha - 3\delta')}{\sigma^2\sqrt[3]{12}}, \quad c_3^S = c_3^U = \frac{2\rho r\theta}{\sigma^4},$$

where parameters (65) and the variable $\chi = \varkappa X^\alpha$ are used. This expansions lead to the following asymptotics of the firm's value function and parameters of the optimal strategy.

Proposition 5.

(i) *If $\chi_0 < p_U^* = c\delta'/(\delta' - \alpha)$, then $\hat{W}(\chi) = W^0(\chi; p_U^*)$ for $\kappa_{S,U} \gg 1$ (cf. (56)–(58)). The firm maximizes its expected pay-off by setting $p = p_U^*$ and $\xi_S = \xi_U = 0$, i.e. zero enrollment/cancellation costs, which leads to the single-threshold behavior of the consumer with the threshold $X_{S,U} = (p_U^*/\varkappa)^{1/\alpha}$; the initial state of the consumer is "unsubscribed".*

(ii) *If $\chi_0 > p_U^*$, then the firm's value function satisfies*

$$\hat{W}(\chi_0) = W_S^0(\chi_0; p_S^*(\chi_0)) + \frac{\alpha p_S^*(\chi_0) k_o^3}{\sigma^2}\kappa_U^{-2} + \frac{6\alpha p_S^*(\chi_0) r k_o^4 \sqrt[3]{12}}{(\alpha - 3\gamma')\sigma^4}\kappa_U^{-3} + O(\kappa_U^{-4}) \tag{83}$$

for $\kappa_{S,U} \gg 1$ (cf. (59)), where

$$k_o = k_o(\chi_0) := \frac{(\alpha - 3\gamma')(p_S^*(\chi_0) - c)}{3\delta'\sqrt[3]{12}((\alpha - \gamma')p_S^*(\chi_0) + \gamma' c)}. \tag{84}$$

The firm maximizes its expected pay-off by setting

$$\xi_S = 0, \tag{85}$$

i.e. zero enrollment cost, while the optimal flow rate and cancellation cost satisfy

$$p = p_S^*(\chi_0) + O(\kappa_U^{-2}), \qquad \xi_U = \frac{2\alpha p_S^*(\chi_0) k_0^3}{\sigma^2}\kappa_U^{-3} + O(\kappa_U^{-4}), \tag{86}$$

leading to the two-threshold behavior of the consumer with the initial state "subscribed". Moreover, setting

$$p = p_S^*(\chi_0), \qquad \xi_S = 0, \qquad \xi_U = \frac{2\alpha p_S^*(\chi_0) k_0^3}{\sigma^2}\kappa_U^{-3} \tag{87}$$

results in the asymptotically the same firm's pay-off as in (83), i.e.

$$W_S(\chi_0) = W_S^0(\chi_0; p_S^*(\chi_0)) + \frac{\alpha p_S^*(\chi_0) k_o^3}{\sigma^2}\kappa_U^{-2} + \frac{6\alpha p_S^*(\chi_0) r k_o^4 \sqrt[3]{12}}{(\alpha - 3\gamma')\sigma^4}\kappa_U^{-3} + O(\kappa_U^{-4}). \tag{88}$$

Proof. Since the coefficients of expansion (82) satisfy $c_1^U = 0$, $c_2^U < 0$, the firm's pay-off W_U decreases in ε near $\varepsilon = 0$ in the case $\chi_0 < p_U^*$. Hence, the firm maximizes its pay-off by setting $\varepsilon = 0$, which corresponds to $\xi_S = \xi_U = 0$, i.e. forgoing both enrollment and cancellation costs as in Proposition 2. This proves **(i)**.

On the other hand, $c_1^S = 0$, $c_2^S > 0$ according to (81), hence the pay-off W_S increases in ε near $\varepsilon = 0$. Therefore, in the case $\chi_0 > p_U^*$ (i.e. under the conditions of part **(ii)**), W_S achieves its maximum at a point where

$$\frac{\partial W_S}{\partial \varepsilon} = \frac{p-c}{\rho(\delta' - \gamma')}\left(\frac{\chi}{p}\right)^{\gamma'/\alpha}\left(\frac{2\rho(\alpha - 3\gamma')}{3\sigma^2\sqrt[3]{12}}\varepsilon^{-1/3} + \frac{2\rho r(\theta_U - \theta_S)}{\sigma^4}\right) + \frac{2p\alpha(\kappa_S\theta_S - \kappa_U\theta_U)}{\sigma^2} + O(\varepsilon^{1/3}) = 0.$$

Since $\kappa_S, \kappa_U \gg 1$, this implies

$$\sqrt[3]{\varepsilon} = \frac{(\alpha - 3\gamma')(p-c)}{3\alpha p\sqrt[3]{12}(\delta' - \gamma')(\kappa_U\theta_U - \kappa_S\theta_S)}\left(\frac{\chi}{p}\right)^{\gamma'/\alpha} + \frac{r(\alpha - 3\gamma')(p-c)^2(\theta_U - \theta_S)}{3\sigma^2\alpha^2 p^2\sqrt[3]{12}(\delta' - \gamma')^2(\kappa_U\theta_U - \kappa_S\theta_S)^2}\left(\frac{\chi}{p}\right)^{2\gamma'/\alpha} + O\big((\kappa_U\theta_U - \kappa_S\theta_S)^{-3}\big). \tag{89}$$

By inspection, for

$$\chi = \chi_0, \quad p = p_S^*(\chi_0),$$

this expansion combined with $\eta(\chi_0; p_S^*(\chi_0)) = 0$ implies

$$\varepsilon = k_o^3(\kappa_U\theta_U - \kappa_S\theta_S)^{-3} + O\big((\kappa_U\theta_U - \kappa_S\theta_S)^{-4}\big) \tag{90}$$

with k_o defined by (84). Moreover, differentiating W_S with respect to θ_S and taking into account that $\theta_U = 1 - \theta_S$, $\theta = 1 - 2\theta_S$, one obtains

$$\begin{aligned}\frac{\partial W_S}{\theta_S} &= \left(-\frac{4r\big(p_S^*(\chi_0) - c\big)}{(\delta' - \gamma')\sigma^4}\left(\frac{\chi_0}{p_S^*(\chi_0)}\right)^{\gamma'/\alpha} + \frac{2\alpha p_S^*(\chi_0)(\kappa_S + \kappa_U)}{\sigma^2}\right)\varepsilon + O(\varepsilon^{4/3}) \\ &= \frac{2\alpha p_S^*(\chi_0)}{\sigma^2}\left(\frac{-2r\big(p_S^*(\chi_0) - c\big)}{\sigma^2\delta'\big((\alpha - \gamma')p_S^*(\chi_0) + \gamma' c\big)} + \kappa_S + \kappa_U\right)\varepsilon + O(\varepsilon^{4/3}),\end{aligned}$$

where

$$\frac{-2r\big(p_S^*(\chi_0) - c\big)}{\sigma^2\delta'\big((\alpha - \gamma)p_S^*(\chi_0) + \gamma' c\big)} + \kappa_S + \kappa_U > \kappa_S + \kappa_U + \frac{2r}{\sigma^2\delta'\gamma'} = \kappa_S + \kappa_U - \frac{r}{\rho}.$$

Therefore, for sufficiently large κ_S, κ_U, the firm's pay-off increases with θ_S, hence $\theta_S \le 0$ implies that the pay-off is maximized by setting $\theta_S = 0$, which corresponds to

$$\xi_S = 0, \quad \xi_U = \frac{2\alpha p_S^*(\chi_0)\varepsilon}{\sigma^2},$$

where due to (90) and $\theta_S = 0$,

$$\varepsilon = k_o^3 \kappa_U^{-3} + O(\kappa_U^{-4}) < -\frac{\kappa_U^{-3}}{324}\left(\frac{\alpha - 3\gamma'}{\delta'\gamma'}\right)^3 = \frac{\kappa_U^{-3}}{324}\left(\frac{\sigma^2(\alpha - 3\gamma')}{2\rho}\right)^3. \tag{91}$$

Now, we check *a posteriori* that the accuracy used for the approximation of p is sufficient to obtain the expansion (83) of the value function. Indeed, by inspection, differentiating (81) with respect to p, using (53) and setting $\varepsilon = O(\kappa_U^{-3})$ gives

$$\frac{\partial W_S}{\partial p} = \frac{\eta(\chi; p)}{\rho} + O(\kappa_U^{-2}).$$

Hence, $\eta(\chi; p_S^*(\chi)) = 0$ implies that the root of the equation $\partial W_S(\chi_0; p)/\partial p = 0$ satisfies $p = p_S^*(\chi_0) + O(\kappa_U^{-2})$. Moreover, the small correction $O(\kappa_U^{-2})$ doesn't affect the asymptotic expansion (91) for ϵ obtained above from $\partial W_S/\partial \varepsilon = 0$ using $p = p_S^*(\chi_0)$; neither it affects the estimate $\partial W_S/\partial \theta_S > 0$ that was established for small ε. Therefore, the optimal parameters satisfy (85)–(86), and the corresponding value of the pay-off can be obtained with the accuracy $O(\kappa_U^{-4})$ by substituting

$$\chi = \chi_0, \quad p = p_S^*(\chi_0), \quad \theta_S = 0, \quad \theta = \theta_U = 1, \quad \varepsilon = k_o^3 \kappa_U^{-3} \tag{92}$$

in (81), which (using $\eta(\chi_0; p_S^*(\chi_0)) = 0$) results in (83) and (88). Finally, parameters (92) correspond to (87). □

Remark 4. *Formulas (89) and (91) combined lead to the next order approximation for ε, namely*

$$\varepsilon = k_o^3 \kappa_U^{-3}\left(1 + \frac{9r\sqrt[3]{12}}{\sigma^2(\alpha - 3\gamma')} k_o \kappa_U^{-1}\right) + O(\kappa_U^{-5}),$$

which can be used to obtain the next order terms in expansions (83) and (86) of the value function and optimal parameters, respectively.

Remark 5. *Proposition 5 doesn't address initial values χ_0 from a small interval of length $O(\kappa_U^{-1})$ centered at p_U^*. An optimal strategy aiming at an increase of the pay-off of the order $o(\kappa_U^{-2})$ over the cost-free enrollment/cancellation benchmark (cf. (58), (59) and (83)) in this more subtle case will be considered elsewhere.*

3.5. Firm Problem: Two Consumers

Continuing the case study, let us consider the firm's problem with two consumers, each optimizing the enrollment/cancellation thresholds $X^i_{S,U}$, $i = 1, 2$, of her two-threshold policy in response to the flow rate p and the enrollment/cancellation costs $\xi_{S,U}$ set by the firm. Consumers' discount rates satisfy

$$0 < r_1 < r_2.$$

Using the notation of Section 2.4, the firm's flow benefit from the i-th subscription is $\beta_i(p - c)$ (i.e., when the i-th consumer's state is "subscribed"), where

$$\beta_1 = \beta \ge 0, \qquad \beta_2 = 1 - \beta \ge 0$$

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due to (41). The system has at most 4 states, and the firm's expected pay-off has the form

$$W_{SS}(X) = \frac{p-c}{\rho} + \hat{B}_{SS}X^{\gamma} + \kappa_S\xi_S - \kappa_U\xi_U, \tag{93}$$

$$W_{UU}(X) = \hat{A}_{UU}X^{\delta} + \kappa_S\xi_S - \kappa_U\xi_U, \tag{94}$$

$$W_{SU}(X) = \frac{\beta(p-c)}{\rho} + \hat{A}_{SU}X^{\delta} + \hat{B}_{SU}X^{\gamma} + \kappa_S\xi_S - \kappa_U\xi_U, \tag{95}$$

$$W_{US}(X) = \frac{(1-\beta)(p-c)}{\rho} + \hat{A}_{US}X^{\delta} + \hat{B}_{US}X^{\gamma} + \kappa_S\xi_S - \kappa_U\xi_U, \tag{96}$$

where the subscript US means that the first consumer is in state "subscribed", while the second consumer is in state "unsubscribed"; the other subscripts have simila meaning; and, the coefficients are determined by the value matching relations at the thresholds (cf. (78), (79)). These value matching conditions depend on the ordering of the thresholds. In particular, if either $X_U^2 < X_S^2 \le X_U^1 < X_S^1$ or $X_U^2 < X_U^1 < X_S^2 < X_S^1$, then there are 3 states, UU, US, SS, and the value matching conditions have the form

$$W_{UU}(X_U^2) = W_{US}(X_U^2);\; W_{UU}(X_S^2) = W_{US}(X_S^2);\; W_{US}(X_U^1) = W_{SS}(X_U^1);\; W_{US}(X_S^1) = W_{SS}(X_S^1). \tag{97}$$

If $X_U^2 < X_U^1 < X_S^1 < X_S^2$, then there are 4 states, UU, SU, US, SS, and the value matching conditions are

$$W_{UU}(X_U^1) = W_{SU}(X_U^1);\;\; W_{UU}(X_S^1) = W_{SU}(X_S^1);\;\; W_{SU}(X_S^2) = W_{SS}(X_S^2);$$
$$W_{US}(X_U^1) = W_{SS}(X_U^1);\;\; W_{US}(X_S^1) = W_{SS}(X_S^1);\;\; W_{UU}(X_U^2) = W_{US}(X_U^2).$$

Due to $r_1 < r_2$, it follows from equations (64)–(70) that for any sufficiently small ε,

$$X_U^2 < X_U^1 < X_S^2 < X_S^1,$$

hence there are 3 states, UU, US and SS.

Let us introduce a consumer with the averaged discount rate

$$r_{2-\beta} = \beta r_1 + (1-\beta)r_2 \tag{98}$$

(i.e. $\beta = 1$ corresponds to r_1 and $\beta = 0$ corresponds to r_2), and consider the firm problem with this one consumer (called the averaged consumer), i.e. the problem discussed in Section 3.4 with $r = r_{2-\beta}$. Denote by $W_{S,U}^{2-\beta}$ the firm's pay-off for this problem (cf. (78)–(80)). We now compare $W_{S,U}^{2-\beta}$ with the firm's pay-off in the market with two consumers.

Proposition 6. *The firm's pay-off in the market with two consumers, which is defined by (93)–(97), satisfies*

$$W_{SS}(\chi) = W_S^{2-\beta}(X) + \frac{d^S(p-c)(r_2-r_1)^2}{\rho(\delta'-\gamma')}\left(\frac{\chi}{p}\right)^{\gamma'/\alpha}(1-\beta)\beta\varepsilon^2 + O(\varepsilon^{7/3}), \tag{99}$$

$$W_{UU}(X) = W_U^{2-\beta}(X) + \frac{d^U(p-c)(r_2-r_1)^2}{\rho(\delta'-\gamma')}\left(\frac{\chi}{p}\right)^{\delta'/\alpha}(1-\beta)\beta\varepsilon^2 + O(\varepsilon^{7/3}), \tag{100}$$

where

$$d^S = \frac{525(1+\gamma')\theta^2 + 246 + 164\delta' - 82\gamma'}{525\sigma^6}, \qquad d^U = \frac{525(1+\delta')\theta^2 + 246 + 164\gamma' - 82\delta'}{525\sigma^6}.$$

Proof. Expansions (99), (100) are obtained by substituting the expansions for the thresholds $X_{S,U}$ developed in Proposition 4 into relations (78)–(80) and (93)–(97). The coefficients $a_1^{S,U}$, $a_2^{S,U}$ of expansions (66) don't depend on consumer's discount rate r, the coefficients $a_3^{S,U}$, $a_4^{S,U}$ depend on r linearly (the

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parameter τ is proportional to r), and the coefficients $a_5^{S,U}$ are quadratic in r (cf. (67)–(74)). The latter nonlinearity induces the discrepancy between W_{UU} and $W_U^{2-\beta}$ of order $O(\varepsilon^2)$ (and between W_{SS} and $W_S^{2-\beta}$, respectively). We omit further details of this direct computation. □

Remark 6. *Comparing the expressions for the firm's pay-off, (78)–(80) and (93)–(97), one obtains*

$$\begin{aligned} W_{SS}(X) =& \beta W_S^1(X) + (1-\beta)W_S^2(X), \\ W_{US}(X) =& \beta W_U^1(X) + (1-\beta)W_S^2(X), \\ W_{UU}(X) =& \beta W_U^1(X) + (1-\beta)W_U^2(X). \end{aligned}$$

An immediate corollary of Propositions 5 and 6 is the following statement.

Proposition 7.

(i) *If $\chi_0 < p_U^* = c\delta'/(\delta' - \alpha)$, then the value function of the firm's problem with two consumers equals $W^0(\chi; p_U^*)$ for $\kappa_{S,U} \gg 1$ (cf. (56)–(58)). The firm maximizes its expected pay-off by setting $p = p_U^*$ and $\xi_S = \xi_U = 0$, while both consumers adopt the single-threshold behavior with the threshold $X_{S,U} = (p_U^*/\varkappa)^{1/\alpha}$; the initial state of the consumers is "unsubscribed".*

(ii) *If $\chi_0 > p_U^*$, then value function of the firm problem with two consumers satisfies equation (83) with $r = r_{2-\beta}$ for $\kappa_{S,U} \gg 1$. The firm maximizes its expected pay-off by setting $\xi_S = 0$, while the optimal flow rate and cancellation cost satisfy (86), leading to the two-threshold behavior of both consumers, each adopting the initial state "subscribed". Moreover, parameters (87) ensure that the firm's pay-off satisfies (88), i.e. it is asymptotically close to the value function.*

3.6. Firm Problem with N Consumers

Adopting the notation of Section 2.1 in the setting of Section 2.5, the states of the market with N consumers are N-tuples $Z = (Z^1, \ldots, Z^N) \in \{0,1\}^N$. Let $W_Z(X)$ denote the firm's expected pay-off corresponding to the state Z. As such, $W_1 = W_S$, $W_0 = W_U$ using the notation of Section 3.4, and $W_{(1,1)} = W_{SS}$, $W_{(0,1)} = W_{US}$, $W_{(0,0)} = W_{UU}$ using the notation of Section 3.5. Since the firm's flow benefit from all the subscriptions is given by (42), the firm's expected pay-off is the weighted sum of the expected pay-offs from individual consumers (cf. Remark 6 for the case of two consumers). More precisely, the following statement holds.

Proposition 8. *The firm's expected pay-off in the market with N consumers equals*

$$W_Z(X) = \sum_{i=1}^{N} \beta_i W_{Z^i}^i(X), \qquad Z = (Z^1, \ldots, Z^N), \tag{101}$$

where W_z^i is the firm's pay-off in the market with one i-th consumer, i.e.

$$W_1^i(X) = \frac{(p-c)(1 + B_i X^{\gamma_i})}{\rho} + \kappa_S \xi_S - \kappa_U \xi_U, \tag{102}$$

$$W_0^i(X) = \frac{(p-c)A_i X^{\delta_i}}{\rho} + \kappa_S \xi_S - \kappa_U \xi_U \tag{103}$$

(cf. (78)–(80)) with

$$A_i = \frac{(X_U^i)^{\gamma_i} - (X_S^i)^{\gamma_i}}{(X_S^i)^{\delta_i}(X_U^i)^{\gamma_i} - (X_S^i)^{\gamma_i}(X_U^i)^{\delta_i}}, \qquad B_i = \frac{(X_U^i)^{\delta_i} - (X_S^i)^{\delta_i}}{(X_S^i)^{\delta_i}(X_U^i)^{\gamma_i} - (X_S^i)^{\gamma_i}(X_U^i)^{\delta_i}}, \tag{104}$$

where $X_{S,U}^i$ are the consumer's thresholds.

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Proof. By inspection, functions (102), (103) solve equations (75), (76), respectively, subject to conditions (77) at zero and at infinity. Also, relations (104) warrant that these functions satisfy the value matching conditions at the thresholds $W_1^i(X_S^i) = W_0^i(X_S^i)$, $W_1^i(X_U^i) = W_0^i(X_U^i)$ of the i-the consumer (cf. (80)). Therefore, for every state Z, function (101) is the solution of the equation

$$\mathcal{L}_\rho W_Z(X) = (p-c)\sum_{i=1}^{N} \beta_i Z^i + \rho(\kappa_S \xi_S - \kappa_U \xi_U)$$

satisfying the same conditions at zero and infinity. Moreover, the value matching conditions $W_Z(X_S^i) = W_{\bar{Z}}^i(X_S^i)$, $W_Z(X_U^i) = W_{\bar{Z}}(X_U^i)$ are satisfied at each pair of states $Z = (Z^1, \dots, Z^N)$, $\bar{Z} = (\bar{Z}^1, \dots, \bar{Z}^N)$ such that $Z^i = 1 - \bar{Z}^i$ and $Z^n = \bar{Z}^n$ for $n \neq i$. □

Recall that the Preisach operator maps continuous inputs X_t to states Z_t (see Section 2.5). According to Proposition 8, the firm's expected pay-off is the function of the state defined by equations (101).

4. Discussion

To summarize a few results obtained above, according to the proposed model, the service provider (firm) increases its expected pay-off by relinquishing any enrollment stimulus for consumers. On the other hand, if the initial consumer's utility from the service is sufficiently high, then the firm's optimal strategy involves setting a non-zero cancellation penalty (see Proposition 5).

The value function of the firm operating in a heterogeneous market where consumers have two distinct discount rates, $r_{1,2}$, can be approximated by the value function of the firm operating in a homogeneous market where all the consumers have the averaged discount rate $r_{2-\beta}$ defined by (98) (see Proposition 7). Moreover, Proposition 6 implies that this approximation underestimates the firm's value function if

$$771 + 164\delta' + 443\gamma' > 0,$$

and overestimates it if the opposite inequality holds (cf. equation (99)).

Equation (81) implies that

$$\frac{\partial W_S}{\partial r} = \frac{2\theta(p-c)}{\sigma^4(\delta'-\gamma')}\left(\frac{X}{p}\right)^{\gamma'/\alpha}\varepsilon + O(\varepsilon^{4/3}) = \frac{(\xi_U-\xi_S)(p-c)}{\alpha\sigma^2(\delta'-\gamma')p}\left(\frac{X}{p}\right)^{\gamma'/\alpha} + O(\varepsilon^{4/3}) > 0.$$

Larger discount rate r is interpreted as higher consumer's impatience. Hence the firm's value function is higher in a market of more impatient consumers or a heterogeneous market with a larger proportion of such consumers. Moreover,

$$\frac{\partial^2 W_S}{\partial r \partial \xi_U} = \frac{p-c}{\alpha\sigma^2(\delta'-\gamma')p}\left(\frac{X}{p}\right)^{\gamma'/\alpha} + O(\varepsilon^{4/3}) > 0,$$

$$\frac{\partial^2 W_S}{\partial r \partial p} = \frac{\xi_U-\xi_S}{\alpha\sigma^2(\delta'-\gamma')p}\left(\frac{X}{p}\right)^{\gamma'/\alpha}\left(1-\left(1-\frac{c}{p}\right)\left(1+\frac{\gamma}{\alpha}\right)\right) + O(\varepsilon^{4/3}) > 0,$$

therefore the sensitivity of the value function to the controls p and ξ_U is also higher in a market with more impatient consumers.

As an example, for the set of parameters $\rho = r_1 = 0.3, r_2 = 0.9$, $\mu = 0$, $\sigma^2 = 0.1$, $\alpha = 1$, $\varkappa = 1$, $p = 100$, $\xi_S = 0$, $\xi_U = 10$, the thresholds of the two consumers are $X_S^1 = 121.26$, $X_U^1 = 80.84$, $X_S^2 = 119.18$, $X_U^2 = 76.88$, and the relative error of the approximation of order $N = 5$ provided by Proposition 4 is 10^{-4}. Assuming $c = 20$ and $X_0 = 150$, the firm's expected pay-off $W_{SS}(X_0)$ increases from $W_{SS}(X_0) = W_S^1(X_0) = 203.89$ to $W_{SS}(X_0) = W_S^2(X_0) = 208.98$ as β varies from 1 to 0. The relative error of the approximation provided by Proposition 6 is also 10^{-4}.

The main premise for the above conclusions is the assumption that the cancellation cost, ξ_U, is sufficiently small compared to the expected total subscription cost, p/r.

5. Conclusions

We showed that the solution to a model of a subscription economy with heterogeneous households naturally contains an equivalent of the Preisach operator. In particular, in the case of non-zero enrollment/cancellation costs, the household's optimal strategy is a two-threshold strategy, and the corresponding firm's expected pay-off is a weighted sum of the expected pay-offs from individual households. Using the ratio of the enrollment/cancellation cost and the expected total subscription cost as a small parameter, we obtained a perturbation expansion of the optimal solution, which relates the parameters of the Preisach operator (i.e., the set of thresholds) with parameters of the households' utility flow (such as the distribution of discount rates) and controls (i.e., enrollment/cancellation and flow costs of subscription imposed by the firm). Within the model constraints, we showed that the firm increases its expected pay-off by relinquishing any enrollment stimulus. On the other hand, if the initial households' flow benefits are sufficiently high, then the firm's optimal strategy involves a (small) non-zero cancellation penalty. The perturbation expansions show that a heterogeneous market can be approximated, within an accuracy range, by an averaged representative household, i.e. a homogeneous market. However, the Preisach operator provides a more accurate solution. The firm's pay-off increases with the increasing proportion of impatient households (i.e., having higher discount rate). Moreover, the sensitivity of the value function to the controls is higher in a more impatient market. Our analysis is limited to the heterogeneity due to a distribution of households' discount rates. Similar analysis can be applied to other types of heterogeneous utility that households derive from a subscription (e.g., $\varkappa$, α in (4)).

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