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Posted Date: 2 October 2025

doi: 10.20944/preprints202510.0104.v1

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Article

On the Semigroup Properties of Fractional Calculus with Respect to a Power Functions and Solutions to Nonlocal Problems in Fractional Differential Equations

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Abstract

This paper investigates the uniform continuity and strong continuity of the semigroups of the fractional integral operators of power functions. Using the Krasnoselskii's fixed point theorem, we have studied the nonlocal problem related to fractional differential equations involving power functions with multi-point integral boundary conditions and obtain the existence of the solution.

Keywords: fractional calculus with respect to a power function; fractional differential equations; nonlocal conditions; strongly continuous semigroup; Krasnoselskii's fixed point

MSC: 26A33; 34L30; 34K10

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1. Introduction

Fractional derivative originated from the initial discussion between L'Hospital and Leibnitz in 1695, but it did not attract enough attention at that time, and was considered a paradox for a long time. Many researchers have cited fractional calculus as the most useful in characterizing materials and processes with memory genetic properties up until 2000s. Until recent decades, many researchers pointed out that fractional calculus is the most effective in characterizing materials and processes with memory genetic properties. For example, the transport of chemical pollutants around rocks through water, viscoelastic material dynamics, cell diffusion process, network flow, and etc. Fractional-order equations can be more accurate than integer-order differential equation while describing the physical change process (cf.[1-3]). As a branch of calculus theory, fractional differential equations have been developed in both theory and application (cf.[4-10]), especially in the modeling abnormal phenomena [11]. There are many forms of fractional calculus, such as Riemann–Liouville, Caputo, and Hadamard fractional calculus. In [12], Erdelyi defined also fractional integration with respect to x^n for any nonzero real n. Recently, a generalized derivative has been considered in [13,14] by Katugampola, which unifies the Riemann-Liouville and Hadamard integrals into a single form. [15] presents the existence and uniqueness results for the solutions to initial value problems of the fractional differential equation with respect to a power function of order $0 < \alpha < 1$.

Usually, initial and boundary conditions cannot describe some information of physical or other processes happening inside the whole area. In order to cope with this situation, Nonlocal conditions are found to be more valuable in modelling many physical change processes and others (cf.[16-21, 23-24]). In [17], by the use of some fixed point index theory on cone, Bai obtain the existence of positive solutions for the equation

$$D^{\alpha}_{0+}x(t) = f(t, x(t))$$

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by employing a fixed-point index theory on the cone with nonlocal boundary value conditions

$$x(0) = 0,\ x(1) = \beta x(\eta),$$

where $0 < \alpha < 1, 0 < \eta < 1, 0 < \beta\eta^{\alpha-1} < 1$, D^{α}_{0+} is the Riemann–Liouville fractional differential operator. N′Guerekata considers the solution to the above problem when the boundary condition becomes

$$x(0) + g(x) = x_0$$

in a Banach space [18], He proved that if f is a jointly continuous function and g is a Lipschitzian function, then the problem has a unique solution. Deng's paper indicated that the above nonlocal condition is better than the initial condition $x(0) = x_0$ in physics [20].

Recently, Ahmad et al. [22] obtained the uniqueness of solutions for boundary value problem

$$\begin{cases} ({}^{\rho}D^{\alpha}_{0+}u)(t) = f(t, u(t)),\ 0 \le t \le T, \\ u(0) = 0,\ \int_0^T u(s)dH(s) = \lambda({}^{\rho}I^{\beta}u)(\xi),\ \xi \in (0, T), \end{cases}$$

where ${}^{\rho}D^{\alpha}_{0+}$ is the fractional differential operator with respect to a power function of order $1 < \alpha \le 2$, ${}^{\rho}I^{\beta}_{0+}$ is the fractional differential operator with respect to a power functions of order β, $\int_0^T u(s)dH(s)$ is the Stieltjes integral with respect to the function H, H is a bounded variation function on $[0, T]$.

In 2015, Chatthai et al. [21] considered the existence and uniqueness of solutions for a problem consisting of nonlinear Langevin equation of Riemann-Liouville type fractional derivatives with the nonlocal Katugampola fractional integral conditions

$$x(0) = 0, \quad x(\eta) = \sum_{i=1}^{n} \alpha_i^{\rho_i} I^{q_i} x(\xi_i).$$

In this paper, we initiate the study of nonlocal boundary value problems of generalized fractional differential equations supplemented with generalized fractional integral boundary conditions

$$({}^{\rho}D^{\alpha}_{0+}u)(t) = f(t, u(t)),\ 0 \le t \le 1, \tag{1.1}$$

$$u(0) = 0,\ u(1) = \sum_{i=1}^{n} k_i({}^{\rho}I^{\alpha_i}_{0+}u)(\eta_i). \tag{1.2}$$

where $\rho > 0$, $J = [0, 1]$, $f \in C(J \times \mathbb{R}, \mathbb{R}) \cap X^p_\nu(0, 1)$, $\nu \in \mathbb{R}, 1 \le p < \infty, 1 < \alpha \le 2$ is a real number, ${}^{\rho}D^{\alpha}_{0+}$ is criterion fractional differential operator with respect to a power functions of order α, ${}^{\rho}I^{\alpha_i}_{0+}$ is the fractional integral with respect to a power functions of order α_i, $\alpha_i > 0$, $\eta_i \in (0, 1)$, and $k_i \in \mathbb{R}$, $i = 1, 2, ..., n$ are real constants such that

$$\sum_{i=1}^{n} k_i \frac{\Gamma(\alpha)}{\rho^{\alpha_i}\Gamma(\alpha + \alpha_i)} \eta_i^{\rho(\alpha+\alpha_i-1)} \ne 1.$$

The rest of the paper is organized as follows. In Section 2, we describe the necessary background material related to our problem, prove operator semigroup ${}^{\rho}I^{\alpha}_{a+}$ uniform continuous and strongly continuous, proves an auxiliary lemma. Section 3 contains the main results on the existence of solutions to nonlocal problems. To demonstrate the validity of the Theorems, Section 4 presents three examples.

2. Preliminaries

In this section, let us review the definitions and certain related theorems regarding the fractional calculus of a function with respect to power functions, and give some lemma which are helpful in next

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section. In [1], Samko et al. provided the definitions of fractional integrals of a function f with respect to another function g on $[a,b]$,

$$(I^{\alpha}_{a+;g}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x \frac{g'(y)}{(g(x)-g(y))^{1-\alpha}} f(y)dy, \ \ (x>a;\ \alpha>0),$$

where g be an increasing and positive monotone function on $(a,b]$, having a continuous derivative g' on (a,b), Γ is the gamma function defined by $\Gamma(x) = \int_0^{\infty} e^{-s}s^{x-1}ds$.

For $\nu \in \mathbb{R}$, $1 \le p < \infty$. Let $X^p_\nu(a,b)$ denote the space of all Lebesgue measurable functions $f:(a,b) \to \mathbb{R}$ for which $\|f\|_{X^p_\nu} < \infty$, where the norm is defined by

$$\|f\|_{X^p_\nu} = \left(\int_a^b |t^\nu f(t)|^p \frac{dt}{t}\right)^{\frac{1}{p}},\ 1 \le p < \infty.$$

$$\|f\|_{X^p_\nu} = ess \sup_{t\in(a,b)} |t^\nu |f(t)||,\ p=\infty.$$

In particular, when $\nu = \frac{1}{p}$, the space $X^p_\nu(a,b) = L_p(a,b)$.

In the above definition of the fractional integral of a function with respect to another function, when select $g(x) = \frac{x^\rho}{\rho}$, we can obtain the following definitions of generalized fractional differential and fractional integral.

Definition 2.1. Let $\rho > 0$, $\alpha > 0$. $-\infty < a < x < b < \infty$, and $f \in X^p_\nu(a,b)$. The fractional integral operator ${}^\rho I^\alpha_{a+}$ with respect to a power function $g(x)$ of order α is defined by

$$({}^\rho I^\alpha_{a+}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^x \frac{y^{\rho-1}}{(x^\rho-y^\rho)^{1-\alpha}} f(y)dy. \tag{2.1}$$

This integral is called the left-sided fractional integral. The right-sided fractional integral ${}^\rho I^\alpha_{b-}$ is defined by

$$({}^\rho I^\alpha_{b-}f)(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_x^b \frac{y^{\rho-1}}{(y^\rho-x^\rho)^{1-\alpha}} f(y)dy. \tag{2.2}$$

Definition 2.2. Let $\rho > 0$, $\alpha > 0$, $n = [\alpha]+1$. $0 \le a < x < b \le \infty$, and $g \in X^p_\nu(a,b)$. The left-sided fractional derivatives operator ${}^\rho D^\alpha_{a+}$ with respect to power function $g(x)$ and right-sided fractional derivatives operator ${}^\rho D^\alpha_{b-}$ with respect to power function $g(x)$ are defined by

$$\begin{aligned}({}^\rho D^\alpha_{a+}g)(x) &= \left(x^{1-\rho}\frac{d}{dx}\right)^n ({}^\rho I^{n-\alpha}_{a+}g)(x) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(x^{1-\rho}\frac{d}{dx}\right)^n \int_a^x \frac{y^{\rho-1}}{(x^\rho-y^\rho)^{\alpha-n+1}} g(y)dy, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned}({}^\rho D^\alpha_{b-}g)(x) &= \left(-x^{1-\rho}\frac{d}{dx}\right)^n ({}^\rho I^{n-\alpha}_{b-}g)(x) \\ &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\left(-x^{1-\rho}\frac{d}{dx}\right)^n \int_x^b \frac{y^{\rho-1}}{(y^\rho-x^\rho)^{\alpha-n+1}} g(y)dy. \end{aligned} \tag{2.4}$$

The properties and related theorems concerning generalized fractional differential operators and generalized fractional integral operators were introduced by Katugampola in 2014 [13].

The generalized differential operators depend on parameter ρ compared with classical fractional derivatives, most of the characteristics of generalized fractional derivatives depend on the value of ρ

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[13]. Infact, we have $({}^{\rho}D^{\alpha}_{a+}f)(x) \to ({}^{L}D^{\alpha}_{a+}f)(x)(\rho \to 1)$, where ${}^{L}D^{\alpha}_{a+}$ is Riemann–Liouville fractional differential operator, and $({}^{\rho}D^{\alpha}_{a+}f)(x) \to ({}^{H}D^{\alpha}_{a+}f)(x)(\rho \to 0^{+})$, ${}^{H}D^{\alpha}_{a+}$ is Hadamard differential fractional operator.

From the definition of the generalized fractional integral operator and by direct computation with respect to z^{λ}, we can find the following proposition.

Proposition 2.3([14, Example 2.10]). Let $\alpha > 0$ and $\rho \in \mathbb{R}$. We have

$$ {}^{\rho}I^{\alpha}_{0+}z^{\lambda} = \frac{\Gamma(\frac{\lambda}{\rho}+1)}{\rho^{\alpha}\Gamma(\frac{\lambda}{\rho}+\alpha+1)}z^{\alpha\rho+\lambda}. \tag{2.5} $$

where $\lambda \in \mathbb{C}$.

Let $AC[0,1]$ be the space of absolutely continuous function on $[0,1]$. In addition, the space $AC^{2}_{\rho}[0,1]$ consists of those functions g that have absolutely continuous $x^{1-\rho}\frac{d}{dx}$ derivative.

$$ AC^{2}_{\rho}[0,1] = \left\{ g : [0,1] \to \mathbb{R} : (x^{1-\rho}\frac{d}{dx})g(x) \in AC[0,1] \right\}. $$

There are the following conclusions regarding the simple properties of the generalized differential operators [13, 14]. Unless otherwise stated, we suppose throughout that $\rho > 0$ and $\alpha > 0$. For $\rho > 0$, $\alpha > 0$ and $h \in X^{p}_{\nu}(a,b)$, we have

$$ ({}^{\rho}D^{\alpha}_{a+}\,{}^{\rho}I^{\alpha}_{a+})h(t) = h(t). \tag{2.6} $$

In particular, the solution of differential equation

$$ {}^{\rho}D^{\alpha}_{a+}f(x) = 0 $$

has the form

$$ f(x) = \sum_{i=1}^{n} a_i \left(\frac{x^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1}, \tag{2.7} $$

where $n = [\alpha] + 1$, $a_i, i = 1, \cdots, n$ are real constants.

For $\rho > 0$, $1 < \alpha \le 2$, $\nu \in \mathbb{R}$ and $1 \le p \le \infty$. If $h \in X^{p}_{\nu}(0,1)$ and ${}^{\rho}I^{2-\alpha}_{0+}h \in AC^{2}_{\rho}[0,1]$, then we have

$$ ({}^{\rho}I^{\alpha}_{0+}\,{}^{\rho}D^{\alpha}_{0+}h)(x) = h(x) + a_1x^{\rho(\alpha-1)} + a_2x^{\rho(\alpha-2)}. $$

On the other hand, we can estimate the $\|{}^{\rho}I^{\alpha}_{a}\|_{X^{p}_{\nu}}$. In [14]. For $0 < a < b < \infty$, $\rho > 0$ and $\nu \in \mathbb{R}$ such that $\rho - 1 \ge \nu$. For any $h \in X^{p}_{\nu}(a,b)$, we have

$$ \|{}^{\rho}I^{\alpha}_{a}h\|_{X^{p}_{\nu}} \le M_0\|h\|_{X^{p}_{\nu}}, \tag{2.8} $$

where

$$ M_0 = \frac{b^{\alpha\rho-1}}{\rho^{\alpha-1}\Gamma(\alpha)} \int_{1}^{\frac{b}{a}} x^{\nu-\alpha\rho-1}(x^{\rho}-1)^{\alpha-1}dx. $$

In order to prove Theorem 2.5 we need the following Theorem 2.4, which is a fundamental result of the fractional integration operator ${}^{\rho}I^{\alpha}_{a+}$ [13].

Theorem 2.4. Let $\alpha, \beta \in \mathbb{C}$, $1 \le p < \infty$, $0 < a < b < \infty$ and let $\rho > 0$, $\nu \in \mathbb{R}$. Then for $f \in X^{p}_{\nu}(a,b)$, the semigroup property holds,

$$ {}^{\rho}I^{\alpha}_{a+}\,{}^{\rho}I^{\beta}_{a+}f = {}^{\rho}I^{\alpha+\beta}_{a+}f. \tag{2.9} $$

For all $g, h \in X^p_\nu(a,b)$,

$$ {}^\rho I^\alpha_{a+}(c_1 g + c_2 h) = c_1\, {}^\rho I^\alpha_{a+} g + c_2\, {}^\rho I^\alpha_{a+} h, \tag{2.10} $$

where c_1, c_2 are arbitrary constants.

Theorem 2.5. If $\alpha > 0$, $p \geq 1$ such that $\min(\rho, \alpha + \frac{1}{p}) \geq \nu + 1$, then the fractional integration operator ${}^\rho I^\alpha_{a+}$ is a uniform continuous semigroup in $X^p_\nu(a,b)$ and which is strongly continuous for all $\alpha \geq 0$.

Proof. By (2.8) and (2.10), ${}^\rho I^\alpha_{a+}$ is the boundary linear operator in $X^p_\nu(a,b)$. Let $\alpha_0 > 0$, we have

$$ \begin{aligned} {}^\rho I^\alpha_{a+} f - {}^\rho I^{\alpha_0}_{a+} f &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s) ds - \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha_0}} f(s) ds \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \left(\frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} - \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha_0}} \right) f(s) ds \\ &\quad + \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\rho^{1-\alpha_0}}{\Gamma(\alpha_0)} \right) \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha_0}} f(s) ds \\ &= \mathcal{P} f + \mathcal{L} f. \end{aligned} $$

First, let us estimate operator norm $\|\mathcal{P} f\|_{X^p_\nu}$ and $\|\mathcal{L} f\|_{X^p_\nu}$. In view of (2.8)

$$ \|\mathcal{L} f\|_{X^p_\nu} \leq M_0 \left| \frac{\Gamma(\alpha_0)}{\Gamma(\alpha)} \rho^{-\alpha+\alpha_0} - 1 \right| \|f\|_{X^p_\nu}. \tag{2.11} $$

Next, since $f(s) \in X^p_\nu$, then $s^{\nu - \frac{1}{p}} f(s) \in L^p(a,b)$, we have

$$ \begin{aligned} \mathcal{P} f &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} |1 - (t^\rho - s^\rho)^{\alpha_0 - \alpha}| s^{\rho-1} f(s) ds \\ &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1} |1 - x^{\alpha_0 - \alpha}| f((t^\rho - x)^{\frac{1}{\rho}}) dx. \end{aligned} $$

Consequently, apply the generalized Minkowski inequality

$$ \begin{aligned} \|\mathcal{P} f\|_{X^p_\nu} &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1} |1 - x^{\alpha_0-\alpha}| x^{\frac{1}{p} - \nu} dx \left(\int_a^b |t^\nu f((t^\rho - x)^{\frac{1}{\rho}})|^p \frac{dt}{t} \right)^{\frac{1}{p}} \\ &\leq \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha-1} |1 - x^{\alpha_0-\alpha}| x^{\frac{1}{p} - \nu} dx \|f\|_{X^p_\nu}. \end{aligned} \tag{2.12} $$

Combining (2.11) and (2.12), we can obtain

$$ \frac{\|({}^\rho I^\alpha_{a+} - {}^\rho I^{\alpha_0}_{a+}) f\|_{X^p_\nu}}{\|f\|_{X^p_\nu}} \leq M_0 \left| \frac{\Gamma(\alpha_0)}{\Gamma(\alpha)} \rho^{-\alpha+\alpha_0} - 1 \right| + \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \int_0^{b^\rho - a^\rho} x^{\alpha - 1 + \frac{1}{p} - \nu} |1 - x^{\alpha_0 - \alpha}| dx. $$

letting $\alpha \to \alpha_0$, take into account that $\Gamma(\alpha)$ is continuous for $\alpha > 0$ and $\Gamma(\alpha) \neq 0$, it follows that

$$ \lim_{\alpha \to \alpha_0} \|{}^\rho I^\alpha_{a+} - {}^\rho I^{\alpha_0}_{a+}\|_{X^p_\nu} = 0. $$

Let $\alpha_0 = 0$, define identity integration operator ${}^\rho I^0_{a+} f = f$. Let us prove that

$$ \lim_{\alpha \to 0} \|{}^\rho I^\alpha_{a+} f - f\|_{X^p_\nu} = 0. \tag{2.13} $$

We have

$$
\begin{aligned}
{}^{\rho}I^{\alpha}_{a+}f - f &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds - f(t) \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds - \frac{\alpha\rho}{(t^\rho - a^\rho)^{\alpha}}\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(t)ds \\
&\leq \left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\alpha\rho}{(b^\rho - a^\rho)^{\alpha}}\right)\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} |f(s) - f(t)|ds \\
&\leq 2\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\alpha\rho}{(b^\rho - a^\rho)^{\alpha}}\right)\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} |f(s)|ds.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\|{}^{\rho}I^{\alpha}_{a+}f - f\|_{X^p_\nu} &\leq 2\left(\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} - \frac{\alpha\rho}{(b^\rho - a^\rho)^{\alpha}}\right)\left(\int_a^b t^{\nu p}\left|\int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}|f(s)|ds\right|^p \frac{dt}{t}\right)^{\frac{1}{p}} \\
&= 2\mathcal{Q}(\alpha)\mathcal{N}(f).
\end{aligned}
$$

Apply the generalized Minkowski inequality in the right-hand side integral

$$
\begin{aligned}
\mathcal{N}(f) &= \left(\int_a^b \left|\int_a^t t^{\nu-\frac{1}{p}} \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}|f(s)|ds\right|^p dt\right)^{\frac{1}{p}} \\
&= \left(\int_a^b \left|\int_1^{\frac{t}{a}} t^{\nu-\frac{1}{p}} (\frac{t}{u})^{\alpha\rho-1}(u^\rho - 1)^{\alpha-1}|f(\frac{t}{u})|\frac{t}{u^2}du\right|^p dt\right)^{\frac{1}{p}} \\
&\leq \int_1^{\frac{b}{a}} \frac{(u^\rho - 1)^{\alpha-1}}{u^{\alpha\rho+1}} b^{\alpha\rho}\left(\int_{ua}^b t^{\nu-\frac{1}{p}}|f(\frac{t}{u})|^p \frac{dt}{t}\right)^{\frac{1}{p}} du. \\
&\leq \int_1^{\frac{b}{a}} b^{\alpha\rho}(u^\rho - 1)^{\alpha-1}u^{\nu-\alpha\rho-1}du\|f\|_{X^p_\nu}.
\end{aligned}
$$

By the Lebesgue-dominated convergence theorem, we obtain

$$
\int_1^{\frac{b}{a}} b^{\alpha\rho}(u^\rho - 1)^{\alpha-1}u^{\nu-\alpha\rho-1}du\|f\|_{X^p_\nu} < \infty,
$$

And, we have

$$
\lim_{\alpha\to 0}\mathcal{N}(f) = 0. \tag{2.14}
$$

Since $\Gamma(\alpha)$ is a continuous function for $\alpha > 0$, and $\Gamma(\alpha) \to +\infty$ when $\alpha \to 0^+$, therefore we have $\lim_{\alpha\to 0}\mathcal{Q}(\alpha) = 0$. Combining the above argument, (2.13) is hold, which completes the estimation and the proof. □

Remark 2.6. When $\nu = \frac{1}{p}$, operator ${}^{\rho}I^{\alpha}_{a+}$ is a semigroup in $L_p(a,b)$. This is the same as standard Riemann-Liouville fractional integration operator I^α. (see [7]).

Lemma 2.7. Assume $\{T_s|s > 0\}$ is a strongly continuous operator semigroup in Banach space X, $0 < a < b < +\infty$, then exist constant $M_1 > 0$ such that

$$
\|T_s\| \leq M_1, \ \ s \in [a,b].
$$

Proposition 2.8. Let $\alpha_i > 0, 1 < \alpha \leq 2$ and $\rho > 0$, $\eta_i \in (0,1)$, $k_i \in \mathbb{R}$, $i = 1,2,...,n$. Let

$$\Delta = 1 - \sum_{i=1}^{n} k_i \frac{\Gamma(\alpha)}{\rho^{\alpha_i}\Gamma(\alpha+\alpha_i)} \eta_i^{\rho(\alpha+\alpha_i-1)} \neq 0.$$

For any $\varphi \in C(0,1)$ and $x \in C([0,1],\mathbb{R})$ with ${}^{\rho}I_{0+}^{2-\alpha}x \in AC_{\rho}^{2}[0,1]$, Then the function x is the solution of nonlocal fractional differential equation boundary-value problem

$$\begin{cases} ({}^{\rho}D_{0+}^{\alpha}x)(t) = \varphi(t), \; 0 < t < 1, \\ x(0) = 0, \; x(1) = \sum\limits_{i=1}^{n} k_i({}^{\rho}I_{0+}^{\alpha_i}x)(\eta_i), \end{cases} \tag{2.15}$$

if and only if

$$x(t) = {}^{\rho}I_{0+}^{\alpha}\varphi(t) + \frac{1}{\Delta}\left(\sum_{i=1}^{n} k_i\, {}^{\rho}I_{0+}^{\alpha+\alpha_i}\varphi(\eta_i) - {}^{\rho}I_{0+}^{\alpha}\varphi(1)\right)t^{\rho(\alpha-1)}. \tag{2.16}$$

Proof. Applying the operator ${}^{\rho}I_{0+}^{\alpha}$ on the linear differential equation (2.15), we have

$${}^{\rho}I_{0+}^{\alpha}({}^{\rho}D_{0+}^{\alpha}x)(t) = ({}^{\rho}I_{0+}^{\alpha}\varphi)(t).$$

Using (2.6) and (2.7), we can obtain

$$x(t) = ({}^{\rho}I_{0+}^{\alpha}\varphi)(t) + c_1 t^{\rho(\alpha-1)} + c_2 t^{\rho(\alpha-2)}, \tag{2.17}$$

where $c_1, c_2 \in \mathbb{R}$. The condition $x(0) = 0$ implies that $c_2 = 0$. Applying the fractional integral operator with respect to a power function $\frac{x^{\rho}}{\rho}$ of order $\alpha_i > 0$ on (2.17) after inserting $c_2 = 0$ in it, and use (2.7), we get

$${}^{\rho}I_{0+}^{\alpha_i}x(t) = {}^{\rho}I_{0+}^{\alpha+\alpha_i}\varphi(t) + c_1 \frac{\Gamma(\alpha)}{\rho^{\alpha_i}\Gamma(\alpha+\alpha_i)} t^{\rho(\alpha+\alpha_i-1)}.$$

which, together with the second condition $x(1) = \sum\limits_{i=1}^{n} k_i({}^{\rho}I_{0+}^{\alpha_i}x)(\eta_i)$, we have

$${}^{\rho}I_{0+}^{\alpha_i}\varphi(1) + c_1 = \sum_{i=1}^{n} k_i({}^{\rho}I_{0+}^{\alpha+\alpha_i}\varphi)(\eta_i) + c_1 \sum_{i=1}^{n} k_i \frac{\Gamma(\alpha)}{\rho^{\alpha_i}\Gamma(\alpha+\alpha_i)} \eta_i^{\rho(\alpha+\alpha_i-1)}.$$

Thus,

$$c_1 = \frac{1}{\Delta}\left(\sum_{i=1}^{n} k_i({}^{\rho}I_{0+}^{\alpha+\alpha_i}\varphi)(\eta_i) - {}^{\rho}I_{0+}^{\alpha_i}\varphi(1)\right).$$

Substituting c_1, c_2 into (2.17), we obtain the solution (2.16). Conversely, it can easily be shown by direct computation that the integral equation (2.16) satisfies the boundary value problem (2.15). □

To prove the main theorems of Section 3, we need the following well-known fixed point theorem [27].

Theorem 2.9. Let E be a nonempty, closed, convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : E \to X$ be two operators such that
(a) A is a contraction,
(b) B is completely continuous, and
(c) $x = Ax + By$ for all $y \in E \Rightarrow x \in E$.
Then the operator equation $Ax + Bx = x$ has a solution in E.

3. Main Results

Let $1 < \alpha \le 2, \rho > 0, C^\alpha(J,\mathbb{R}) = \{u \in C([0,1],\mathbb{R}) : {}^\rho I_{0+}^{2-\alpha} u \in AC[0,1]\}$. For $u \in C^\alpha(J,\mathbb{R})$, define the norm $\|u\|_{C^\alpha} = \sup_{t\in[0,1]} |u(t)|$. When $\rho(2-\alpha) > 1$, $C^\alpha(J,\mathbb{R})$ is a Banach space.

Lemma 3.1. The space $C^\alpha(J,\mathbb{R})$ is a Banach space.

Proof. Set $J = [0,1]$. Let $1 < \alpha \le 2$ and $\rho > 0$ such that $\rho(2-\alpha) > 1$. Given a Cauchy sequence $\{u_n\}$ in $C^\alpha(J,\mathbb{R})$, then $\{{}^\rho I_{0+}^{2-\alpha} u_n\}$ is a Cauchy sequence in $AC[0,1]$. Since $AC[0,1]$ is complete, there exist a function $u \in AC[0,1]$ such that

$$ {}^\rho I_{0+}^{2-\alpha} u_n \to u (n \to \infty). $$

Assume function $u_0 \in AC[0,1]$ such that

$$ u_n \to u_0 (n \to \infty). $$

We will prove that

$$ u_0 \in C^\alpha(J,\mathbb{R}), \qquad {}^\rho I_{0+}^{2-\alpha} u_0 = u. \tag{3.1} $$

In order to prove $u_0 \in C^\alpha(J,\mathbb{R})$, we need to prove ${}^\rho I_{0+}^{2-\alpha} u_0(x) \in AC[0,1]$. Since u_0 is a continuous function on $[0,1]$, there exists a constant K_α such that $\|u_0\|_{L_p} \le K_\alpha$ for all $x \in [0,1]$. For any $\varepsilon > 0$, we take $\delta = \min\{1, \frac{\rho^{1-\alpha}\Gamma(2-\alpha)\varepsilon}{4K_\alpha}\}$, $\{(t_i, s_i) : i = 1, ..., n\}$ is any finite collection of mutually disjoint subintervals of $[0,1]$, such that $\Sigma_{i=1}^n |s_i - t_i| < \delta$ holds.

Based on the above results, we have

$$
\begin{aligned}
\sum_{i=1}^n |{}^\rho I_{0+}^{2-\alpha} u_0(s_i) - {}^\rho I_{0+}^{2-\alpha} u_0(t_i)| &= \sum_{i=1}^n \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \left| \int_0^{s_i} \frac{t^{\rho-1} u_0(t)}{(s_i^\rho - t^\rho)^{\alpha-1}} dt - \int_0^{t_i} \frac{t^{\rho-1} u_0(t)}{(t_i^\rho - t^\rho)^{\alpha-1}} dt \right| \\
&\le \sum_{i=1}^n \frac{\rho^{\alpha-2}}{\Gamma(2-\alpha)} \left(\int_0^{s_i^\rho} v^{1-\alpha} |u_0((s_i^\rho - v)^{\frac{1}{\rho}}) - u_0((t_i^\rho - v)^{\frac{1}{\rho}})| dv \right. \\
&\quad \left. + \int_{t_i^\rho}^{s_i^\rho} v^{1-\alpha} |u_0((t_i^\rho - v)^{\frac{1}{\rho}})| dv \right) \\
&\le \sum_{i=1}^n \frac{4\|u_0\|_{L_p}}{(2-\alpha)\rho^{2-\alpha}\Gamma(2-\alpha)} |s_i^{\rho(2-\alpha)} - t_i^{\rho(2-\alpha)}| \\
&\le \frac{4\|u_0\|_{L_p}}{\rho^{1-\alpha}\Gamma(2-\alpha)} \sum_{i=1}^n |s_i - t_i| < \varepsilon.
\end{aligned}
$$

Which implies that ${}^\rho I_{0+}^{2-\alpha} u_0(x) \in AC[0,1]$.

Furthermore, since $u_n - u_0 \to 0 (n \to \infty)$, exists $N \in \mathbb{N}$ such that for $n > N$,

$$ \|u_n - u_0\| < \frac{\varepsilon}{K_{\alpha-1}}, $$

where $K_{\alpha-1} = \frac{1}{(2-\alpha)\rho^{2-\alpha}\Gamma(2-\alpha)}$. Which yields that $\|{}^\rho I_{0+}^{2-\alpha} u_n(x) - {}^\rho I_{0+}^{2-\alpha} u_0(x)\| \to 0 (n \to \infty)$ in $C[0,1]$. Consequently, we have

$$ {}^\rho I_{0+}^{2-\alpha} u_0 = \lim_{n\to\infty} {}^\rho I_{0+}^{2-\alpha} u_n = u. $$

Thus (3.1) is valid. Since ${}^\rho I_{0+}^{2-\alpha} u_n \to {}^\rho I_{0+}^{2-\alpha} u_0$ in $AC[0,1]$. It shows that $u_n \to u_0$ in $C^\alpha(J,\mathbb{R})$. Therefore, $C^\alpha(J,\mathbb{R})$ is a Banach space. □

Theorem 3.2. Let $1 < \alpha \le 2$. Assume $f \in C(J \times \mathbb{R}, \mathbb{R}) \cap X_\nu^p(0,1)$, and the following conditions

 doi:10.20944/preprints202510.0104.v1

hold:
(H_1): exist constant $0 < \theta_1 < 1$ and $\theta \in X_\nu^p((0,1),[0,\infty))$ such that

$$\sup_{0\le t\le 1}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}2\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}\theta(s)ds \le 1-\theta_1, \tag{3.2}$$

and

$$|f(t,x_1) - f(t,x_2)| \le \theta(t)|x_1 - x_2|,\ \ \forall t\in J,\ x_1,x_2\in\mathbb{R}. \tag{3.3}$$

(H_2): $\alpha_i > 0$, $\eta_i \in (0,1)$, and $k_i \in \mathbb{R}$, $i = 1,2,...,n$ satisfy

$$0 < r_0 := \frac{\rho^{1-\alpha}}{|\Delta|}\left(\sum_{i=1}^{n+1}\frac{|k_i|}{\rho^{\alpha_i}\Gamma(\alpha+\alpha_i)}\int_0^{\eta_i}\frac{s^{\rho-1}}{(\eta_i^\rho - s^\rho)^{1-\alpha-\alpha_i}}\theta(s)ds\right) < 1, \tag{3.4}$$

where $\alpha_{n+1} = 0$, $k_{n+1} = \eta_{n+1} = 1$.

Then problem (1.1)-(1.2) has at least one solution.

Proof. Let $1 < \alpha \le 2, \rho > 0$, set $f_1 = \max_{0\le s\le 1}|f(s,u(1))| < \infty$. For $\lambda > \frac{f_1}{\rho^\alpha\Gamma(\alpha+1)}$, define the space S by

$$S = \{u \in C^\alpha(J,\mathbb{R}) : \|u(t)\| \le \lambda\}.$$

Define an operator $\mathcal{T}$ on S as follows

$$\mathcal{T}u(t) = {}^\rho I_{0+}^\alpha f(t,u(t)) + \frac{1}{\Delta}\left(\sum_{i=1}^n k_i\, {}^\rho I_{0+}^{\alpha+\alpha_i} f(\eta_i,u(\eta_i)) - {}^\rho I_{0+}^\alpha f(1,u(1))\right)t^{\rho(\alpha-1)}. \tag{3.5}$$

Let

$$\mathcal{T}_1u(t) = {}^\rho I_{0+}^\alpha f(t,u(t)).$$

$$\mathcal{T}_2u(t) = \frac{1}{\Delta}\left(\sum_{i=1}^n k_i\, {}^\rho I_{0+}^{\alpha+\alpha_i} f(\eta_i,u(\eta_i)) - {}^\rho I_{0+}^\alpha f(1,u(1))\right)t^{\rho(\alpha-1)}.$$

It is clear that $u(t)$ is a solution of (1.1) if it is a fixed point of the operator $\mathcal{T}$. Then, we will prove $\mathcal{T}_1$ is a completely continuous operator and $\mathcal{T}_2$ is a contractor operator. For $u(t) \in S$,

$$\begin{aligned}
\mathcal{T}_1u(t) &\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}|f(s,u(s))|ds \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}\Big(|f(s,u(s)) - f(s,u(1))| + |f(s,u(1))|\Big)ds \\
&\le \lambda(1-\theta_1) + \frac{1}{\rho^\alpha\Gamma(\alpha+1)}f_1 \\
&< \lambda.
\end{aligned} \tag{3.6}$$

which implies that $\mathcal{T}_1 S \subset S$. In order to show that the operator $\mathcal{T}_1$ is continuous, for any $u_n, u_0 \in S, n = 1,2,\cdots$ with $u_n \to u_0 (n\to\infty)$, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
|(\mathcal{T}_1u_n)(t) - (\mathcal{T}_1u_0)(t)| &\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}|f(s,u_n(s)) - f(s,u_0(s))|ds \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}\theta(s)|u_n(s) - u_0(s)|ds \\
&\to 0 (n\to\infty),
\end{aligned} \tag{3.7}$$

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Next, we prove that $\mathcal{T}_1 S$ is equicontinuous. Let $t_1, t_2 \in J$, $t_1 > t_2$. For given $\varepsilon > 0$, we take

$$\delta = \min\left\{1, \left(\frac{\varepsilon\rho^{\alpha}\Gamma(\alpha+1)}{N_0 2^{\alpha\rho+1}}\right)^{\frac{1}{\alpha\rho}}\right\},$$

where $N_0 = \left(\int_0^1 |f(s,u(s))|^p ds\right)^{\frac{1}{p}} < \infty$. Then, when $|t_1 - t_2| < \delta$, for each $u \in S$, we will get

$$\begin{aligned}
|(\mathcal{T}_1 u)(t_1) - (\mathcal{T}_1 u)(t_2)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left|\int_0^{t_1} \frac{s^{\rho-1}}{(t_1^{\rho} - s^{\rho})^{1-\alpha}} f(s,u(s))ds - \int_0^{t_2} \frac{s^{\rho-1}}{(t_2^{\rho} - s^{\rho})^{1-\alpha}} f(s,u(s))ds\right| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2} \left(\frac{s^{\rho-1}}{(t_1^{\rho} - s^{\rho})^{1-\alpha}} - \frac{s^{\rho-1}}{(t_2^{\rho} - s^{\rho})^{1-\alpha}}\right)|f(s,u(s))|ds \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{t_2}^{t_1} \frac{s^{\rho-1}}{(t_1^{\rho} - s^{\rho})^{1-\alpha}}|f(s,u(s))|ds \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}$$

For $\mathcal{I}_1$, Consider the function $g(x) = \frac{x^{\rho-1}}{(t_1^{\rho}-x^{\rho})^{1-\alpha}} - \frac{x^{\rho-1}}{(t_2^{\rho}-x^{\rho})^{1-\alpha}}$, $x \in [0, t_2)$, we can obtain that $\|g\|_{L_1} < \frac{1}{\alpha\rho}(t_1^{\alpha\rho} - t_2^{\alpha\rho})$. Case 1: Let $\delta \leq t_2 < t_1 < 1$, then $t_1^{\alpha\rho} - t_2^{\alpha\rho} \leq \alpha\rho\delta^{\alpha\rho}$. Case 2: Let $0 < t_2 < \delta$, $t_1 < 2\delta$, then $t_1^{\alpha\rho} - t_2^{\alpha\rho} < (2\delta)^{\alpha\rho}$. Combining above two cases, $\|g\|_{L_1} < \frac{1}{\alpha\rho}(2\delta)^{\alpha\rho}$. Apply the generalized Minkowski's inequality

$$\begin{aligned}
\mathcal{I}_1 &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2} \left(\frac{s^{\rho-1}}{(t_1^{\rho} - s^{\rho})^{1-\alpha}} - \frac{s^{\rho-1}}{(t_2^{\rho} - s^{\rho})^{1-\alpha}}\right)\|f\|_{L_p} ds \\
&< \frac{2^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}|t_1 - t_2|^{\alpha\rho}\|f\|_{L_p},
\end{aligned} \tag{3.8}$$

and similarly,

$$\begin{aligned}
\mathcal{I}_2 &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1-t_2} \frac{(x+t_2)^{\rho-1}}{(t_1^{\rho} - (x+t_2)^{\rho})^{1-\alpha}}\|f\|_{L_p} dx \\
&\leq \frac{2^{\alpha\rho}}{\rho^{\alpha}\Gamma(\alpha+1)}|t_1 - t_2|^{\alpha\rho}\|f\|_{L_p}.
\end{aligned} \tag{3.9}$$

Consequently, together with (3.8) and (3.9) give

$$|(\mathcal{T}_1 u)(t_1) - (\mathcal{T}_1 u)(t_2)| < \varepsilon. \tag{3.10}$$

Therefore, $\mathcal{T}_1$ is a completely continuous operator.

Finally, we show that $\mathcal{P}_2$ is contractive operator. For any $u, v \in S$,

$$\begin{aligned}
|(\mathcal{T}_2 u)(t) - (\mathcal{T}_2 v)(t)| &\leq \frac{1}{|\Delta|}\left(\sum_{i=1}^{n} k_i^{\rho} I_{0+}^{\alpha+\alpha_i}|f(\eta_i, u(\eta_i)) - f(\eta_i, v(\eta_i))| + {}^{\rho} I_{0+}^{\alpha}|f(1,u(1)) - f(1,v(1))|\right) \\
&\leq \frac{1}{|\Delta|}\left(\sum_{i=1}^{n} k_i^{\rho} I_{0+}^{\alpha+\alpha_i}\theta(\eta_i)|u(\eta_i) - v(\eta_i)| + {}^{\rho} I_{0+}^{\alpha}\theta(1)|u(1) - v(1)|\right) \\
&\leq \frac{1}{|\Delta|}\left(\sum_{i=1}^{n} |\lambda_i|\, {}^{\rho} I_{0+}^{\alpha+\alpha_i}\theta(\eta_i) + {}^{\rho} I_{0+}^{\alpha}\theta(1)\right)\|u - v\|_{C^{\alpha}} \\
&< r_0\|u - v\|_{C^{\alpha}}.
\end{aligned}$$

Which implies that $\mathcal{T}_2$ is a contraction by using (H_2). Thus, according to Theorem 2.9, there exists a $u \in S$ such that $u = \mathcal{T}_1 u + \mathcal{T}_2 u$, So operator $\mathcal{T}$ has a fixed point implies that the problem (1.1) has at least one solution on $[0, 1]$. □

Remark 3.3. If $\theta(t) = L$ is a constant, then condition (3.2) reduces to

$$L < \rho^{\alpha}\Gamma(\alpha+1),$$

where L and λ satisfies $L\lambda > f_1$.

Remark 3.4. In the case that the generalized fractional integral boundary condition reduces to

$$u(1) = k\int_0^{\eta} u(s)ds, \quad \eta \in (0,1).$$

Then, the value Δ is found to be

$$\Delta = 1 - k\frac{\eta^{\rho(\alpha-1)+1}}{\rho(\alpha-1)+1},$$

(3.5) modifies the form

$$\mathcal{T}u(t) = {}^{\rho}I^{\alpha}_{0+}f(t,u(t)) + \frac{1}{\Delta}\left(k\int_0^{\eta}({}^{\rho}I^{\alpha}_{0+}f(t,u(t)))dt - {}^{\rho}I^{\alpha}_{0+}f(1,u(1))\right)t^{\rho(\alpha-1)}. \tag{3.11}$$

Then, we consider the existence of a solution for the differential equation (1.1) with boundary condition

$$u(0) = 0, \ \ ({}^{\rho}I^{\beta}_{0+}u)(\zeta) = \sum_{i=1}^{n}\lambda_i u(\zeta_i), \tag{3.12}$$

Where $\beta > 0, 1 < \alpha \le 2$ and $\rho > 0, 0 < \zeta \le 1, \zeta_i \in (0,1), \lambda_i \in \mathbb{R}, i = 1,2,...,n$.

Proposition 3.5. Let $\beta > 0, 1 < \alpha \le 2$ and $\rho > 0, 0 < \zeta \le 1, \zeta_i \in (0,1), \lambda_i \in \mathbb{R}, i = 1,2,...,n$. Let

$$\sigma = \frac{\Gamma(\alpha)}{\rho^{\beta}\Gamma(\alpha+\beta)}\zeta^{\rho(\alpha+\beta-1)} - \sum_{i=1}^{n}\lambda_i\zeta_i^{\rho(\alpha-1)} \neq 0.$$

For any $\varphi \in C(0,1)$ and $x \in C([0,1],\mathbb{R})$ with ${}^{\rho}I^{2-\alpha}_{0+}x \in AC^2_{\rho}[0,1]$, Then the function x is the solution of nonlocal fractional differential equation boundary-value problem

$$\begin{cases} ({}^{\rho}D^{\alpha}_{0+}x)(t) = \varphi(t), \ 0 < t < 1, \\ x(0) = 0, \ \ ({}^{\rho}I^{\beta}_{0+}x)(\zeta) = \sum\limits_{i=1}^{n}\lambda_i x(\zeta_i). \end{cases} \tag{3.13}$$

if and only if

$$x(t) = {}^{\rho}I^{\alpha}_{0+}\varphi(t) + \frac{1}{\sigma}\left(\sum_{i=1}^{n}\lambda_i\,{}^{\rho}I^{\alpha}_{0+}\varphi(\zeta_i) - {}^{\rho}I^{\alpha+\beta}_{0+}\varphi(\zeta)\right)t^{\rho(\alpha-1)}. \tag{3.14}$$

Proof. This Proposition is a special case of Proposition 2.8. so we will not prove it again. □

Theorem 3.6. Let $1 < \alpha \le 2$. Assume $f \in C(J\times\mathbb{R},\mathbb{R}) \cap X^p_{\nu}(0,1)$, and the following conditions hold:
(H_3): exist constant $0 < \delta < 1$ and $\theta \in X^p_{\nu}((0,1),[0,\infty))$ such that

$$\sup_{0\le t\le 1}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^t \frac{s^{\rho-1}}{(t^{\rho}-s^{\rho})^{1-\alpha}}\theta(s)ds \le 1-\delta, \tag{3.15}$$

and

$$|f(t,x_1) - f(t,x_2)| \le \theta(t)|x_1 - x_2|, \ \ \forall t \in J, \ x_1, x_2 \in \mathbb{R}. \tag{3.16}$$

(H_4): $\beta > 0, 0 < \zeta \leq 1, \zeta_i \in (0,1)$, and $\lambda_i \in \mathbb{R}$, $i = 1, 2, ..., n$ satisfy

$$0 < \varepsilon_0 := \frac{1}{|\sigma|}\left(\sum_{i=1}^{n} |\lambda_i| + {}^{\rho}I_{0+}^{\alpha+\beta}\theta(\zeta)\right) < 1. \tag{3.17}$$

Then problem (1.1)-(3.12) has at least one solution.

Proof. Let $1 < \alpha \leq 2$, $\rho > 0$, set $f_\zeta = \max_{0\leq s\leq 1} |f(s, u(\zeta))| < \infty$, λ is a constant, $\lambda > \frac{f_\zeta}{2\rho^\alpha\Gamma(\alpha+1)}$. $S = \{u \in C^\alpha(J, \mathbb{R}) : \|u(t)\| \leq \lambda\}$. Define an operator $\mathcal{P}$ on S as follows

$$\mathcal{P}u(t) = {}^{\rho}I_{0+}^{\alpha} f(t, u(t)) + \frac{1}{\sigma}\left(\sum_{i=1}^{n} \lambda_i \, {}^{\rho}I_{0+}^{\alpha} f(\zeta_i, u(\zeta_i)) - {}^{\rho}I_{0+}^{\alpha+\beta} f(\zeta, u(\zeta))\right) t^{\rho(\alpha-1)}. \tag{3.18}$$

Let

$$\mathcal{P}_1 u(t) = {}^{\rho}I_{0+}^{\alpha} f(t, u(t)).$$

$$\mathcal{P}_2 u(t) = \frac{1}{\sigma}\left(\sum_{i=1}^{n} \lambda_i \, {}^{\rho}I_{0+}^{\alpha} f(\zeta_i, u(\zeta_i)) - {}^{\rho}I_{0+}^{\alpha+\beta} f(\zeta, u(\zeta))\right) t^{\rho(\alpha-1)}.$$

It is clear that $u(t)$ is a solution of (1.1) and (3.12) if it is a fixed point of the operator $\mathcal{P}$. Similar to the proof of Theorem 3.2, we may deduce that $\mathcal{P}_1$ is a completely continuous operator and $\mathcal{P}_2$ is a contractor operator. Therefore, according to Proposition 2.3, $\mathcal{P}$ has a fixed point in $\mathcal{S}$. □

Remark 3.7. In Deng′s paper [20], nonlocal condition

$$x(s, 0) + \sum_{i=1}^{n} p_i(s)x(s, t_i) = q(x), \tag{3.19}$$

with $t_i \in (0, T](i = 1, 2, \cdots, n)$ can be applied to describe the diffusion phenomenon of a small amount of gas in a transparent tube. Obviously the boundary condition (3.12) in Theorem 3.6 is a special form of condition (3.19).

4. Examples

This section, to illustrate the application of the Theorems, we constructed the following examples.

Example 4.1. Let us consider the following fractional differential equation boundary value problem

$$\begin{cases} ({}^{\rho}D_{0+}^{\frac{3}{2}} u)(t) = t^\beta u(t), \ 0 \leq t \leq 1, \\ u(0) = 0, \ u(1) = \int_0^\eta u(s)ds, \ \eta \in (0,1). \end{cases} \tag{4.1}$$

Where $\alpha = \frac{3}{2}$, $\rho > 0$, $\beta \in \mathbb{R}$ and $f(t, u) = t^\beta u$. $f(t, u)$ satisfy $|f(t, x_1) - f(t, x_2)| \leq t^\beta |x_1 - x_2|$. By Theorem 3.2, if the continuous solution to problem (4.1) exists, λ and β must satisfy certain conditions.

In fact, since

$$\frac{2\rho^{-\frac{1}{2}}}{\Gamma(\frac{3}{2})}\int_0^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{-\frac{1}{2}}} s^\beta ds = \frac{2t^{\frac{3}{2}\rho+\beta}}{\rho^{\frac{3}{2}}\Gamma(\frac{3}{2})}\int_0^1 \tau^{\frac{\beta}{\rho}}(1-\tau)^{\frac{1}{2}} d\tau = \frac{2\Gamma(\frac{\beta}{\rho}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\beta}{\rho}+\frac{5}{2})} t^{\frac{3}{2}\rho+\beta}$$

and

$$\frac{1}{\Delta}\left(\int_0^\eta {}^{\rho}I_{0+}^{\alpha} t^\beta dt + {}^{\rho}I_{0+}^{\alpha} 1\right) = \frac{1}{1 - \frac{\eta^{\frac{\rho}{2}+1}}{\frac{\rho}{2}+1}}\left(\frac{\Gamma(\frac{\beta}{\rho}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\beta}{\rho}+\frac{5}{2})}\left(\frac{\eta^{\frac{3}{2}\rho+\beta+1}}{\frac{\rho}{2}+\beta+1}+1\right)\right).$$

Thus, conditions H_1 and H_2 now are

$$\frac{\Gamma(\frac{\beta}{\rho}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\beta}{\rho}+\frac{5}{2})}<\frac{1}{2},\qquad \frac{\Gamma(\frac{\beta}{\rho}+1)}{\rho^{\frac{3}{2}}\Gamma(\frac{\beta}{\rho}+\frac{5}{2})}(\frac{\eta^{\frac{3}{2}\rho+\beta+1}}{\frac{3}{2}\rho+\beta+1}+1)<1-\frac{\eta^{\frac{\rho}{2}+1}}{\frac{\rho}{2}+1},\eta\in(0,1). \tag{4.2}$$

Combining the above two inequalities, we have

$$\frac{1-\int_0^\eta t^{\frac{\rho}{2}}dt}{1+\int_0^\eta t^{\frac{3}{2}\rho+\beta}dt}\leq\frac{1}{2}.$$

Let us choose $\rho=1$, $\beta=\frac{1}{2}$, the first inequality in (4.2) becomes $\sqrt{\pi}<2.25$. η satisfies inequality $\eta^3+4\eta^{\frac{3}{2}}-3\geq 0$, by solving this inequality, we get $\eta\geq 0.5867862$. When $\rho\to 0^+$, the inequality stated above is not true. Therefore, for $\rho=1$, $\beta=\frac{1}{2}$, $\eta\geq 0.5867862$, the boundary value problem (4.1) has at least one solution on $[0,1]$.

Example 4.2. Consider the following fractional differential equation boundary value problem

$$\begin{cases} ({}^{\frac{1}{2}}D^{\frac{3}{2}}_{0+}u)(t)=\frac{1}{7(2e^t+1)}(\frac{|u(t)|}{|u(t)|+1}+sint),\ 0\leq t\leq 1,\\ u(0)=0,\ u(1)=({}^{\frac{1}{2}}I^{\frac{1}{2}}_{0+}u)(\frac{1}{2})+2({}^{\frac{1}{2}}I^{\frac{5}{2}}_{0+}u)(\frac{1}{3}).\end{cases} \tag{4.3}$$

Where $\alpha=\frac{3}{2}$, $\rho=\frac{1}{2}$, $n=2$, $\alpha_1=\frac{1}{2}$, $\alpha_2=\frac{5}{2}$, $\eta_1=\frac{1}{2}$, $\eta_2=\frac{1}{3}$, $k_1=1$, $k_2=2$ and

$$f(t,u(t))=\frac{1}{7(2e^t+1)}(\frac{|u(t)|}{|u(t)|+1}+sint),\ \ t\in J.$$

Using the given values, we can calculate $|\Delta|\approx 0.2078275$ and $\rho^\alpha\Gamma(\alpha+1)\approx 0.4699928$. It is easy to check that $f(t,u(t))$ is continuous and

$$|f(t,u(t))-f(t,v(t))|\leq\frac{1}{7(2e^t+1)}|u(t)-v(t)|,$$

select $L=\frac{1}{21}$, Also $r_0\approx 0.97587238<1$ satisfy the condition (H_2) of Theorem 3.1, for any $0<\theta_1<1$, $0.20266529<1-\theta_1$ satisfy the condition (H_1).

Therefore, by the conclusion of Theorem 3.2, the nonlocal boundary value problem (4.3) has at least one solution on $[0,1]$.

Example 4.3. Consider the following problem

$$\begin{cases} ({}^{2}D^{\frac{5}{3}}_{0+}u)(t)=\xi(t)(t-u(t))+\xi_0,\ 0\leq t\leq 1,\\ u(0)=0,\quad ({}^{2}I^{\frac{4}{3}}_{0+}u)(\frac{1}{2})=\frac{1}{2}u(\frac{1}{4})+\frac{1}{3}u(\frac{3}{4})+\frac{1}{8}u(\frac{4}{5}).\end{cases} \tag{4.4}$$

Where $\alpha=\frac{5}{3}$, $\rho=2$, $\beta=\frac{4}{3}$, $n=3$, $\zeta=\frac{1}{2}$, $\zeta_1=\frac{2}{3}$, $\zeta_2=\frac{3}{4}$, $\zeta_3=\frac{4}{5}$ $\lambda_1=-\frac{1}{15}$, $\lambda_2=\frac{1}{30}$, $\lambda_3=\frac{1}{60}$. ξ_0 is a constant, $\xi(t)\in C^\alpha(J,\mathbb{R})$ such that

$$\xi(t)\leq t^{-\frac{1}{2}},\ \ t\in J.$$

Set

$$f(t,x)=\xi(t)(t-x)+\xi_0,\ \ (t,x)\in J\times\mathbb{R}.$$

Using the given values, we can calculate $|\sigma|\approx 0.1828616$, $\epsilon_0\approx 0.7959415$. Select $\theta(t)=t^{-\frac{1}{2}}$, we have

$$\frac{2^{-\frac{2}{3}}}{\Gamma(\frac{5}{3})}\int_0^t\frac{s}{(t^2-s^2)^{-\frac{2}{3}}}s^{-\frac{1}{2}}ds=\frac{\Gamma(\frac{3}{4})}{2^{\frac{5}{3}}\Gamma(\frac{29}{12})}t^{\frac{17}{6}}.$$

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It is easy to check that $f(t,u(t))$ is continuous and

$$|f(t,u(t)) - f(t,v(t))| \le t^{-\frac{1}{2}}|u(t) - v(t)|,$$

$\rho^\alpha\Gamma(\alpha+1) \approx 4.776279$, $\rho^\alpha\Gamma(\alpha+\beta) \approx 6.349604$. The calculation results satisfy condition (H_3) and (H_4) in Theorem 3.6.

Therefore, by the conclusion of Theorem 3.6, the nonlocal boundary value problem (4.4) has at least one solution on $[0,1]$.

5. Conclusions

In this paper, we investigate the definitions and properties of fractional integrals with respect to a power functions. We proved the strong continuity properties of the associated semigroups and obtained an existence Theorem for solutions of differential equations under non-local boundary conditions when the order is $1 < \alpha \le 2$. Notably, the definition contains a special parameter ρ, which influences the results of the integrals. Furthermore, the existence conditions of solutions to the non-local problem are impacted by the selection of the parameter $\rho^\alpha\Gamma(\alpha+1)$.

Funding: This work sponsored by Gansu Province Philosophy and Social Science Planning Project, Grant/Award Number: 2024YB063; Gansu Province Higher School Innovation Fund Project, Grant/Award Number: 2022B-104.

Conflicts of Interest: This work does not have any conflicts of interest.

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