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Symmetria: Trinary Symmetry, Trions, and the Geometry of Neutrality and Polarity

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Symmetria: Trinary Symmetry, Trions, and the Geometry of Neutrality and Polarity

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Abstract

This paper introduces Symmetria, a new mathematical framework for analyzing systems composed of two opposing components mediated by a neutral element. Such triadic structures appear in opinion dynamics, physics, classification theory, and information geometry, yet they lack a unified representation. We formalize these systems as Trions, ordered triples (L, N, R), and define their invariants: mass, commit, and lean. We prove that all admissible Trions project into a Transition Cone, a convex polyhedral domain characterized by normalization, edge, and mirror symmetries. Within this setting, we derive structural identities linking invariants, including the quadratic coexistence law and purity-balance relation, and introduce alternative coordinate systems that reveal neutrality as a contraction operator. Geometrically, we establish convexity, boundary characterizations, and a rectangular representation of the state space. Dynamically, we analyze trajectories confined to the cone, oscillatory states described by spectral ellipses, and the role of the Symmetria Laplacian in harmonic analysis on cones. Case studies demonstrate applications to political polarization, statistical classification with abstention, two-phase physics with disorder, and noisy communication channels. Together, these results show that Symmetria is elementary in definition yet fertile in consequence: a coherent analytic framework uniting neutrality, polarity, and symmetry.

Keywords: symmetria; trions; neutrality; polarity; geometry; dynamics

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1. Introduction

Symmetry is one of the oldest and most universal ideas in mathematics. The Pythagoreans regarded proportion and harmony as the structure of reality itself. Euclid's geometry rested on balanced ratios. Centuries later, algebraic invariants and group theory formalized symmetry into the language of modern mathematics. From these beginnings to the present day, symmetry remains both a guiding principle and a tool of discovery: the classification of Lie groups, the central role of invariance in physics, the convex cones of optimization, and the entropy function of information theory are all consequences of the search for hidden balance.

But balance is only half of the story. Just as important is asymmetry: the small deviation from balance that generates structure, drives dynamics, or signals instability. A perfect sphere may represent symmetry, but it is asymmetry, a tilt, a rotation, or a distortion that tells us something about forces at play. In probability theory, the uniform distribution embodies balance, while skew or bias encodes asymmetry. In game dynamics, equilibrium reflects balance, while selection pressure produces asymmetry. The two concepts are inseparable.

The probability simplex has long been the central arena for this interplay. A system with k categories is represented by a nonnegative vector whose entries sum to one, a point in the $(k-1)$-simplex [6,14]. This simple geometry is extraordinarily powerful: it supports the definition of entropy and divergence [6], underlies evolutionary game dynamics [6,12], and serves as the stage for algorithms in optimization and learning. Within the simplex, balance corresponds to points near the barycenter, while asymmetry pushes states toward the vertices.

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Yet the simplex framework treats all categories as interchangeable. This works well when categories are symmetric, but there are many systems where one category plays a qualitatively different role. Consider opinion dynamics: individuals may be left-leaning, right-leaning, or undecided [12]. The "undecided" category is not symmetric with the others; it mediates between them, dilutes polarization, and vanishes under commitment. In physics, competing ordered phases may coexist with an intermediate disordered phase [15]. Again, the disordered state is not just "one more option"; it is the neutral ground against which polarity is measured. In combinatorial or algebraic decompositions, two opposing parts may exist only in relation to a balancing middle.

These are not accidents. They point to a structural truth: in many three-component systems, neutrality is privileged, while the other two components stand in polarity to each other. The full permutation symmetry of the simplex obscures this fact. What is needed is a new framework where neutrality and polarity are built into the geometry itself.

We call such a system a *Trion*. Formally, a Trion is an ordered triple (L, N, R) of nonnegative quantities, representing left, neutral, and right components. Unlike a general probability triple, a Trion is not just three numbers summing to one. Its structure encodes two symmetries only: scaling invariance (multiplying by a constant does not change the normalized state) and mirror symmetry (swapping L and R reverses polarity but preserves neutrality). The other symmetries of the simplex are broken. This reduction from full symmetry to restricted symmetry is what defines *Symmetria*, the mathematical theory we develop here.

To capture this structure, we introduce three invariants. The mass

$$\mu = L + N + R \tag{1}$$

records the total scale of the Trion. The commit

$$\kappa = \frac{L + R}{\mu}, \tag{2}$$

measures the proportion of non-neutral elements. The lean

$$\lambda = \frac{L - R}{\mu}, \tag{3}$$

measures directional imbalance between the polarized components. These two normalized quantities reduce the three-dimensional description to a two-dimensional representation (κ, λ).

The striking fact is that all admissible states fall within a sharply defined region:

$$0 \leq \kappa \leq 1, \qquad |\lambda| \leq \kappa. \tag{4}$$

As shown in Figure 1, this triangular domain, which we call the *Transition Cone*, is the fundamental state space of Symmetria. Its apex $(0, 0)$ corresponds to pure neutrality, its edges $|\lambda| \leq \kappa$ to one-sided states, and its central axis $\lambda = 0$ to balanced opposition.

This paper develops Symmetria systematically. Within the Transition Cone we establish a Normalization Theorem bounding the invariants, an Edge Theorem identifying the precise boundary where one side vanishes, and a Mirror Theorem showing that reflection flips lean while preserving commit and mass. These theorems make clear that the cone is not an arbitrary construction but the exact image of all possible Trions under normalization.

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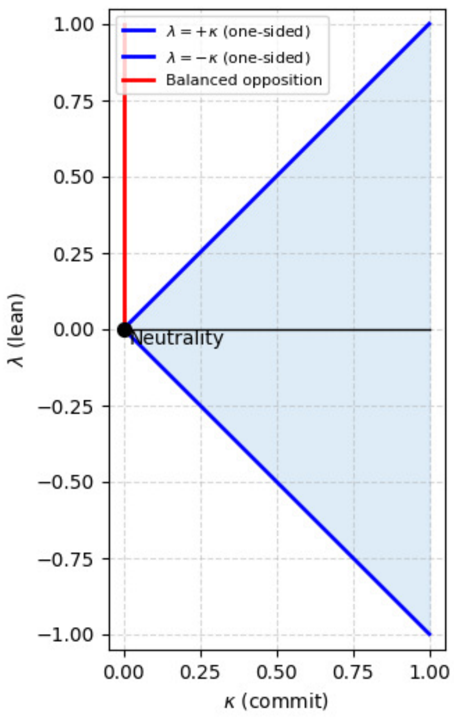

Figure 1. The Transition Cone is bounded by the one-sided edges $\lambda = \pm\kappa$ with balanced opposition along $\lambda = 0$.

Beyond these basics, we uncover structural identities linking the invariants. A quadratic relation ties κ and λ to the coexistence product LR. A purity-balance law connects the side ratio L/R to normalized invariants. Additional measures, such as slack and purity, create alternative coordinate systems and show how neutrality contracts states toward balance. Geometric consequences include convexity of the cone and invariance under unions of Trions.

Finally, we extend Symmetria into dynamics and analysis. Considering trajectories $(\kappa(t), \lambda(t))$ evolving in time, we introduce the Symmetria Laplacian

$$\Delta_{\text{sym}} f = \frac{\partial^2 f}{\partial \kappa^2} + \frac{\partial^2 f}{\partial \lambda^2}. \tag{5}$$

The Euclidean Laplacian restricted to the Transition Cone. This operator connects Symmetria to harmonic analysis, allowing oscillations and constraints to be studied with spectral tools [5,11].

The aim of this work is to present Symmetria as a new mathematical theory: elementary in its construction but rich in its consequences. Section 2 introduces Trions and their invariants. Section 3 establishes the fundamental theorems. Section 4 develops structural identities and alternative coordinate systems. Section 5 analyzes the geometry of the Transition Cone. Section 6 explores dynamics and spectral aspects. Section 7 concludes with a discussion, applications, and open problems.

2. Preliminaries and Definitions

The framework of Symmetria begins with the definition of a Trion and the introduction of its invariants. At the most basic level, a Trion is just a triple of nonnegative numbers. But, as in projective geometry or probability theory, the meaning lies not in the raw coordinates but in the relations among them. The distinctive contribution of Symmetria is to recognize that the three roles, left, right, and neutral, are not interchangeable. Scale, polarity, and neutrality interact in precise ways, producing invariants and geometric constraints that distinguish Symmetria from the general simplex.

This section lays the foundations. We begin by defining Trions and their invariants, then derive secondary quantities, provide worked examples, compare with the simplex, describe the geometry of projection into the Transition Cone, and formalize the symmetry principles that act on this space.

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2.1. Trions

Definition 1 (Trion)**.** *A Trion is an ordered triple*

$$(L, N, R) \in \mathbb{R}^3_{\geq 0}$$

with mass

$$\mu = L + N + R. \tag{6}$$

Unless otherwise stated, we assume $\mu > 0$.

Interpretation

L and R represent two opposing components, while N represents a mediating neutral component. The ordering is crucial: unlike points in the 2-simplex, the roles of L, N, and R cannot be permuted without altering the structural meaning.

Scalar multiples of a Trion encode the same normalized state, since multiplying by a positive constant rescales all entries but does not change their proportions. Thus, the essential geometry lies not in the raw triple but in its invariant ratios. This is conceptually parallel to projective normalization in algebraic geometry, where equivalence classes of proportional vectors define the same point in projective space [11].

2.2. Primary Invariants

The first step is to reduce the raw data of a Trion to two scale-free descriptors.

Definition 2 (Commit and Lean)**.** *For a Trion with total mass μ:*

$$\kappa = \frac{L + R}{\mu}, \qquad \lambda = \frac{L - R}{\mu}. \tag{7}$$

- κ (commit) measures the fraction of mass lying outside the neutral component.
- λ (lean) measures the normalized imbalance between left and right.

Every Trion thus projects to a pair (κ, λ). These quantities are close relatives of barycentric coordinates [15], but specialized to encode neutrality and polarity rather than treating all vertices symmetrically.

2.3. Derived Invariants

Beyond κ and λ, several secondary invariants refine the description.

Definition 3 (Slack, Purity, Balance ratio)**.** *For a Trion (L, N, R) with $\mu > 0$:*

- ***Slack:***
$$\sigma = \frac{N}{\mu} = 1 - \kappa, \tag{8}$$
quantifying the neutral share.
- ***Purity:***
$$\pi = \frac{|\lambda|}{\kappa}, \qquad (\kappa > 0), \tag{9}$$
measuring the one-sidedness of the committed portion.
- ***Balance ratio:***
$$\beta = \frac{L}{R}, \qquad (R > 0), \tag{10}$$
measuring multiplicative asymmetry. If $R = 0$ and $L > 0$, we set $\beta = +\infty$.

These derived functionals are analogous to normalized divergences and bias measures in information geometry [14] and to bias indicators used in the statistical physics of social dynamics [12].

2.4. Examples

To illustrate, consider the following Trions (normalized to mass $\mu = 1$ for convenience):

1. **Pure neutral** $(0,1,0)$.
   - $\kappa = 0,\ \lambda = 0,\ \sigma = 1$.
   - Lies at the apex of the Transition Cone.
2. **Pure left** $(1,0,0)$.
   - $\kappa = 1,\ \lambda = 1,\ \sigma = 0,\ \pi = 1$.
   - Lies on the extreme boundary of the cone.
3. **Balanced committed** $(1,0,1)$.
   - $\kappa = 1,\ \lambda = 0,\ \sigma = 0,\ \pi = 0$.
   - Represents a perfect balance between L and R.
4. **Mixed state** $(2,3,1)$.
   - $\mu = 6$.
   - $\kappa = 0.5,\ \lambda \approx 0.167,\ \sigma = 0.5,\ \pi \approx 0.333$.
   - Lies strictly inside the cone, illustrating coexistence with moderate asymmetry.

Worked examples like these are standard in treatments of normalized coordinates [14,15], and they clarify the geometric meaning of invariants.

2.5. Comparison with the Probability Simplex

The normalized triple $\dfrac{(L,N,R)}{\mu}$ lies in the 2-simplex

$$\Delta^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_{\geq 0} : x_1 + x_2 + x_3 = 1\}. \tag{11}$$

In the simplex, all coordinates are symmetric [6]. In contrast, Symmetria preserves only:

- **Scaling symmetry** (rescaling does not change normalized invariants), and
- **Mirror symmetry** (swapping L and R flips polarity).

Neutrality is privileged and cannot be permuted with the other coordinates without breaking the structure. This restricted symmetry parallels symmetry-breaking frameworks in physics [15], where one component acquires a qualitatively different role.

2.6. Geometry of the Projection

The projection map

$$\phi : (L,N,R) \mapsto (\kappa, \lambda)$$

is homogeneous of degree zero (scale invariant), continuous, and surjective onto a two-dimensional region. As will be proved in Section 3, its image is the *Transition Cone*

$$\mathcal{C} = \{(\kappa, \lambda) \in \mathbb{R}^2 : 0 \leq \kappa \leq 1,\ |\lambda| \leq \kappa\}. \tag{12}$$

Geometrically:

- Neutrality pulls states toward the apex $(0,0)$.
- Perfect balance corresponds to the vertical axis $\lambda = 0$.
- Pure asymmetry pushes states to the cone's edges $|\lambda| = \kappa$.

This construction is directly related to convex cone geometry in optimization [6] and to state spaces in dynamical systems [6,12].

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2.7. Symmetry Principles

Two transformations act naturally on Trions:

1. **Scaling:** multiplying (L, N, R) by $c > 0$ leaves $\mu, \kappa, \lambda, \sigma, \pi, \beta$ unchanged.
2. **Mirror symmetry:** swapping L and R preserves μ, κ, σ, π but maps $\lambda \mapsto -\lambda$ and $\beta \mapsto 1/\beta$.

Together, these generate the automorphism group of Symmetria. Unlike the full permutation group S_3 of the simplex [6], this group reflects the privileged role of neutrality and the binary polarity between left and right.

2.8. Notation Summary

Table 1. Summary of key symbols, definitions, ranges, and interpretations used in the Trion framework.

Symbol	Definition	Range	Interpretation
μ	$L + N + R$	$(0, \infty)$	Mass (scale)
κ	$(L + R)/\mu$	$[0, 1]$	Commit (non-neutral share)
λ	$(L - R)/\mu$	$[-1, 1]$	Lean (directional imbalance)
σ	N/μ	$[0, 1]$	Slack (neutral share)
π	$\lvert L - R\rvert/(L + R)$, equivalently $\lvert\lambda\rvert/\kappa$ when $\kappa > 0$	$[0, 1]$	Purity (one-sidedness of commitment)
β	L/R	$(0, \infty]$	Balance ratio

3. Fundamental Theorems

The invariants introduced in Section 2 are not independent. Instead, they are bound by exact inequalities and symmetries that carve out a highly structured state space. These constraints do more than just restrict possible values: they reveal the geometry of Symmetria and establish the Transition Cone as its fundamental domain.

3.1. Normalization

Theorem 1 (Normalization Bounds)**.** *For any Trion* (L, N, R) *with* $\mu > 0$*, the invariants satisfy*

$$0 \le \kappa \le 1, \qquad -1 \le \lambda \le 1.$$

Proof. Since $L, N, R \ge 0$, we have $0 \le L + R \le \mu$.
Dividing by μ yields $0 \le \kappa \le 1$.
For $\lambda = \frac{L-R}{\mu}$, note that $|L - R| \le L + R \le \mu$.
Dividing by μ gives $|\lambda| \le 1$, hence $-1 \le \lambda \le 1$. □

Narrative

This theorem is the most basic constraint: it shows that commit κ is a normalized share (like a probability between 0 and 1), and lean λ is a normalized imbalance (between -1 and 1). But note the difference: while probabilities must sum to one, here neutrality and polarity compete within bounded ratios. This already hints that Symmetria is not just probability theory in disguise but a geometry shaped by opposition.

Figure 2 shows the raw bounding box imposed by Theorem 1: $0 \le \kappa \le 1$, $-1 \le \lambda \le 1$. Later theorems will sharpen this into the Transition Cone.

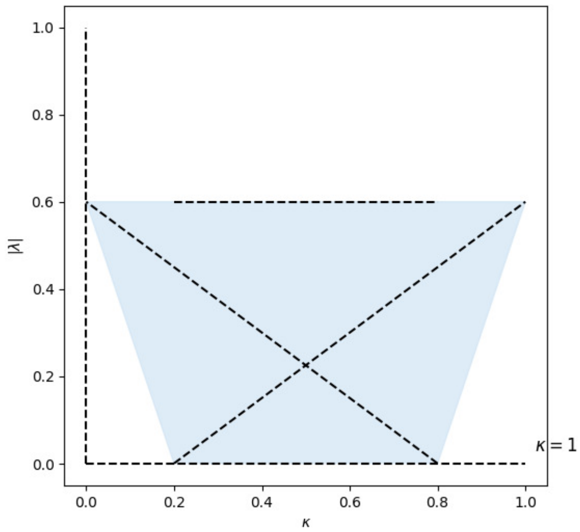

Figure 2. The Normalization Theorem bounds Trions within a trapezoidal region in the $(\kappa, |\lambda|)$-plane.

3.2. Edge Theorem

Theorem 2 (Edge Law)**.** *For any Trion,*

$$\lambda| = \kappa \iff \min(L, R) = 0. \tag{13}$$

Proof. Suppose $L = 0$, then

$$\kappa = \frac{R}{\mu}, \qquad \lambda = \frac{-R}{\mu}, \tag{14}$$

so $|\lambda| = \kappa$.
Similarly, if $R = 0$.
Conversely, suppose $|\lambda| = \kappa$, then

$$\frac{|L - R|}{\mu} = \frac{L + R}{\mu}, \tag{15}$$

hence $|L - R| = L + R$.
Because $L, R \geq 0$, this equality holds only if $\min(L, R) = 0$. □

Narrative

The Edge Law identifies the extreme rays of the *Transition Cone*: the points where one polarized side is extinguished. In opinion dynamics, this means all commitment lies on one side. In physics, it corresponds to one phase collapsing while the other dominates. The edges, therefore, represent pure polarity, with no coexistence.

In Figure 3, the dashed diagonals mark the edges where one component vanishes. The interior between them is the zone of coexistence.

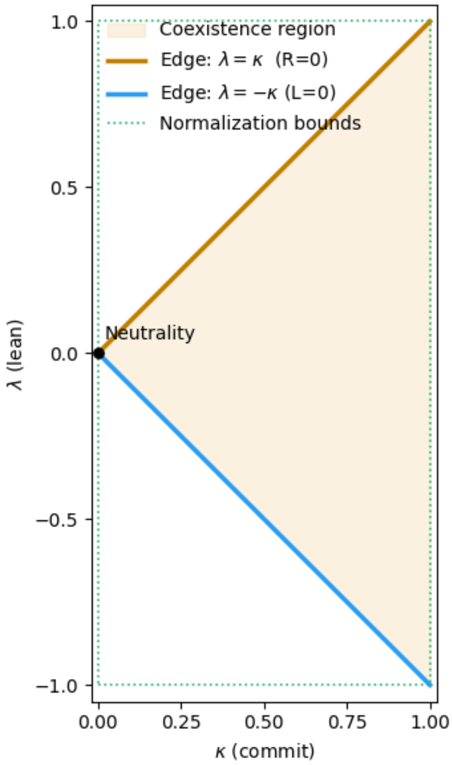

Figure 3. The Edge Law also identifies the coexistence region bounded by the lines $\lambda = \pm\kappa$.

3.3. Mirror Symmetry

Theorem 3 (Mirror Symmetry)**.** *Let* (L, N, R) *be a Trion. Its mirror image* (R, N, L) *satisfies*

$$\mu' = \mu, \qquad \kappa' = \kappa, \qquad \lambda' = -\lambda. \tag{16}$$

Proof. Interchanging L and R leaves

$$\mu = L + N + R,\ \kappa = \frac{L+R}{\mu} \tag{17}$$

unchanged, but flips the sign of

$$\lambda = \frac{L-R}{\mu}. \tag{18}$$

□

Narrative

Mirror symmetry formalizes the intuition that "left" and "right" are structurally identical, except for orientation. In the (κ, λ)-plane, this symmetry is a reflection across the vertical axis. It is analogous to parity transformation in physics [15], where a system behaves identically under spatial reflection. The cone, therefore, has a built-in mirror, making neutrality the axis of symmetry.

In Figure 4, every point at lean $+\lambda$ has a symmetric partner at lean $-\lambda$, with commit preserved.

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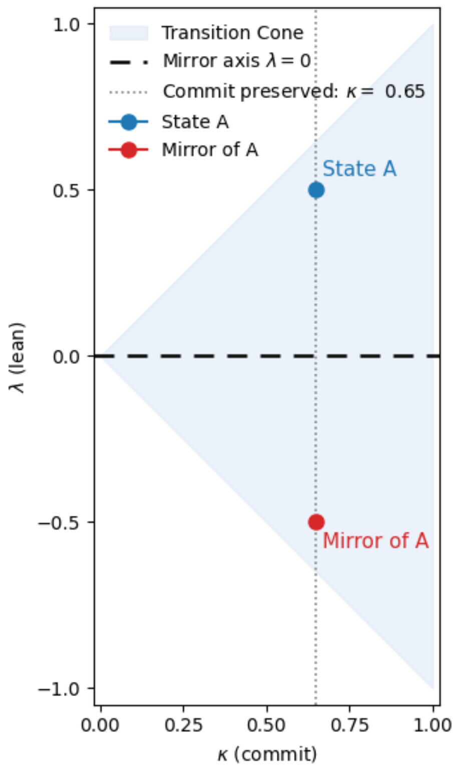

Figure 4. Mirror symmetry reflects a state across $\lambda = 0$, preserving commit while flipping lean.

3.4. Transition Cone

Theorem 4 (Transition Cone Theorem). *The set of all possible pairs* (κ, λ) *arising from Trions is*

$$\mathcal{C} = \{(\kappa, \lambda) \in \mathbb{R}^2 : 0 \leq \kappa \leq 1, \ |\lambda| \leq \kappa\}. \tag{19}$$

This set is convex, closed, and pointed with an apex at the origin.

Proof. Theorem 1 gives the bounding box:

$$0 \leq \kappa \leq 1, \qquad |\lambda| \leq 1.$$

Theorem 2 shows that the diagonals $|\lambda| = \kappa$ are attainable boundaries.

The interior corresponds to both $L, R > 0$.

Hence, the image is exactly the isosceles triangle bounded by these lines. Convexity follows from linearity: convex combinations of Trions map to convex combinations of invariants. □

Narrative

This result ties everything together: Symmetria's state space is not the whole square $[0,1] \times [-1,1]$ but a triangular cone. Its apex $(0,0)$ is neutrality; its vertical axis $\lambda = 0$ is perfect balance; its diagonals are pure extremes. The geometry forces coexistence into the interior. Unlike the probability simplex, which treats all coordinates symmetrically, the *Transition Cone* encodes neutrality as contraction and polarity as expansion.

In Figure 5, the shaded cone is the exact image of all Trions.

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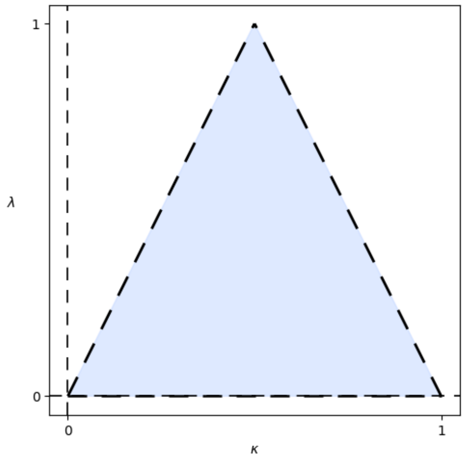

Figure 5. The Transition Cone is the triangular state space $\mathcal{C} = \{(\kappa, \lambda) \in \mathbb{R}^2 : 0 \leq \kappa \leq 1, |\lambda| \leq \kappa\}$. with the apex at neutrality.

3.5. Worked Examples

- Neutral: $(0, 1, 0) \mapsto (0, 0)$.
- Extreme left: $(1, 0, 0) \mapsto (1, 1)$.
- Balanced: $(1, 0, 1) \mapsto (1, 0)$.
- Mixed: $(2, 3, 1) \mapsto (0.5, 0.167)$.

These examples illustrate the apex, edges, axis, and interior points of the cone.

3.6. Summary

The four theorems establish Symmetria's foundations:

1. **Normalization:** invariants are bounded.
2. **Edge Law:** edges correspond to one-sided dominance.
3. **Mirror Symmetry:** left and right are structurally identical.
4. **Transition Cone:** the admissible state space is triangular, convex, and symmetric.

This structure is the skeleton on which all later identities, metrics, and dynamics will rest.

4. Derived Measures and Identities

Beyond the constraints of Section 3, the invariants of a Trion are tied by further relations. These relations, though simple in algebraic form, expose hidden structure: coexistence laws, projective invariance, and dualities between neutrality and polarity. This section develops four such results, each with proof, commentary, and a visual interpretation.

4.1. Quadratic Identity

Proposition 1 (Quadratic Identity)**.** *For any Trion (L, N, R) with mass μ:*

$$\kappa^2 - \lambda^2 = \frac{4LR}{\mu^2}. \tag{20}$$

Proof. By definition,

$$\kappa = \frac{L+R}{\mu}, \quad \lambda = \frac{L-R}{\mu}. \tag{21}$$

Hence

$$\kappa^2 - \lambda^2 = \frac{(L+R)^2}{\mu^2} - \frac{(L-R)^2}{\mu^2} = \frac{(L+R)^2 - (L-R)^2}{\mu^2} = \frac{4LR}{\mu^2}. \tag{22}$$

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□

Narrative

This identity is striking because it ties the two invariants back to the raw coexistence term LR. When one side vanishes, the product LR is zero, and the expression collapses exactly at the cone's edges. The quadratic form resembles discriminants in algebra [15], and, just like a discriminant, it detects the presence or absence of coexistence.

In Figure 6, the curve shows $\kappa^2 - \lambda^2$ as L varies with R fixed. The red dashed line indicates where the identity vanishes, marking the edges $L = 0$ or $R = 0$.

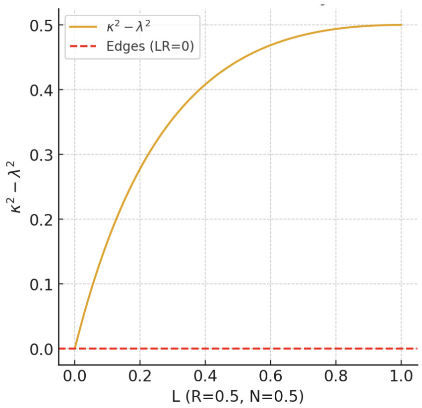

Figure 6. The function $\kappa^2 - \lambda^2$ vanishes on the edges and remains positive in the cone interior.

4.2. Purity–Balance Relation

Proposition 2 (Purity–Balance Law). *For a Trion with $\kappa > 0$:*

$$\kappa^2 - \lambda^2 \pi = \frac{|\beta - 1|}{\beta + 1}, \tag{23}$$

where

$$\beta = \frac{L}{R}, \qquad (\textit{with } \beta = \infty \textit{ if } R = 0). \tag{24}$$

Proof. From Definition 3,

$$\pi = \frac{|L - R|}{L + R}. \tag{25}$$

Dividing numerator and denominator by R (when $R > 0$):

$$\pi = \frac{\left|\frac{L}{R} - 1\right|}{\frac{L}{R} + 1} = \frac{|\beta - 1|}{\beta + 1}. \tag{26}$$

If $R = 0$, then $\beta = \infty$, and the limit gives $\pi = 1$. □

Narrative

Purity depends only on the ratio L/R, not on scale. This makes it a projective invariant: rescaling both L and R leaves it unchanged, much like ratios in projective geometry [11]. The function itself is symmetric: $\pi(\beta) = \pi(1/\beta)$, reflecting the mirror symmetry of Symmetria.

In Figure 7, the curve plots π as a function of $\beta = L/R$. Purity is zero at balance ($\beta = 1$) and approaches one as the imbalance grows ($\beta \to 0$ or ∞).

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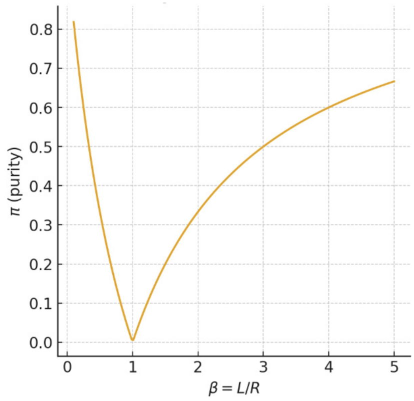

Figure 7. Purity π is minimized at balance $\beta = 1$ and increases monotonically with imbalance.

4.3. Slack–Commit Relation

Proposition 3 (Slack–Commit Law)**.** *For any Trion:*

$$\sigma = 1 - \kappa. \tag{27}$$

Proof. By definition,

$$\sigma = \frac{N}{\mu}, \qquad \kappa = \frac{L+R}{\mu}. \tag{28}$$

Since $\mu = N + (L + R)$, we obtain $\sigma + \kappa = 1$. □

Narrative

This is a conservation law: every Trion splits exactly into neutral and committed fractions. In information-theoretic terms, slack measures abstention, while commit measures expressed polarity. In dynamical models, the growth of one directly reduces the other.

In Figure 8, the line shows the perfect trade-off: as commitment increases, slack decreases, meeting at the midpoint.

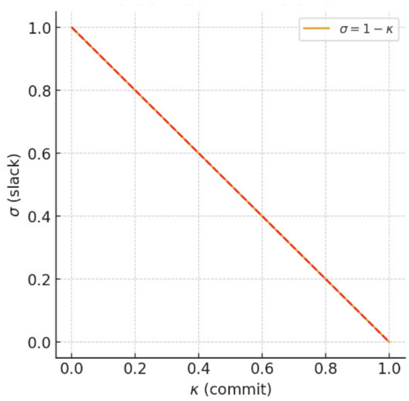

Figure 8. Slack decreases linearly with commit as $\sigma = 1 - \kappa$.

4.4. Slack–Purity Coordinates

Proposition 4 (Slack–Purity Coordinates)**.** *Every interior point* (κ, λ) *in the Transition Cone corresponds uniquely (except at* $\kappa = 0$*) to a pair* (σ, π) *with*

$$\sigma = 1 - \kappa, \qquad \pi = \frac{|\lambda|}{\kappa}. \tag{29}$$

Proof. Given (κ, λ), define

$$\sigma = 1 - \kappa, \qquad \pi = \frac{|\lambda|}{\kappa}. \tag{30}$$

Then

$$0 \leq \sigma \leq 1, \qquad 0 \leq \pi \leq 1.$$

Conversely, given (σ, π), we reconstruct

$$\kappa = 1 - \sigma, \qquad \lambda = \pm \pi \kappa, \tag{31}$$

where the sign encodes left versus right lean. □

Narrative

This change of coordinates unwraps the triangular *Transition Cone* into a rectangle. In (σ, π)-space, neutrality and purity vary independently in $[0, 1]$. The polarity sign provides a binary choice. Such rectangularizations are common in information geometry [14], where coordinate systems are chosen to simplify analysis.

As shown in Figure 9, the *Transition Cone* maps to the rectangle $[0, 1] \times [0, 1]$ in (σ, π). Neutrality (σ) and purity (π) vary freely, while lean's sign distinguishes left from right.

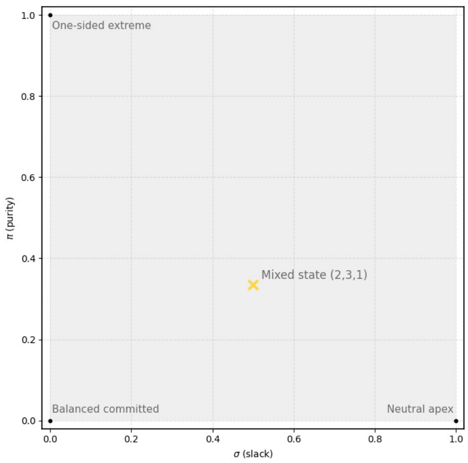

Figure 9. The slack-purity rectangle maps Trions into (σ, π) with corners marking neutrality, balance, and extremes.

4.5. Summary

The derived measures extend Symmetria's structure:

- The **Quadratic Law** captures the coexistence of opposing sides.
- The **Purity–Balance Relation** isolates bias as a projective invariant.
- The **Slack–Commit Law** expresses the conservation of neutral versus polarized mass.
- The **Slack–Purity Coordinates** provide a rectangular chart for the cone.

These results enrich the framework beyond the basic theorems, offering alternative viewpoints and tools that will be critical in analyzing the geometry (Section 5) and dynamics (Section 6).

5. Geometry of Symmetria Space

The Transition Cone $\mathcal{C}$, introduced in Theorem 4, is the natural state space of Symmetria. Its geometry encodes the interplay of neutrality, polarity, and scale. In this section, we analyze four aspects: convexity, neutrality contraction, axes and boundaries, and alternative coordinates.

5.1. Convexity

Theorem 5 (Convexity of the Transition Cone). *If $(\kappa_1, \lambda_1), (\kappa_2, \lambda_2) \in \mathcal{C}$, then for any $0 \leq t \leq 1$, the convex combination*

$$(\kappa, \lambda) = t(\kappa_1, \lambda_1) + (1-t)(\kappa_2, \lambda_2) \tag{32}$$

also lies in $\mathcal{C}$.

Proof. Each pair (κ_i, λ_i) arises from a Trion (L_i, N_i, R_i).

Consider the convex combination

$$(L, N, R) = t(L_1, N_1, R_1) + (1-t)(L_2, N_2, R_2). \tag{33}$$

This is again a Trion with mass $\mu = t\mu_1 + (1-t)\mu_2$.

Its invariants are

$$\kappa = \frac{L+R}{\mu}, \qquad \lambda = \frac{L-R}{\mu}. \tag{34}$$

By construction, (κ, λ) is exactly the convex combination above.

Since Trions always map into $\mathcal{C}$, the result follows. □

Narrative

Convexity means mixtures of states remain valid states. In opinion dynamics, this corresponds to mixing two populations without creating impossible proportions. In optimization language, $\mathcal{C}$ is a convex cone, a structure central to convex analysis [14].

In Figure 10, two points in the cone (blue, red) and their convex combinations (black line segment) remain inside the shaded region.

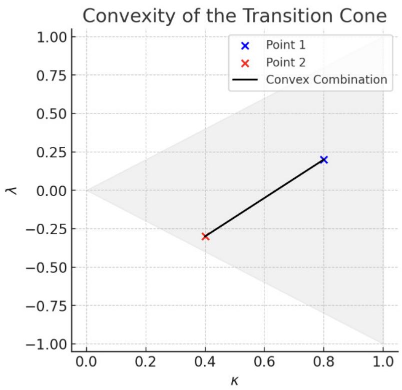

Figure 10. Convexity is demonstrated as any line segment between two states remains within the Transition Cone.

5.2. Neutrality Contraction

Theorem 6 (Neutrality Contraction). *Fix $L, R \geq 0$. As $N \to \infty$,*

$$\kappa \to 0, \qquad \lambda \to 0.$$

Proof.

$$\kappa = \frac{L+R}{\mu}, \qquad \lambda = \frac{L-R}{\mu}, \tag{35}$$

where $\mu = L + N + R$.

As $N \to \infty$, we have $\mu \to \infty$ while L, R are fixed, so both ratios tend to zero. □

Narrative

Neutrality acts as a contraction operator. Increasing the neutral component overwhelms polarity and compresses the state toward the apex $(0,0)$. This is reminiscent of dissipative flows in dynamical systems [14], where trajectories collapse toward an attractor.

In Figure 11, as N grows, both commit κ and lean λ decay to zero, pulling the state toward neutrality.

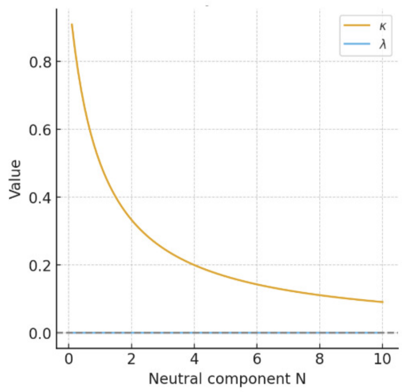

Figure 11. Increasing neutrality N reduces commitment κ while keeping lean λ fixed at zero.

5.3. Axis and Boundaries

The geometry of the *Transition Cone* has three structural features:

- The **central axis** $\lambda = 0$ corresponds to balance ($L = R$).
- The **edges** $|\lambda| = \kappa$ correspond to pure one-sided states (Theorem 2).
- The **apex** $(0,0)$ corresponds to pure neutrality.

Thus, the cone is polyhedral: bounded by straight lines and anchored at a sharp apex.

In Figure 12, the shaded region is the cone. The dotted blue line is the axis of balance. Dashed red lines are the edges. The black point at the origin is the apex of neutrality.

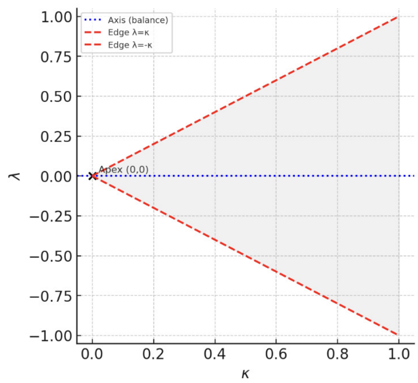

Figure 12. The Transition Cone is bounded by the edges $\lambda = \pm\kappa$ with the balance axis at $\lambda = 0$

5.4. Alternative Coordinates

As shown in Proposition 4, the map

$$(\kappa, \lambda) \mapsto (\sigma, \pi), \quad \sigma = 1 - \kappa, \quad \pi = \frac{|\lambda|}{\kappa}, \qquad (\kappa > 0),$$

transforms the Transition Cone into the rectangle

$$[0,1] \times [0,1],$$

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with an additional binary choice of sign for λ.

Proposition 5 (Rectangular Representation). *The Transition Cone $\mathcal{C}$ is equivalent, under the change of variables, to a prism*

$$\{(\sigma,\pi,\varepsilon) : \sigma,\pi \in [0,1],\ \varepsilon \in \{-1,+1\}\},$$

where $\varepsilon = \mathrm{sgn}(\lambda)$.

Proof. Immediate from the definitions:

- σ encodes neutrality,
- π encodes one-sidedness, and
- ε encodes direction.

□

Narrative

This rectangularization simplifies analysis by turning the triangular cone into a box. Neutrality and purity vary independently between 0 and 1, while lean is reduced to a binary choice of sign. Such reparametrizations are standard in information geometry [14], where alternative coordinate systems allow for analytic tractability.

In Figure 13, the shaded square is the (σ,π) rectangle. Key Trions map cleanly: neutrality at $(1,0)$, balance at $(0,0)$, extremes at $(0,1)$, and mixed states inside.

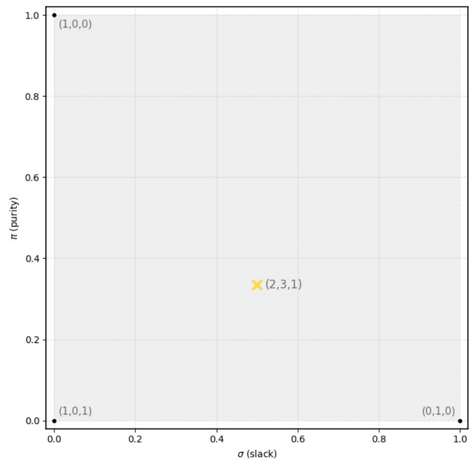

Figure 13. The slack-purity rectangle maps extreme and mixed states, with the point $(2,3,1)$ lying in the interior.

5.5. Geometric Interpretation

The Transition Cone can be viewed as a restricted cross-section of the probability simplex. Unlike the simplex, where coordinates are interchangeable [6], Symmetria distinguishes:

- neutrality as unique,
- left/right as mirror-symmetric but not permutable.

Thus, Symmetria space is closer to cones in convex optimization [14] and to symmetry-breaking frameworks in physics [15] than to the symmetric simplex.

5.6. Summary

The geometry of Symmetria is governed by:

- Convexity under mixtures of Trions,
- Neutrality contraction pulling states to the apex,
- A polyhedral structure with clear axis and boundaries,
- A rectangular coordinate representation in (σ,π).

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This geometric framework prepares the ground for dynamical and spectral analysis in Section 6.

6. Dynamics and Spectral Aspects

The static geometry of the Transition Cone is only the beginning. Real systems evolve in time, and Trions can change as $L(t), N(t), R(t)$ vary. This section studies how trajectories behave inside the cone and what happens near its boundaries, how oscillatory dynamics fit within it, and how harmonic analysis enters through the Symmetria Laplacian.

6.1. Trajectories in the Cone

A trajectory in Symmetria space is a continuous map

$$t \mapsto (\kappa(t), \lambda(t)) \in \mathcal{C}, \quad t \in I \subseteq \mathbb{R}.$$

Such trajectories may arise from time-dependent Trions $(L(t), N(t), R(t))$ with positive mass $\mu(t)$.

Proposition 6 (Invariance of the Cone). *If $(L(t), N(t), R(t)) \in \mathbb{R}^3_{\geq 0}$ for all t, then the associated trajectory $(\kappa(t), \lambda(t))$ remains in $\mathcal{C}$.*

Proof. By Theorem 3.4, every Trion projects into $\mathcal{C}$.
Since $(L(t), N(t), R(t))$ always defines a valid Trion, its projection remains in $\mathcal{C}$ for all t. □

Narrative

This invariance parallels the forward invariance of simplices in replicator dynamics [12]. It guarantees that time evolution never "escapes" the cone: the geometry itself acts as a natural constraint.

As shown in Figure 14, a decaying oscillatory trajectory remains entirely inside the cone, moving from a high-commitment state (green) toward the neutral apex (red).

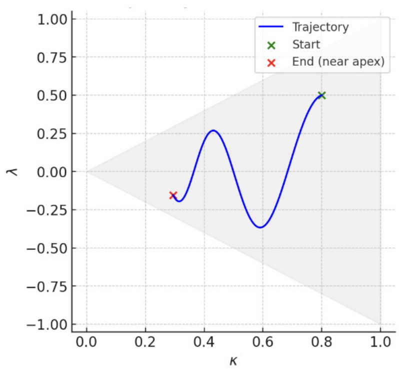

Figure 14. A sample trajectory within the Transition Cone shows oscillatory lean λ converging toward higher commitment κ.

6.2. Boundary Behavior

The cone's boundaries impose sharp dynamical constraints:

- If $|\lambda(t)| = \kappa(t)$, one side is absent: the trajectory runs along an edge.
- If $\kappa(t) = 0$, the trajectory collapses to the apex (pure neutrality).
- If $|\lambda(t)| < \kappa(t)$, both sides coexist, keeping the trajectory interior.

Narrative

Boundaries are not arbitrary; they represent degenerate configurations. Once a trajectory hits an edge, it encodes complete dominance of one side. The apex is even more restrictive: total loss of polarity.

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In Figure 15, dashed red lines show the edges, the dotted blue axis shows balance, and the black dot marks the neutral apex.

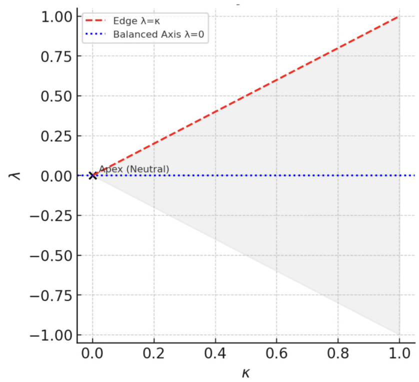

Figure 15. The Transition Cone highlights the edge $\lambda = \kappa$ and the balanced axis $\lambda = 0$ meeting at the neutral apex.

6.3. Spectral Ellipses

Consider oscillatory trajectories of the form

$$\lambda(t) = b + c\sin(t), \quad \kappa(t) = a\cos(t). \tag{36}$$

For admissibility, parameters must satisfy $|\lambda(t)| \leq \kappa(t)$ for all t.

Proposition 7 (Cone Constraint on Oscillations)**.** *Elliptical trajectories are admissible only if amplitudes a, b, c are such that the inequality above holds globally.*

Proof. By Theorem 3.4, all admissible points must remain inside $\mathcal{C}$.

For ellipses, this requires that the oscillation never crosses the boundaries $|\lambda| = \kappa$.

Thus, feasibility reduces to amplitude inequalities. □

Narrative

These spectral ellipses highlight how oscillatory dynamics interact with geometry: not all harmonic motions are allowed, only those confined to the cone. The conditions echo positivity constraints in oscillatory dynamical systems [14].

In Figure 16, an oscillatory loop (purple) remains entirely inside the cone when amplitudes are chosen carefully.

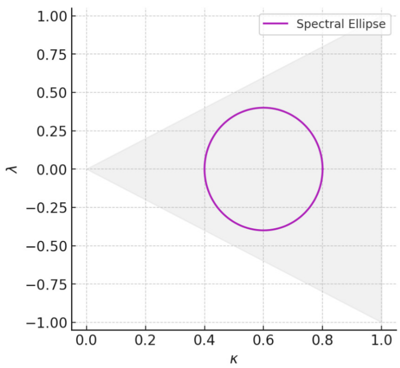

Figure 16. A spectral ellipse illustrates bounded oscillations within the Transition Cone.

6.4. The Symmetria Laplacian

To study harmonic functions on $\mathcal{C}$, we define the operator:

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$$\Delta_{\text{sym}} f(\kappa, \lambda) = \frac{\partial^2 f}{\partial \kappa^2} + \frac{\partial^2 f}{\partial \lambda^2}. \tag{37}$$

This operator is the Euclidean Laplacian restricted to the cone's domain.

Theorem 7 (Boundary Constraints for Harmonic Functions). *If $f : \mathcal{C} \to \mathbb{R}$ satisfies $\Delta_{sym} f = 0$, then boundary values along $\partial\mathcal{C}$ determine f uniquely inside $\mathcal{C}$.*

Proof. Standard uniqueness theorems for harmonic functions on convex domains apply [8].

Since $\mathcal{C}$ is convex and bounded in λ for each fixed κ, the boundary values determine the solution uniquely. □

Narrative

This theorem connects Symmetria to Dirichlet problems on cones [11]. Just as harmonic functions on a disk are fixed by their boundary values, harmonic functions on the Transition Cone are governed entirely by its edges.

Contours of a mock harmonic function inside the cone, illustrating the oscillatory patterns constrained by the geometry, are shown in Figure 17.

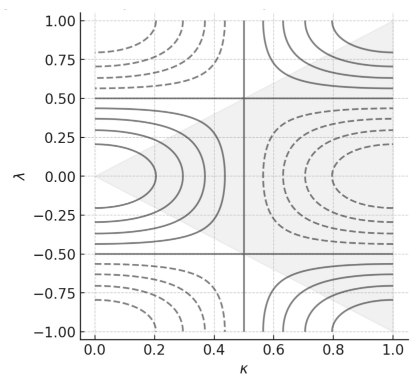

Figure 17. Contour plot of oscillatory modes over the Transition Cone, highlighting symmetry and periodicity in (κ, λ).

6.5. Dynamical Interpretations

- **Neutrality growth** ($N(t)$ large) contracts states toward the apex.
- **Balanced reinforcement** ($L = R$) keeps trajectories on the vertical axis.
- **Spectral analysis** with Δ_{sym} suggests an eigenbasis adapted to the cone, analogous to spherical harmonics on spheres.

6.6. Summary

Dynamics in Symmetria space are governed by:

1. **Invariance:** trajectories always remain in $\mathcal{C}$.
2. **Boundaries:** edges and apex encode degenerate dynamics.
3. **Spectral ellipses:** oscillations are constrained by the cone's inequalities.
4. **Symmetria Laplacian:** boundary values determine harmonic states.

This framework bridges geometry and analysis, opening the way for a spectral theory of Symmetria.

7. Discussion and Applications

The preceding sections built Symmetria as a mathematical theory. To ground the framework, we now show how Trions and the Transition Cone arise naturally in real-world systems. These case studies highlight both the interpretive power and predictive potential of Symmetria.

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7.1. *Opinion Dynamics: Polarization with a Neutral Middle*

In political science, populations are often divided into left, right, and undecided [12]. Standard approaches collapse undecideds into statistical noise or treat them as a third "party," but this ignores their mediating role.

- A Trion (L, N, R) directly encodes left-leaning (L), right-leaning (R), and neutral (N) populations.
- **Commit** κ measures the fraction of polarized individuals, distinguishing highly engaged societies from disengaged ones.
- **Lean** λ quantifies the net directional tilt.
- **Purity** π measures the dominance of one side among those who are committed.

Case Study: Survey Data

Suppose a poll yields

$$L = 40\%, \quad R = 30\%, \quad N = 30\%.$$

Normalizing:

$$\kappa = L + R = 0.7, \quad \lambda = L - R = 0.1, \quad \pi = \frac{|\lambda|}{\kappa} \approx 0.14. \tag{38}$$

Interpretation

Most individuals are engaged ($\kappa = 0.7$), society leans slightly left ($\lambda = 0.1$), but purity is low ($\pi = 0.14$), meaning both sides are still coexisting without extreme asymmetry.

Here, Symmetria provides a precise, geometry-based polarization index, more informative than a raw percentage split.

7.2. *Statistical Classification with Abstention*

In machine learning, classifiers sometimes include a "reject" option, abstaining when uncertainty is high. This naturally yields three categories: Class A, Class B, and Abstain.

- A Trion (L, N, R) can represent fractions of predictions in each category.
- **Slack** $\sigma = \frac{N}{\mu}$ measures the abstention rate.
- **Purity** π quantifies bias when the classifier does commit.

Case Study: Medical Diagnosis

Imagine an AI diagnostic system where:

$$L = 600 \quad \text{patients classified positive,}$$

$$R = 550 \quad \text{classified negative,}$$

$$N = 350 \quad \text{flagged as "uncertain."}$$

Then:

$$\kappa = \frac{600 + 550}{1500} = 0.77, \quad \lambda = \frac{600 - 550}{1500} = 0.03, \quad \sigma = 0.23. \tag{39}$$

Interpretation

The system commits 77% of the time, is nearly balanced (λ small), and abstains in 23% of cases. The geometry of Symmetria lets us visualize abstention as a contraction toward the apex, rather than treating it as an external error mode.

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7.3. Physics: Competing Phases and Disorder

In condensed matter physics, systems often exist in one of two ordered phases, with an intermediate disordered state [11].

- Example: magnetization in a ferromagnet. Left/right corresponds to spin-up/spin-down order, while neutrality corresponds to high-temperature disorder.
- The Transition Cone model shows how increasing disorder (temperature) contracts states toward neutrality, while cooling sharpens polarity.

Case Study: Magnetization

At low temperature,

$$L = 0.9, \quad R = 0.1, \quad N = 0.$$

Then

$$\kappa = 1, \quad \lambda = 0.8, \quad \pi = 0.8$$

strong order, right at the cone's edge.

At high temperature,

$$L = 0.45, \quad R = 0.45, \quad N = 0.1.$$

Then

$$\kappa = 0.9, \quad \lambda = 0, \quad \pi = 0$$

high commitment but perfectly balanced.

At very high temperature,

$$L = 0.33, \quad R = 0.33, \quad N = 0.34.$$

Then

$$\kappa = 0.66, \quad \lambda = 0, \quad \pi = 0$$

neutrality dominates, dragging the state toward the apex.

This simple mapping shows how phase transitions appear geometrically as trajectories within $\mathcal{C}$.

7.4. Information Geometry and Communication Systems

In communication theory, signals may fall into bit 0, bit 1, or undecided/noise [6]. Trions provide a normalized structure:

- Commit κ measures effective transmission.
- Slack σ measures erasures.
- Purity π captures whether noise skews toward false 0s or false 1s.

Case Study: Noisy Channel

If a channel produces

$$L = 480, \quad R = 470, \quad N = 50$$

out of 1000 bits:

$$\kappa = 0.95, \quad \lambda = 0.01, \quad \sigma = 0.05.$$

Interpretation

Transmission is highly reliable (κ close to 1), slack is small (few erasures), and lean is nearly neutral. Compared to Shannon entropy, Symmetria provides a geometric bias measure that highlights even small asymmetries.

7.5. Open Problems

- **Spectral theory:** Develop cone harmonics to analyze oscillatory political polarization or phase oscillations in physics.
- **Higher-dimensional generalizations:** Extend Trions to quadruple systems (two polarized axes + neutrality), capturing multipolar societies.
- **Dynamical models:** Apply replicator-like equations to evolving polarization in elections or shifting phase states in materials.
- **Metric geometry:** Define distances that compare polarization intensities across countries or order-disorder transitions across materials.

7.6. Summary

Symmetria's strength lies in turning messy three-way states into clean geometry. Whether in politics, machine learning, physics, or communication systems, the framework offers invariants that are interpretable, scale-free, and mathematically tractable. In each domain, the Transition Cone acts as both a constraint and a map of possibilities, showing that triadic balance is not incidental but structural.

8. Conclusion

This paper introduced Symmetria, the mathematical framework of Trions, as a geometry tailored to systems with two opposing poles and a mediating neutral state. Starting from the elementary decomposition (L, N, R), we defined the invariants of mass, commit, and lean and showed that their normalized representation always lies within the Transition Cone. Through the Normalization, Edge, and Mirror Theorems, we established that all admissible states form a closed convex cone bounded by sharp linear constraints and governed by reflection symmetry.

Beyond these fundamentals, we developed derived measures (slack, purity, and balance ratio) that refine the description of Trions. We proved algebraic identities such as the quadratic coexistence law and the purity-balance relation, showing that these quantities are not arbitrary but part of a tightly interwoven structure. By introducing coordinate transformations, we mapped the cone into rectangular domains, clarifying the complementary roles of neutrality and polarity.

On the geometric side, we demonstrated convexity, characterized neutrality as a contraction toward the apex, and described the central axis and extreme boundaries. Extending from static to dynamic, we studied trajectories evolving inside the cone, showed how oscillatory motions trace spectral ellipses subject to strict feasibility constraints, and introduced the Symmetria Laplacian, which opens a spectral and harmonic perspective on triadic systems.

Taken together, these results demonstrate that Symmetria is both elementary and fertile. Elementary, because it emerges from nothing more than three nonnegative quantities. Fertile, because it yields a rich interplay of algebra, geometry, and analysis that resonates with established domains: convex geometry, probability simplices, information geometry, and symmetry-breaking physics.

The framework also shows promise far beyond mathematics. In opinion dynamics, Symmetria offers clean measures of polarization that respect the role of undecided populations. In classification theory, it provides a geometry for abstention rather than treating it as a statistical error. In physics, it models two-phase systems with disordered intermediates. In information theory, it suggests new divergence-like coordinates rooted in neutrality and polarity.

Future directions include the development of a spectral theory of the Symmetria Laplacian, complete with eigenfunction expansions analogous to spherical harmonics; higher-dimensional generalizations to multipolar systems with layered neutrality; dynamical models that explore attractors and bifurcations inside the Transition Cone; and the design of natural metrics that measure distances between polarized states.

In closing, Symmetria offers a new lens: it formalizes the intuition that between opposites lies not emptiness but structure. Neutrality, polarity, and symmetry combine into a coherent analytic framework, one that is mathematically rigorous, geometrically elegant, and potentially transformative across disciplines.

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