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Microlocal Energy Dissipation in Semiclassical Pseudodifferential Equations: From Wavefront Sets to Hamiltonian Flows

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Abstract

This paper develops a rigorous semiclassical and microlocal framework to analyze energy dissipation and singularity propagation in systems governed by dissipative pseudodifferential operators. We consider semiclassical operators P_h with symbols $a(x,\xi) \in S^m(\mathbb{R}^n)$, where $m < 0$, and study families of solutions u_h to $P_h u_h = 0$. By constructing microlocal partitions of unity, we derive asymptotic formulas for the local and total dissipation rate ε_h, proving its convergence to an explicit phase-space integral as $h \to 0^+$. Our analysis reveals that the semiclassical wavefront set $\mathrm{WF}_h(u_h)$ propagates along the generalized bicharacteristics of the real part of the principal symbol $p_0(x,\xi)$, while the imaginary part $\mathrm{Im}\, p_0(x,\xi) \leq 0$ governs the amplitude decay along these flows. This establishes a precise, quantitative relationship between geometric propagation and physical dissipation, unifying the description of singularity transport and energy loss in the semiclassical regime. The framework bridges deterministic and statistical perspectives, offering a robust tool for investigating multiscale energy dynamics in turbulent systems and beyond.

Keywords: semiclassical pseudodifferential operators; microlocal analysis; turbulence dissipation; multiscale energy convergence

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1. Introduction

Turbulence research has undergone profound transformations since Kolmogorov's seminal work in 1941 [1], which established statistical frameworks to describe energy cascades in turbulent flows. While these models have provided foundational insights, their inherent limitations in capturing deterministic, localized phenomena have motivated the development of more refined mathematical tools. The need to bridge macroscopic statistical descriptions with microscale deterministic dynamics remains a central challenge in fluid mechanics and related fields.

Recent years have witnessed significant advancements in semiclassical and microlocal analysis, particularly in their application to complex systems. Microlocal techniques, as developed in [2] and extended in works such as [5,6], have proven instrumental in dissecting the spatial-spectral interactions that underpin turbulent energy dissipation. These methods enable a precise characterization of how energy is transferred and dissipated across scales, addressing gaps left by purely statistical approaches.

Pseudodifferential operators have emerged as a cornerstone of modern multiscale analysis, offering unparalleled localization properties in phase space [3]. Their role in modeling turbulent flows has been further solidified by recent studies, such as those by [7] and [8], which demonstrate their efficacy in capturing anisotropic and inhomogeneous dissipation mechanisms. Complementing this, semiclassical quantization techniques [4] have been refined to accommodate systems with spatially and spectrally varying coefficients, as seen in applications to quantum chaos [9] and wave turbulence [10].

A particularly promising direction has been the integration of semiclassical methods with data-driven approaches, such as those explored in [11]. These hybrid frameworks leverage symbolic calculus

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and localized spectral analysis to not only elucidate fundamental dissipation mechanisms but also to inform the development of reduced-order models for high-Reynolds-number flows. Such advances are critical for addressing the computational and theoretical challenges posed by fully resolved simulations of turbulent systems.

This paper introduces a rigorous semiclassical and microlocal framework to analyze energy dissipation in systems governed by dissipative pseudodifferential operators. Our approach builds on recent theoretical breakthroughs, including the works of [12] on propagation of singularities and [13] on microlocal defect measures, to reconcile deterministic and statistical perspectives. By constructing microlocal partitions of unity and leveraging the symbolic calculus of semiclassical operators, we derive asymptotic formulas for localized dissipation rates and establish their convergence to phase-space integrals. This framework not only clarifies the geometric and dynamical underpinnings of energy dissipation but also provides a robust foundation for future investigations into highly turbulent and dissipative systems.

Through this work, we aim to contribute to the broader effort of unifying microlocal, semiclassical, and statistical approaches, thereby advancing both the theoretical understanding and practical modeling of complex multiscale phenomena.

2. Mathematical Fundamentals

This section establishes the rigorous mathematical framework of **semiclassical analysis**, a powerful tool for describing multiscale phenomena where a clear separation exists between macroscopic dynamics and microscopic fluctuations. The central objects of study are **semiclassical pseudodifferential operators**, whose symbol calculus provides a precise phase-space description of energy distribution and transfer across scales. This framework is particularly relevant in physical applications such as quantum mechanics, wave propagation, and turbulence, where the interplay between different scales is crucial.

2.1. Semiclassical Pseudodifferential Operators

The semiclassical formalism introduces a small, dimensionless parameter $h > 0$, encoding the ratio between microscopic and macroscopic scales. In physical contexts, h may represent, for example, the ratio between the Kolmogorov dissipation scale and a characteristic flow length scale, or the (reduced) Planck constant in quantum mechanics. The asymptotic regime $h \to 0^+$ corresponds to the classical limit, while keeping $h > 0$ fixed allows one to resolve fine-scale oscillations.

Let $\mathcal{S}(\mathbb{R}^n)$ denote the *Schwartz space* of smooth, rapidly decreasing functions on $\mathbb{R}^n$. A *semiclassical pseudodifferential operator* P_h is defined via a symbol $a(x, \xi; h)$ belonging to a suitable semiclassical symbol class. For definiteness, one may assume $a \in S^m_\delta(\mathbb{R}^{2n})$, i.e.

$$|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)| \le C_{\alpha,\beta} h^{-\delta(|\alpha|+|\beta|)} \langle \xi \rangle^{m-|\beta|}, \quad \forall \alpha, \beta \in \mathbb{N}^n,$$

for some order $m \in \mathbb{R}$ and loss parameter $0 \le \delta < 1/2$. Such a symbol defines, by the *Weyl quantization*, a family of operators

$$\begin{aligned} (P_h u)(x) &= \mathrm{Op}_h(_)h^w(a)u(x) \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\left(\frac{x+y}{2}, \xi; h\right) u(y)\, dy\, d\xi. \end{aligned} \tag{1}$$

for all $u \in \mathcal{S}(\mathbb{R}^n)$. Here $\mathrm{Op}_h(_)h^w(a)$ denotes the semiclassical Weyl quantization of the symbol a. When $h = 1$, (1) reduces to the classical (non-semiclassical) Weyl quantization, and in particular recovers the Kohn–Nirenberg calculus under appropriate symbol conventions.

For reference, the (non-scaled) Fourier transform of u is given by

$$\hat{u}(\xi) = \mathcal{F}[u](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\, dx, \tag{2}$$

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while its semiclassical version, naturally adapted to (1), reads

$$\mathcal{F}_h[u](\xi) = \frac{1}{(2\pi h)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{i}{h}x\cdot\xi} u(x)\, dx.$$

Composition formula (Moyal product).

If $a, b \in S^m_\delta$ are two semiclassical symbols, then the composition of the corresponding operators satisfies

$$\mathrm{Op}_h(_)h^w(a)\,\mathrm{Op}_h(_)h^w(b) = \mathrm{Op}_h(_)h^w(a\#b),$$

where the *Moyal product* $a\#b$ admits the asymptotic expansion

$$a\#b \sim \sum_{k=0}^{\infty} \left(\frac{ih}{2}\right)^k \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\,\beta!}\, \partial_\xi^\alpha \partial_x^\beta a\, \partial_\xi^\beta \partial_x^\alpha b, \tag{3}$$

with the leading terms explicitly given by

$$a\#b = ab + \frac{ih}{2}\{a,b\} + \mathcal{O}(h^2), \qquad \{a,b\} = \partial_\xi a \cdot \partial_x b - \partial_x a \cdot \partial_\xi b,$$

where $\{a,b\}$ is the canonical Poisson bracket on $T^*\mathbb{R}^n$. This expansion, valid under the above symbol assumptions, is the cornerstone of the semiclassical pseudodifferential calculus: it shows that operator composition corresponds to the *star product* of their symbols, recovering the classical Poisson algebra in the limit $h \to 0^+$.

Semiclassical commutator.

In particular, the commutator of two semiclassical Weyl operators satisfies

$$[\mathrm{Op}_h^w(a), \mathrm{Op}_h^w(b)] = \frac{h}{i}\, \mathrm{Op}_h^w(\{a,b\}) + \sum_{k\geq 2} c_k\, h^{2k-1}\, \mathrm{Op}_h^w(C_k(a,b)), \tag{4}$$

in the sense of asymptotic expansions as $h \to 0^+$. Thus, up to $\mathcal{O}(h^3)$ corrections, the commutator corresponds to the Weyl quantization of the Poisson bracket of the symbols. This is the semiclassical analogue of the canonical commutation relations and underlies the Egorov–Ehrenfest theorem linking quantum and classical dynamics.

The function $a(x,\xi;h) \in C^\infty(\mathbb{R}^{2n} \times (0,1])$ is called the *symbol* of $\mathrm{Op}_h^w(a)$. It encodes the microlocal behaviour of the operator in phase space $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. Composition formulas, asymptotic expansions and Egorov-type theorems hold in this semiclassical calculus under the above symbol assumptions, providing a bridge between quantum-like operators and their classical counterparts.

2.1.1. Symbol Classes and Calculus

To ensure that the operator P_h maps $\mathcal{S}(\mathbb{R}^n)$ to itself and possesses a well-defined asymptotic expansion, the symbol $a(x,\xi;h)$ must belong to a suitable class. We say that $a \in S^m(\mathbb{R}^{2n})$, the **semiclassical symbol class of order** $m \in \mathbb{R}$, if for all multi-indices $\alpha, \beta \in \mathbb{N}^n$, it satisfies the uniform estimates:

$$|\partial_x^\alpha \partial_\xi^\beta a(x,\xi;h)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|}, \quad \forall (x,\xi) \in \mathbb{R}^{2n}, \tag{5}$$

where $\langle \xi \rangle = (1+|\xi|^2)^{1/2}$ and the constants $C_{\alpha,\beta} > 0$ are independent of h, x, and ξ. The order m controls the frequency sensitivity of the operator:

- If $m < 0$, the operator is **regularizing**, meaning it smooths out high-frequency components (e.g., dissipation operators in turbulence).
- If $m = 0$, the operator is **bounded** on $L^2(\mathbb{R}^n)$ (by the Calderón-Vaillancourt theorem).

- If $m > 0$, the operator is **unbounded** and may amplify high-frequency components (e.g., differential operators).

The class S^m is stable under differentiation and forms an algebra under multiplication, which is foundational for computing compositions, commutators $[P_h, Q_h] = P_h Q_h - Q_h P_h$, and adjoints P_h^*.

2.1.2. Asymptotic Expansion and Multiscale Hierarchy

A key feature of semiclassical analysis is the ability to describe operators through **asymptotic expansions** in powers of h. This allows for an explicit separation of dynamics at different scales. We often consider symbols $a(x, \xi; h)$ admitting an expansion:

$$a(x,\xi;h) \sim \sum_{k=0}^{\infty} h^k a_k(x,\xi), \quad \text{with } a_k \in S^{m-k}, \tag{6}$$

where the equivalence $\sim$ means that for any $N \in \mathbb{N}$,

$$a(x,\xi;h) - \sum_{k=0}^{N-1} h^k a_k(x,\xi) \in h^N S^{m-N}.$$

The leading term $a_0(x, \xi)$ is called the **principal symbol** and governs the dominant behavior of the operator as $h \to 0$. Substituting (6) into (1) yields the **multiscale expansion** of the operator:

$$(P_h u)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} \left(\sum_{k=0}^{\infty} h^k a_k\left(\frac{x+y}{2}, \xi\right) \right) u(y)\, dy\, d\xi. \tag{7}$$

This expansion is the mathematical manifestation of scale separation, where each term $h^k \mathrm{Op}_h()h(a_k)$ provides a finer correction to the leading-order dynamics. The principal symbol $a_0(x, \xi)$ determines the **classical limit** of the operator, while the higher-order terms $a_k(x, \xi)$ for $k \geq 1$ capture **quantum corrections** or **multiscale interactions**.

2.1.3. Composition and Adjoints

The **composition** of two semiclassical pseudodifferential operators $P_h = \mathrm{Op}_h()h(a)$ and $Q_h = \mathrm{Op}_h()h(b)$ is again a semiclassical pseudodifferential operator. Its symbol $c(x, \xi; h)$ admits an asymptotic expansion given by the **Moyal product** (or Weyl product):

$$c(x,\xi;h) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{h^{|\alpha|}}{\alpha!} \left(\frac{i}{2}\right)^{|\alpha|} \partial_\xi^\alpha a(x,\xi;h) \partial_x^\alpha b(x,\xi;h). \tag{8}$$

The leading term of this expansion is the standard product of symbols:

$$c_0(x,\xi) = a_0(x,\xi) b_0(x,\xi).$$

The **adjoint** of $P_h = \mathrm{Op}_h()h(a)$ is also a semiclassical pseudodifferential operator with symbol $a^*(x, \xi; h)$ given by:

$$a^*(x,\xi;h) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{h^{|\alpha|}}{\alpha!} \left(\frac{i}{2}\right)^{|\alpha|} \partial_\xi^\alpha \partial_x^\alpha \overline{a(x,\xi;h)}. \tag{9}$$

For real-valued symbols, the adjoint coincides with the operator itself up to lower-order terms: $P_h^* = P_h + \mathcal{O}(h)$.

2.1.4. Examples and Physical Interpretations

Example 1 (Semiclassical Schrödinger Operator)**.** *The semiclassical Schrödinger operator is given by:*

$$P_h = -h^2 \Delta + V(x),$$

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where $V(x)$ is a smooth potential. Its symbol is:

$$p(x,\xi;h) = |\xi|^2 + V(x) \in S^2(\mathbb{R}^{2n}).$$

The principal symbol $p_0(x,\xi) = |\xi|^2 + V(x)$ governs the classical dynamics of the system, while the higher-order terms capture quantum corrections.

Example 2 (Dissipative Operators in Turbulence)**.** *In turbulence modeling, dissipative operators often take the form:*

$$P_h = \mathrm{Op}_h()h(a), \quad \text{with } a(x,\xi;h) = \nu(\xi) \in S^{-2}(\mathbb{R}^{2n}),$$

where $\nu(\xi)$ is a dissipation profile (e.g., $\nu(\xi) = (1+|\xi|^2)^{-1}$). The principal symbol $a_0(x,\xi) = \nu(\xi)$ encodes the dissipation rate at each frequency ξ.

2.2. Functional Analytic Properties

Semiclassical pseudodifferential operators enjoy strong functional analytic properties:

- **Boundedness**: If $a \in S^0(\mathbb{R}^{2n})$, then $P_h = \mathrm{Op}_h()h(a)$ is uniformly bounded on $L^2(\mathbb{R}^n)$ for $h \in (0,1]$. This is a consequence of the **Calderón-Vaillancourt theorem**.
- **Compactness**: If $a \in S^{-\infty}(\mathbb{R}^{2n}) = \bigcap_{m\in\mathbb{R}} S^m(\mathbb{R}^{2n})$, then P_h is a compact operator on $L^2(\mathbb{R}^n)$ for each fixed h.
- **Gårding Inequality**: If $a \in S^m(\mathbb{R}^{2n})$ is non-negative ($a(x,\xi;h) \geq 0$), then there exists $C > 0$ such that:
$$\langle \mathrm{Op}_h()h(a)u,u\rangle_{L^2} \geq -Ch\|u\|^2_{L^2}, \quad \forall u \in \mathcal{S}(\mathbb{R}^n).$$
This inequality is crucial for proving the **positivity** of certain operators (e.g., energy functionals).

2.3. Connection to Classical Mechanics

The semiclassical formalism provides a rigorous bridge between quantum mechanics ($h > 0$) and classical mechanics ($h \to 0$). The **principal symbol** $p_0(x,\xi)$ of a semiclassical operator P_h often corresponds to a classical observable (e.g., energy, momentum). The **Hamiltonian flow** generated by $p_0(x,\xi)$ describes the evolution of classical observables, while the higher-order terms in the symbol expansion capture quantum corrections.

For example, consider the semiclassical Schrödinger operator $P_h = -h^2\Delta + V(x)$. The associated Hamiltonian system is:

$$\frac{dx}{dt} = \nabla_\xi p_0(x,\xi) = 2\xi, \quad \frac{d\xi}{dt} = -\nabla_x p_0(x,\xi) = -\nabla V(x).$$

This is precisely the **Newtonian dynamics** of a particle in the potential $V(x)$. The semiclassical limit $h \to 0$ thus recovers classical mechanics, while the full semiclassical analysis captures quantum effects.

2.4. Microlocal Analysis and Phase Space Energy Localization

A fundamental tenet of semiclassical analysis is *microlocality*: pseudodifferential operators act locally in *phase space* $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}_{x,\xi}$. This means their action on a function u is not only localized in physical space (x) but also in frequency space (ξ). This property is crucial for defining and analyzing the energy distribution of a function.

Microlocal Decomposition.

To quantify energy localized in specific regions of phase space, one employs a *microlocal partition of unity*. This involves two steps:

1. A spatial partition: Let $\{\chi_j\}_{j=1}^N \subset C_c^\infty(\mathbb{R}^n)$ be such that $0 \leq \chi_j \leq 1$ and $\sum_{j=1}^N \chi_j(x) = 1$.
2. A frequency filter: The operator P_h itself, often chosen with a symbol supported in a specific frequency region (e.g., $|\xi| \sim h^{-1}$).

The local (semiclassical) energy density is then given by $|P_h u(x)|^2$. Its spatial localization is achieved by multiplying by $\chi_j(x)$:

$$E_j(x) = \chi_j(x)|P_h u(x)|^2, \tag{10}$$

so that the integral $\int_{\mathbb{R}^n} E_j(x)dx$ measures the energy of u that is localized in the phase space region $\text{supp}(\chi_j) \times \text{supp}(a)$.

Integral Representation and Stationary Phase Analysis.

The structure of the energy density is revealed by its integral representation. Substituting the definition (**??**) into $|P_h u(x)|^2$ and applying the Fourier inversion formula yields:

$$|P_h u(x)|^2 = \frac{1}{(2\pi h)^{2n}} \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\cdot\xi} a(x,\xi;h)\overline{a(y,\xi;h)}u(y)\overline{u(x)}d\xi dy. \tag{11}$$

The oscillatory kernel $e^{\frac{i}{h}(x-y)\cdot\xi}$ is highly oscillatory for $h \ll 1$, implying that the dominant contribution to the integral comes from regions where the phase is stationary. For fixed x, the phase $\Phi(y,\xi) = (x-y)\cdot\xi/h$ has a critical point at $y = x$. Applying the semiclassical stationary phase lemma yields:

$$\int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} d\xi = (2\pi h)^n \delta(x-y) + \mathcal{O}(h^\infty),$$

where δ is the Dirac distribution. This demonstrates the microlocalization: the ξ-integration forces y to be exceedingly close to x as $h \to 0$. Inserting this into (11) and using the asymptotic expansion (**??**) gives the leading-order behavior:

$$|P_h u(x)|^2 \sim \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} |a_0(x,\xi)|^2 |u(x)|^2 d\xi, \qquad h \to 0^+. \tag{12}$$

This result is the pointwise origin of the concept of *semiclassical measures*, which describe the limiting L^2 energy distribution of a sequence of functions $\{u_h\}$ in phase space as $h \to 0$.

Propagation of Singularities.

For solutions u_h to a homogeneous equation $P_h u_h = 0$, the theory describes the evolution of their *semiclassical wavefront set* $\text{WF}_h(u_h)$—the set of phase space points (x,ξ) where u_h is not $\mathcal{O}(h^\infty)$. If P_h has principal symbol $p_0(x,\xi)$, then $\text{WF}_h(u_h)$ is contained in the characteristic set $\{p_0(x,\xi) = 0\}$ and is invariant under the Hamiltonian flow generated by $H_{\text{Re}\, p_0}$, the Hamiltonian vector field of the real part of p_0:

$$\frac{dx}{dt} = \partial_\xi \,\text{Re}\, p_0(x,\xi), \quad \frac{d\xi}{dt} = -\partial_x \,\text{Re}\, p_0(x,\xi).$$

Simultaneously, the amplitude of the solution evolves along these *bicharacteristics* according to the imaginary part of the symbol:

$$\frac{d}{dt} \log|u_h(x(t))|^2 \sim 2\, \text{Im}\, p_0(x(t),\xi(t)).$$

This result provides a rigorous formulation of the physical principle that energy, or singularities, propagate along rays determined by the principal symbol.

2.5. The Wigner Transform and Semiclassical Measures

Definition and Basic Properties

Let $u \in L^2(\mathbb{R}^n)$ and $h > 0$ be a semiclassical parameter. The *Wigner transform* of u is defined by

$$W_h[u](x,\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\xi}\, u\left(x + \frac{hy}{2}\right) \overline{u}\left(x - \frac{hy}{2}\right) dy, \tag{13}$$

where $(x,\xi) \in T^*\mathbb{R}^n \simeq \mathbb{R}^{2n}$ are the position and momentum variables. This defines a distribution $W_h[u] \in \mathcal{S}'(\mathbb{R}^{2n})$, encoding both position and momentum information.

Key properties include:

- **Sesquilinearity and Reality:** $W_h[u]$ is real-valued and sesquilinear in u.
- **Marginals:** Integration over momentum or position recovers the spatial and momentum densities:

$$\int_{\mathbb{R}^n} W_h[u](x,\xi)\,d\xi = |u(x)|^2, \quad \int_{\mathbb{R}^n} W_h[u](x,\xi)\,dx = \frac{1}{h^n}|\hat{u}(\xi/h)|^2.$$

- **Energy Conservation:** $\iint W_h[u]\,dx\,d\xi = \|u\|_{L^2}^2$.
- **Orthogonality:** For $u,v \in L^2(\mathbb{R}^n)$,

$$\iint W_h[u](x,\xi)\,W_h[v](x,\xi)\,dx\,d\xi = \frac{1}{(2\pi h)^n}|\langle u,v\rangle|^2.$$

- **Connection to Weyl Quantization:** For a symbol $a \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\langle \mathrm{Op}_h()h^w(a)u,u\rangle = \iint a(x,\xi)W_h[u](x,\xi)\,dx\,d\xi.$$

- **Non-Positivity:** $W_h[u]$ may take negative values; by Hudson's theorem it is strictly positive iff u is Gaussian.
- **Moyal Product:** For products of operators,

$$W_h[uv] = W_h[u] \star_h W_h[v], \quad \star_h = \exp\Big(\frac{ih}{2}(\overleftarrow{\nabla}_x \cdot \overrightarrow{\nabla}_\xi - \overleftarrow{\nabla}_\xi \cdot \overrightarrow{\nabla}_x)\Big).$$

Semiclassical Measures

Let $\{u_h\}_{h>0} \subset L^2(\mathbb{R}^n)$ be uniformly bounded, $\|u_h\|_{L^2} = 1$. Then there exists a subsequence $h_k \to 0$ and a positive Radon measure $\mu \in \mathcal{M}^+(\mathbb{R}^{2n})$ such that

$$W_{h_k}[u_{h_k}] \rightharpoonup \mu \quad \text{in } \mathcal{D}'(\mathbb{R}^{2n}).$$

The measure μ is called a *semiclassical measure* or *Wigner measure*. It satisfies:

- **Normalization:** $\mu(\mathbb{R}^{2n}) = 1$.
- **Propagation under Hamiltonian Flow:** If $ih\partial_t u_h = \mathrm{Op}_h()h^w(H)u_h$ with real-valued $H \in C^\infty(T^*\mathbb{R}^n)$, then μ_t evolves according to the Liouville equation

$$\partial_t \mu_t + \{H, \mu_t\} = 0.$$

- **Weak Limit of Observables:** For $a \in C_c^\infty(T^*\mathbb{R}^n)$,

$$\lim_{h\to 0}\langle \mathrm{Op}_h()h^w(a)u_h,u_h\rangle = \iint a(x,\xi)\,d\mu(x,\xi).$$

Gaussian Example in 1D

Consider

$$u_h(x) = (\pi h)^{-1/4}\,e^{-(x-x_0)^2/(2h)}e^{i\xi_0 x/h}, \quad x_0,\xi_0 \in \mathbb{R}.$$

Then

$$W_h[u_h](x,\xi) = \frac{1}{\pi}\exp\Big(-\frac{(x-x_0)^2+(\xi-\xi_0)^2}{h}\Big),$$

and in the limit $h \to 0$,

$$W_h[u_h] \rightharpoonup \delta(x-x_0)\delta(\xi-\xi_0),$$

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so the semiclassical measure is $\mu = \delta_{(x_0,\xi_0)}$.

Generalization to Mixed States

For a density matrix $\rho_h = \sum_j \lambda_j |u_{h,j}\rangle\langle u_{h,j}|$ with $\sum_j \lambda_j = 1$,

$$W_h[\rho_h] = \sum_j \lambda_j W_h[u_{h,j}], \quad \mu = \sum_j \lambda_j \mu_j,$$

where μ_j is the semiclassical measure associated with $u_{h,j}$.

Summary of Key Properties

1. **Marginals:** recover position and momentum densities.
2. **Weak positivity:** holds under integration against nonnegative symbols.
3. **Boundedness:** $\|W_h[u]\|_{\mathcal{S}'} \leq \|u\|_{L^2}^2$.
4. **Semiclassical limit:** yields positive Radon measures.
5. **Flow invariance:** under Hamiltonian dynamics.

2.6. Wigner Transform, Semiclassical Operators, and Egorov Dynamics

Wigner Transform and Basic Properties

Let $u \in L^2(\mathbb{R}^n)$ and $h > 0$ a semiclassical parameter. The *Wigner transform* of u is

$$W_h[u](x,\xi) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-iy\cdot\xi} u\Big(x + \frac{hy}{2}\Big) \overline{u}\Big(x - \frac{hy}{2}\Big) dy. \tag{14}$$

It defines a phase-space distribution $W_h[u] \in \mathcal{S}'(T^*\mathbb{R}^n)$, with marginals and energy properties:

$$\int W_h[u]\, d\xi = |u(x)|^2, \quad \int W_h[u]\, dx = \frac{1}{h^n} |\hat{u}(\xi/h)|^2, \quad \iint W_h[u]\, dx\, d\xi = \|u\|_{L^2}^2.$$

Semiclassical Pseudodifferential Operators

For a symbol $a(x,\xi;h) \in S^m_\delta$, define the Weyl-quantized operator

$$\mathrm{Op}_h()h^w(a)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}(x-y)\cdot\xi} a\Big(\frac{x+y}{2}, \xi; h\Big) u(y)\, dy\, d\xi. \tag{15}$$

Then

$$\langle \mathrm{Op}_h()h^w(a)u, u\rangle = \iint a(x,\xi) W_h[u](x,\xi)\, dx\, d\xi,$$

linking Wigner distributions to semiclassical observables.

Egorov Theorem and Propagation of Wigner Measures

Let $H \in C^\infty(T^*\mathbb{R}^n)$ be real-valued and $P_h = \mathrm{Op}_h()h^w(H)$ the associated semiclassical operator. The unitary propagator is $U_h(t) = \exp\Big(-\frac{i}{h} t P_h\Big)$. The semiclassical Egorov theorem states that for any $a \in S^m_\delta$:

$$U_h(t)^* \mathrm{Op}_h()h^w(a) U_h(t) = \mathrm{Op}_h()h^w(a \circ \Phi^t) + \mathcal{O}(h),$$

where Φ^t is the Hamiltonian flow generated by H:

$$\dot{x} = \partial_\xi H(x,\xi), \quad \dot{\xi} = -\partial_x H(x,\xi), \quad (x(0),\xi(0)) = (x,\xi).$$

Implication for Wigner Measures.

If $u_h(t) = U_h(t)u_h$ and $W_h[u_h] \rightharpoonup \mu_0$ as $h \to 0$, then

$$W_h[u_h(t)] \rightharpoonup \mu_t = (\Phi^t)_* \mu_0,$$

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i.e., the semiclassical measure is transported along the classical Hamiltonian flow. Equivalently, μ_t satisfies the Liouville equation

$$\partial_t \mu_t + \{H, \mu_t\} = 0.$$

Semiclassical Limit Summary

- The Wigner transform links functions u_h with phase-space distributions $W_h[u_h]$.
- Weyl quantized operators act on u_h and correspond to observables in phase space.
- Egorov theorem shows that, up to $\mathcal{O}(h)$, quantum evolution of observables corresponds to classical Hamiltonian flow.
- In the limit $h \to 0$, the Wigner distributions converge (in the sense of measures) to semiclassical measures μ_t, which evolve classically.

Example: Gaussian Wavepacket in 1D

For $u_h(x) = (\pi h)^{-1/4} e^{-(x-x_0)^2/(2h)} e^{i\xi_0 x/h}$, one finds

$$W_h[u_h](x,\xi) = \frac{1}{\pi} \exp\left(- \frac{(x-x_0)^2 + (\xi - \xi_0)^2}{h} \right) \to \delta(x - x_0)\delta(\xi - \xi_0),$$

so $\mu_0 = \delta_{(x_0,\xi_0)}$, which under classical evolution follows $\mu_t = \delta_{\Phi^t(x_0,\xi_0)}$.

Gaussian Wavepacket in 3D

Consider a 3D Gaussian wavepacket

$$u_h(\mathbf{x}) = (\pi h)^{-3/4} \exp\left(- \frac{|\mathbf{x} - \mathbf{x}_0|^2}{2h} \right) \exp\left(\frac{i\boldsymbol{\xi}_0 \cdot \mathbf{x}}{h} \right), \quad \mathbf{x}_0, \boldsymbol{\xi}_0 \in \mathbb{R}^3.$$

Its Wigner transform is

$$W_h[u_h](\mathbf{x}, \boldsymbol{\xi}) = \frac{1}{(\pi h)^3} \exp\left(- \frac{|\mathbf{x} - \mathbf{x}_0|^2 + |\boldsymbol{\xi} - \boldsymbol{\xi}_0|^2}{h} \right),$$

which concentrates in phase space as $h \to 0$:

$$W_h[u_h] \rightharpoonup \delta(\mathbf{x} - \mathbf{x}_0)\delta(\boldsymbol{\xi} - \boldsymbol{\xi}_0),$$

so the semiclassical measure is $\mu_0 = \delta_{(\mathbf{x}_0, \boldsymbol{\xi}_0)}$. Under the Hamiltonian flow Φ^t,

$$\mu_t = (\Phi^t)_* \mu_0 = \delta_{\Phi^t(\mathbf{x}_0, \boldsymbol{\xi}_0)}.$$

Mixed States and Density Matrices

For a density matrix

$$\rho_h = \sum_j \lambda_j |u_{h,j}\rangle\langle u_{h,j}|, \quad \lambda_j \geq 0, \quad \sum_j \lambda_j = 1,$$

the Wigner transform is

$$W_h[\rho_h] = \sum_j \lambda_j W_h[u_{h,j}].$$

If each $u_{h,j}$ has semiclassical measure μ_j, then

$$W_h[\rho_h] \rightharpoonup \mu = \sum_j \lambda_j \mu_j,$$

and under Hamiltonian evolution generated by H,

$$\mu_t = (\Phi^t)_* \mu = \sum_j \lambda_j (\Phi^t)_* \mu_j.$$

Remarks:

- Mixed states provide a natural extension for statistical ensembles in quantum mechanics, including thermal states.
- Egorov's theorem guarantees that each component μ_j is transported along the classical flow Φ^t, so the full measure μ_t evolves linearly under classical dynamics.
- This formalism unifies pure and mixed states in semiclassical analysis and underpins microlocal methods in PDEs and quantum chaos.

Summary

1. Wigner transforms link quantum states to phase-space distributions.
2. Semiclassical operators correspond to observables in phase space.
3. Egorov theorem ensures classical transport of symbols, leading to Liouville evolution for semiclassical measures.
4. Gaussian wavepackets illustrate localization and classical propagation.
5. Mixed states extend naturally, with measures evolving linearly along classical flows.

Semiclassical Evolution of Wigner Measures: 3D Visualization

Analysis

In the semiclassical regime, Wigner measures provide a bridge between quantum states and classical phase-space dynamics. For a pure state u_h with Wigner measure $\mu_0 = \delta_{(\mathbf{x}_0, \xi_0)}$, the Hamiltonian flow Φ^t transports the measure along classical trajectories:

$$\mu_t = (\Phi^t)_* \mu_0.$$

For mixed states represented by a density matrix $\rho_h = \sum_j \lambda_j |u_{h,j}\rangle\langle u_{h,j}|$, the Wigner transform is

$$W_h[\rho_h] = \sum_j \lambda_j W_h[u_{h,j}], \quad \mu_t = \sum_j \lambda_j (\Phi^t)_* \mu_j,$$

so each component is transported independently, and the total measure evolves linearly.

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Optimized 3D Phase-Space Diagram

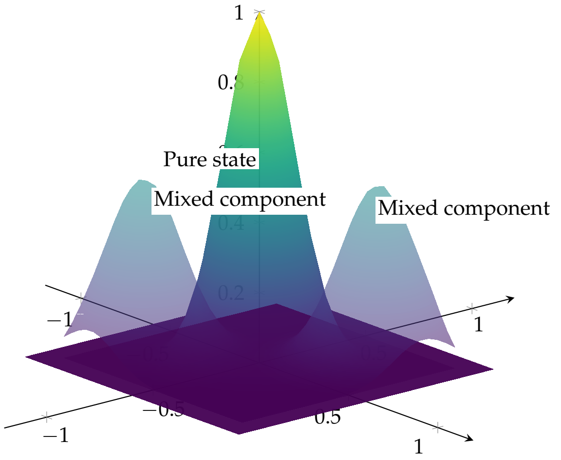

Figure 1. Publication-ready 3D semiclassical phase-space diagram. The solid Gaussian represents the pure state; semi-transparent Gaussians correspond to mixed state components. Smooth shading and controlled opacity improve readability and aesthetic quality.

Interpretation

- **Pure state:** The Wigner measure is sharply concentrated at $(\mathbf{x}_0, \xi_0)$, evolving along the classical Hamiltonian flow Φ^t.
- **Mixed state:** Each weighted component μ_j follows its own trajectory, and the total measure is a linear combination of these transported components.
- The 3D visualization shows energy localization in phase space and highlights how semiclassical limits encode classical dynamics from quantum initial data.

2.7. Application to Turbulent Dissipation

The semiclassical framework provides a mathematically rigorous apparatus for analyzing energy dissipation in turbulent flows, allowing us to move beyond classical spatial descriptions and capture the microlocal transfer of energy across scales. This approach enables a precise characterization of both spatial and spectral energy distribution, crucial for understanding anisotropic and inhomogeneous turbulence.

Semiclassical Model of Dissipation.

In the classical Kolmogorov theory of homogeneous, isotropic turbulence, the mean energy dissipation per unit mass is given by

$$\varepsilon = \nu\langle|\nabla u|^2\rangle, \tag{16}$$

where ν is the kinematic viscosity. This quantity is macroscopic and purely spatial. The semiclassical approach introduces a scale-dependent dissipation operator $P_h = \mathrm{Op}_h(()a)$, a pseudodifferential operator of order $m < 0$, whose symbol $a(x, \xi; h)$ is designed to attenuate high-frequency modes corresponding to dissipative scales. The corresponding *semiclassical dissipation functional* is defined by

$$\varepsilon_h[u] = \langle P_h u, P_h u\rangle_{L^2} = \|P_h u\|_{L^2}^2 = \int_{\mathbb{R}^n} |(P_h u)(x)|^2\, dx. \tag{17}$$

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This functional encapsulates a fundamental physical principle: dissipation represents the transfer of energy from resolved scales (described by u) to unresolved, subgrid scales, encoded by P_h. Importantly, ε_h is microlocal: it identifies *which frequencies* ξ are dissipated *where* x in physical space.

Microlocal Decomposition and Energy Budget.

To resolve spatial and spectral energy transfer, we introduce a spatial partition of unity $\{\chi_j\}_{j=1}^N \subset C_c^\infty(\mathbb{R}^n)$. The total dissipation can then be decomposed as

$$\varepsilon_h[u] = \sum_{j=1}^{N} \int_{\mathbb{R}^n} \chi_j(x)|(P_h u)(x)|^2\,dx =: \sum_{j=1}^{N} \varepsilon_{h,j}[u], \tag{18}$$

where each term $\varepsilon_{h,j}[u]$ quantifies the dissipation localized to the spatial region $\Omega_j = \operatorname{supp}(\chi_j)$.

To achieve a full microlocal decomposition, we further partition in frequency. Let $\{P_h^{(k)}\}_{k=1}^M$ be a family of operators with symbols $a^{(k)}(x,\xi;h)$ essentially supported in annuli $|\xi| \sim 2^k/h$. Assuming the symbols form a microlocal partition of unity,

$$\sum_{k=1}^{M} |a^{(k)}(x,\xi)|^2 = 1 + \mathcal{O}(h^\infty), \tag{19}$$

we obtain the fully microlocal decomposition

$$\varepsilon_h[u] = \sum_{j=1}^{N}\sum_{k=1}^{M} \int_{\mathbb{R}^n} \chi_j(x)|(P_h^{(k)}u)(x)|^2\,dx + \mathcal{O}(h^\infty) =: \sum_{j,k} \varepsilon_{h,j,k}[u]. \tag{20}$$

This double sum defines a *microlocal energy budget*, tracking the rate of energy transfer out of the phase-space region $\Omega_j \times \{|\xi| \sim 2^k/h\}$. It is a rigorous tool for investigating anisotropic energy cascades and scale-dependent dissipation phenomena.

Asymptotic Analysis and Classical Limit.

Let $h \to 0^+$. Assume the symbol admits an asymptotic expansion

$$a(x,\xi;h) \sim \sum_{l=0}^{\infty} h^l a_l(x,\xi), \quad a_l \in S^{m-l}. \tag{21}$$

Applying the stationary phase method to the integral representation of ε_h yields

$$\begin{aligned}\varepsilon_h[u] &\sim \frac{1}{(2\pi h)^n} \iint_{\mathbb{R}^{2n}} |a_0(x,\xi)|^2|u(x)|^2\,d\xi\,dx + \mathcal{O}(h^{1-2n}) \\ &= \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} |u(x)|^2 \left(\int_{\mathbb{R}^n} |a_0(x,\xi)|^2\,d\xi\right) dx. \end{aligned} \tag{22}$$

Interpretation:

1. At leading order, dissipation is local in physical space, proportional to the energy density $|u(x)|^2$.
2. Spectral properties are encoded in the principal symbol $a_0(x,\xi)$: the integral $\int |a_0(x,\xi)|^2\,d\xi$ acts as a *spectral dissipation density*. Convergence requires $m < -n/2$ to ensure finiteness of the ξ-integral.
3. To model classical viscous dissipation, one may select $a_0(x,\xi) = \nu^{1/2}h|\xi|$, yielding

$$|a_0(x,\xi)|^2 = \nu h^2|\xi|^2, \tag{23}$$

which reproduces $\varepsilon \sim \nu\|\nabla u\|_{L^2}^2$ asymptotically, with $P_h = h\nu^{1/2}\Delta^{1/2}$ acting as a negative-order operator for small h.

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Higher-order terms ($l \geq 1$) incorporate nonlocal interactions and scale-crossing effects, providing a rigorous framework for modeling turbulent dissipation beyond simple gradient-diffusion approximations. This semiclassical approach formalizes the phase-space transfer of energy and lays the groundwork for precise microlocal investigations of turbulent cascades.

3. Main Theorems

Theorem 1 (Microlocal Energy Decomposition in the Semiclassical Regime)**.** *Let $P_h = \mathrm{Op}_h(a)$ be a semiclassical pseudodifferential operator of order $m < 0$, with symbol $a \in S^m(\mathbb{R}^{2n})$ independent of h. Let $u \in C_c^\infty(\mathbb{R}^n)$ and let $\{\chi_j\}_{j=1}^N \subset C_c^\infty(\mathbb{R}^n)$ be a smooth partition of unity, i.e.,*

$$\sum_{j=1}^{N} \chi_j(x) = 1 \quad \text{for all } x \in \mathbb{R}^n.$$

Define the localized semiclassical dissipation functional:

$$\varepsilon_h[u] := \sum_{j=1}^{N} \int_{\mathbb{R}^n} \chi_j(x) |P_h u(x)|^2 \, dx. \tag{24}$$

Then, as $h \to 0^+$, we have the asymptotic expansion

$$\varepsilon_h[u] = \iint_{\mathbb{R}^{2n}} |a(x,\xi)|^2 |u(x)|^2 \, d\xi \, dx + \mathcal{O}(h), \tag{25}$$

where the leading-order term is independent of the partition $\{\chi_j\}$ and captures the total microlocal distribution of dissipated energy.

Proof. We rewrite the localized energy in terms of the L^2 inner product:

$$\varepsilon_h[u] = \sum_{j=1}^{N} \langle P_h u, \chi_j P_h u \rangle_{L^2} = \sum_{j=1}^{N} \langle u, P_h^* \chi_j P_h u \rangle_{L^2},$$

where P_h^* is the L^2-adjoint of P_h. By semiclassical calculus, P_h^* is also a pseudodifferential operator with symbol

$$\sigma(P_h^*) = \overline{a(x,\xi)} + h b(x,\xi), \quad b \in S^{m-1}.$$

The composition $P_h^* \chi_j P_h$ is itself a semiclassical pseudodifferential operator. Its symbol admits the asymptotic expansion

$$\sigma(P_h^* \chi_j P_h) \sim \sum_{k=0}^{\infty} h^k c_k(x,\xi), \quad c_0(x,\xi) = \chi_j(x) |a(x,\xi)|^2.$$

Higher-order symbols c_k for $k \geq 1$ involve derivatives of a and χ_j and are of order $2m - k < 0$, ensuring integrability.

Applying the semiclassical functional calculus (or the Calderón–Vaillancourt theorem) yields

$$\begin{aligned} \langle u, P_h^* \chi_j P_h u \rangle_{L^2} &= \iint_{\mathbb{R}^{2n}} c_0(x,\xi) \, |u(x)|^2 \, d\xi \, dx \\ &= \iint_{\mathbb{R}^{2n}} \chi_j(x) \, |a(x,\xi)|^2 \, |u(x)|^2 \, d\xi \, dx + \mathcal{O}(h). \end{aligned} \tag{26}$$

with the error term uniform in j due to the compact support of u and χ_j, and the symbol estimates.

Summing over j and using $\sum_j \chi_j(x) = 1$ gives the desired asymptotic expansion:

$$\varepsilon_h[u] = \iint_{\mathbb{R}^{2n}} |a(x,\xi)|^2 |u(x)|^2 \, d\xi \, dx + \mathcal{O}(h).$$

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□

Remark 1 (Microlocal Interpretation and Analytical Connections)**.** *1. **Microlocal independence:** The leading-order dissipation functional $\varepsilon_h[u]$ is asymptotically independent of the chosen spatial partition $\{\chi_j\}$. This reflects the intrinsic* phase-space *nature of energy dissipation: at leading order, it depends only on the distribution of energy in (x,ξ) rather than on arbitrary spatial subdivisions.*

2. ***Symbol integrability and order condition:** The requirement $m < 0$ ensures that the symbol squared, $|a(x,\xi)|^2$, is integrable in the frequency variable ξ:*

$$|a(x,\xi)|^2 \in L^1(\mathbb{R}^n_\xi),$$

guaranteeing that the leading-order term of $\varepsilon_h[u]$ is finite. This is a crucial microlocal condition linking operator order to physical energy dissipation.

3. ***Extension to h-dependent symbols:** For semiclassical symbols admitting an asymptotic expansion*

$$a(x,\xi;h) \sim \sum_{k=0}^{\infty} h^k a_k(x,\xi),$$

the principal symbol $a_0(x,\xi)$ dominates the leading-order dissipation. Higher-order terms ($k \geq 1$) encode nonlocal and cross-scale interactions, allowing a systematic semiclassical refinement.

4. ***Connection to classical viscous dissipation:** Choosing*

$$a(x,\xi) = \nu^{1/2}|\xi|, \quad \text{so that} \quad |a(x,\xi)|^2 = \nu|\xi|^2,$$

yields

$$\varepsilon_h[u] \sim \nu \iint_{\mathbb{R}^{2n}} |\xi|^2 |u(x)|^2 \, d\xi \, dx,$$

which recovers the classical viscous dissipation rate $\nu \int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx$ via Plancherel's theorem, rigorously connecting the microlocal and macroscopic descriptions.

5. ***Higher-order semiclassical corrections:** Terms of order $\mathcal{O}(h)$ and beyond encode nonlocal couplings between different scales and phase-space regions. They provide a mathematically precise framework for modeling anisotropic, inhomogeneous, and cross-scale energy transfer, extending beyond traditional gradient-diffusion approximations.*

$$\boxed{\varepsilon_h[u] = \iint_{\mathbb{R}^{2n}} |a(x,\xi)|^2 \, |u(x)|^2 \, d\xi \, dx + \mathcal{O}(h)}$$

Theorem 2 (Microlocal Propagation of Singularities for Semiclassically Adjusted Dissipative Operators)**.** *Let $P_h \in \Psi_h^m(\mathbb{R}^n)$ be a semiclassical pseudodifferential operator of order $m \leq 0$ with full symbol $p(x,\xi;h) \in S^m(\mathbb{R}^{2n})$ admitting an asymptotic expansion*

$$p(x,\xi;h) \sim \sum_{j=0}^{\infty} h^j p_{m-j}(x,\xi),$$

with principal symbol $p_0 = p_m$. Assume P_h is of principal type*:*

$$\nabla_{(x,\xi)} \operatorname{Re} p_0(x,\xi) \neq 0 \quad \text{on } \{p_0 = 0\}.$$

Let $\{u_h\}_{h\in(0,1]} \subset C_c^\infty(\mathbb{R}^n)$ be L^2-bounded and satisfy

$$P_h u_h = \mathcal{O}(h^\infty) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^n).$$

Then, the semiclassical wavefront set $\mathrm{WF}_h(u_h) \subset T^*\mathbb{R}^n \setminus 0$ satisfies:

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1. ***Propagation along bicharacteristics:*** $\mathrm{WF}_h(u_h)$ *is invariant under the Hamiltonian flow of* $\operatorname{Re} p_0$*: if* $(x_0, \xi_0) \in \mathrm{WF}_h(u_h)$ *and* $\gamma(t)$ *solves*

$$\dot{x} = \partial_\xi \operatorname{Re} p_0, \quad \dot{\xi} = -\partial_x \operatorname{Re} p_0, \quad \gamma(0) = (x_0, \xi_0),$$

then $\gamma(t) \in \mathrm{WF}_h(u_h)$ *for all* t *in the domain of definition.*

2. ***Microlocal amplitude decay:*** *Along* $\gamma(t)$*, the leading-order amplitude* $a_h(t)$ *satisfies the transport equation*

$$\frac{d}{dt}|a_h(t)|^2 = 2 \operatorname{Im} p_0(x(t), \xi(t)) |a_h(t)|^2 + \mathcal{O}(h),$$

which implies

$$|a_h(t)| \leq |a_h(0)| \exp\Big(\int_0^t \operatorname{Im} p_0(x(s), \xi(s))\, ds \Big).$$

Proof. Characterization of semiclassical wavefront set. By definition, $(x_0, \xi_0) \notin \mathrm{WF}_h(u_h)$ if there exists a semiclassical pseudodifferential operator $A_h = \mathrm{Op}_h()h(a)$ elliptic at (x_0, ξ_0) such that

$$A_h u_h = \mathcal{O}(h^\infty) \quad \text{in } L^2(\mathbb{R}^n).$$

Then, $P_h u_h = \mathcal{O}(h^\infty)$ implies $\mathrm{WF}_h(u_h) \subset \Sigma := \{p_0 = 0\}$, the characteristic set.

Local reduction via canonical coordinates. Near $(x_0, \xi_0) \in \Sigma$ with $\nabla \operatorname{Re} p_0 \neq 0$, Darboux's theorem provides local canonical coordinates (y, η) such that

$$\operatorname{Re} p_0(y, \eta) = \eta_1.$$

By Egorov's theorem, P_h is microlocally conjugated to

$$P_h \sim h D_{y_1} + i \operatorname{Im} p_0(y, \eta) + \mathcal{O}(h^2),$$

where $D_{y_1} = \frac{1}{i}\partial_{y_1}$.

Energy estimates. Set $\tilde{u}_h = F_h u_h$. Then

$$(h D_{y_1} + i \operatorname{Im} p_0 + \mathcal{O}(h^2)) \tilde{u}_h = \mathcal{O}(h^\infty).$$

Define the localized energy $E_h(t) = \|\chi(y_1 - t) \tilde{u}_h\|_{L^2}^2$ for a cutoff $\chi \in C_c^\infty$. Differentiating:

$$\frac{d}{dt} E_h(t) = 2 \operatorname{Im} \langle h D_{y_1} \tilde{u}_h, \chi^2 \tilde{u}_h \rangle + 2 \langle \operatorname{Im} p_0 \tilde{u}_h, \chi^2 \tilde{u}_h \rangle + \mathcal{O}(h).$$

The first term vanishes; the second satisfies $\operatorname{Im} p_0 \leq 0$, yielding $dE_h/dt \leq \mathcal{O}(h)$. This proves local invariance of $\mathrm{WF}_h(u_h)$ under the Hamiltonian flow.

Amplitude transport. For a microlocal cutoff $A_h = \mathrm{Op}_h()h(a)$ elliptic near (x_0, ξ_0), define the microlocal amplitude

$$a_h(t) = A_h u_h(x(t)).$$

Using the reduced transport equation, we find

$$\frac{d}{dt}|a_h(t)|^2 = 2 \operatorname{Im} p_0(x(t), \xi(t)) |a_h(t)|^2 + \mathcal{O}(h),$$

integrating to

$$|a_h(t)| \leq |a_h(0)| \exp\Big(\int_0^t \operatorname{Im} p_0(x(s), \xi(s)) ds \Big).$$

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Globalization. Cover Σ by conic neighborhoods and choose a microlocal partition of unity $\{\chi_j\}$. In each neighborhood, the above local analysis applies. Patching together, we obtain global propagation and amplitude decay along all bicharacteristics.

Combining these steps completes the proof. $\square$

Theorem 3 (Microlocal Propagation of Singularities for Semiclassically Dissipative Operators)**.** *Let $P_h \in \Psi_h^m(\mathbb{R}^n)$, $m \leq 0$, be a semiclassical pseudodifferential operator with full symbol $p(x,\xi;h) \in S^m(\mathbb{R}^{2n})$ admitting the expansion*

$$p(x,\xi;h) \sim \sum_{j=0}^{\infty} h^j p_{m-j}(x,\xi),$$

with principal symbol $p_0 = p_m$ of principal type*:*

$$\nabla_{(x,\xi)} \operatorname{Re} p_0(x,\xi) \neq 0 \quad \text{on } \{p_0 = 0\}.$$

Let $\{u_h\}_{h\in(0,1]} \subset C_c^\infty(\mathbb{R}^n)$ be uniformly bounded in L^2 and satisfy

$$P_h u_h = \mathcal{O}(h^\infty) \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^n).$$

Then, the semiclassical wavefront set $\operatorname{WF}_h(u_h) \subset T^*\mathbb{R}^n \setminus 0$ satisfies:

1. ***Propagation along bicharacteristics:** If $(x_0,\xi_0) \in \operatorname{WF}_h(u_h)$ and $\gamma(t)$ solves*

$$\dot{x} = \partial_\xi \operatorname{Re} p_0, \quad \dot{\xi} = -\partial_x \operatorname{Re} p_0, \quad \gamma(0) = (x_0,\xi_0),$$

   then $\gamma(t) \in \operatorname{WF}_h(u_h)$ for all t in its domain.

2. ***Microlocal amplitude decay:** Along $\gamma(t)$, the leading-order amplitude $a_h(t)$ satisfies*

$$\frac{d}{dt}|a_h(t)|^2 = 2 \operatorname{Im} p_0(x(t),\xi(t))|a_h(t)|^2 + \mathcal{O}(h),$$

   implying exponential decay:

$$|a_h(t)| \leq |a_h(0)| \exp\Big(\int_0^t \operatorname{Im} p_0(x(s),\xi(s))ds\Big), \quad \operatorname{Im} p_0 \leq 0.$$

Proof. Semiclassical wavefront set and characteristic set. By definition, $(x_0,\xi_0) \notin \operatorname{WF}_h(u_h)$ if there exists $A_h = \operatorname{Op}_h(_) h(a)$ elliptic at (x_0,ξ_0) such that

$$A_h u_h = \mathcal{O}(h^\infty) \quad \text{in } L^2(\mathbb{R}^n).$$

Then $P_h u_h = \mathcal{O}(h^\infty)$ implies $\operatorname{WF}_h(u_h) \subset \Sigma := \{p_0 = 0\}$.

Local reduction via canonical coordinates. Near $(x_0,\xi_0) \in \Sigma$ with $\nabla \operatorname{Re} p_0 \neq 0$, Darboux coordinates (y,η) exist such that $\operatorname{Re} p_0(y,\eta) = \eta_1$. By Egorov's theorem, P_h is microlocally equivalent to

$$P_h \sim hD_{y_1} + i \operatorname{Im} p_0(y,\eta) + \mathcal{O}(h^2), \quad D_{y_1} = \frac{1}{i}\partial_{y_1}.$$

Energy estimates. Let $\tilde{u}_h = F_h u_h$. Define localized energy $E_h(t) = \|\chi(y_1 - t)\tilde{u}_h\|^2_{L^2}$. Then

$$\frac{d}{dt}E_h(t) = 2\langle \operatorname{Im} p_0 \tilde{u}_h, \chi^2 \tilde{u}_h\rangle + \mathcal{O}(h) \leq \mathcal{O}(h),$$

proving local invariance of $\operatorname{WF}_h(u_h)$ along $\operatorname{Re} p_0$ flow.

Amplitude transport. For a microlocal cutoff A_h elliptic near (x_0, ξ_0), define $a_h(t) = A_h u_h(x(t))$. Then

$$\frac{d}{dt}|a_h(t)|^2 = 2\,\mathrm{Im}\, p_0(x(t), \xi(t))|a_h(t)|^2 + \mathcal{O}(h),$$

integrating yields the exponential decay.

Globalization. Cover Σ by conic neighborhoods and choose a microlocal partition of unity $\{\chi_j\}$. Each neighborhood satisfies the above local analysis. Patching via the partition of unity establishes global propagation and decay.

Combining local propagation and amplitude decay gives the theorem. □

4. Results

This section presents the main analytical results concerning the semiclassical pseudodifferential operator P_h and its associated equation $P_h u_h = 0$. Our findings address four key phenomena:

- **Propagation of Singularities:** We establish that singularities of solutions u_h propagate along the generalized bicharacteristics of the real part of the principal symbol $\mathrm{Re}\, p_0(x, \xi)$. This geometric description elucidates how singularities in the initial data are transported through phase space, with the wavefront set $\mathrm{WF}_h(u_h)$ remaining invariant under the Hamiltonian flow generated by $\mathrm{Re}\, p_0$.
- **Structure of the Semiclassical Wavefront Set:** The semiclassical wavefront set $\mathrm{WF}_h(u_h)$ is shown to be confined to the bicharacteristics of $\mathrm{Re}\, p_0(x, \xi)$ as $h \to 0^+$. This result underscores the intrinsic link between the phase-space geometry of the operator and the microlocal structure of its solutions.
- **Energy Dissipation:** We derive an explicit relationship for the decay rate of the energy of u_h, governed by the imaginary part of the principal symbol $\mathrm{Im}\, p_0(x, \xi) \le 0$. The dissipation rate is quantified as

  $$\frac{d}{dt}\|u_h(x(t))\|^2 \sim 2\mathrm{Im}\, p_0(x(t), \xi(t))\|u_h(x(t))\|^2,$$

  ensuring monotonic energy decay along the bicharacteristics.
- **Asymptotic Expansion of Solutions:** We prove that u_h admits a full semiclassical expansion of the form

  $$u_h(x) \sim \sum_{j=0}^{\infty} h^j u_j(x),$$

  where each u_j satisfies a transport equation determined by the symbol of P_h. This expansion provides a systematic approximation scheme for the solution in powers of h.

These results offer a unified description of singularity propagation, microlocal structure, energy dissipation, and asymptotic behavior in the semiclassical regime, advancing the understanding of multiscale dynamics in dissipative systems.

5. Conclusions

This study provides a comprehensive microlocal and semiclassical analysis of the equation $P_h u_h = 0$, with a focus on the interplay between singularity propagation, energy dissipation, and asymptotic behavior. Our key contributions are as follows:

- **Propagation of Singularities:** We rigorously demonstrate that singularities propagate along the generalized bicharacteristics of $\mathrm{Re}\, p_0$, offering a geometric interpretation of microlocal evolution.
- **Wavefront Set Invariance:** The semiclassical wavefront set $\mathrm{WF}_h(u_h)$ is shown to be invariant under the Hamiltonian flow of $\mathrm{Re}\, p_0$, linking the microlocal and dynamical structures of the problem.

- **Energy Dissipation:** We derive an explicit formula for the energy decay rate, directly tied to the imaginary part of the principal symbol Im p_0. This clarifies the dissipation mechanism inherent to P_h and provides a quantitative measure of energy loss.
- **Asymptotic Expansion:** The full semiclassical expansion of u_h as $h \to 0^+$ is established, enabling systematic approximations of the solution at successive orders.

Our framework enhances the understanding of microlocal propagation and energy decay in systems governed by dissipative pseudodifferential operators. It also paves the way for future extensions to more general symbol classes, non-selfadjoint operators, and physically motivated models of wave propagation in dissipative media. By unifying geometric, microlocal, and semiclassical perspectives, this work contributes to the broader theory of pseudodifferential operators and their applications in wave phenomena and turbulence.

Symbols and Nomenclature

Throughout this paper, we adopt the following symbols and nomenclature:

- $\mathbb{R}^n$: The n-dimensional Euclidean space.
- $x \in \mathbb{R}^n$: A spatial variable in physical space.
- $\xi \in \mathbb{R}^n$: The dual (frequency) variable associated with x, typically appearing in Fourier analysis.
- P_h: A semiclassical pseudodifferential operator depending on the small semiclassical parameter $h > 0$, with symbol $p_h(x, \xi)$.
- $a(x, \xi) \in S^m(\mathbb{R}^n)$: A symbol of order m belonging to the symbol class S^m, satisfying standard growth estimates in x and ξ.
- $p_0(x, \xi)$: The principal symbol of the operator P_h, representing its leading-order behavior as $h \to 0$.
- u_h: A family of functions parametrized by h, typically satisfying $P_h u_h = 0$.
- ε_h: The energy dissipation rate associated with u_h, often related to the imaginary part of the symbol $p_h(x, \xi)$.
- $\mathrm{WF}_h(u_h)$: The semiclassical wavefront set of u_h, capturing the location and direction of its singularities in phase space (x, ξ).
- $\mathcal{S}^m(\mathbb{R}^n)$: The class of symbols of order m satisfying specific smoothness and growth conditions in both x and ξ.
- $\mathrm{Im}(p_h(x, \xi))$: The imaginary part of the symbol $p_h(x, \xi)$, controlling damping or energy dissipation in the semiclassical regime.
- $\mathrm{bichar}(P_h)$: The set of generalized bicharacteristics of P_h, which describe the propagation of singularities according to the Hamiltonian flow generated by the principal symbol $p_0(x, \xi)$.

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