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Abstract

In laminar flow with disturbances, the instantaneous flow is decomposed into the mean flow and the disturbances. At a sufficiently large Reynolds number, the nonlinear evolution alters the mean flow profile. As the amplitude of the disturbances grows, at certain points, the viscous term of the disturbances cancels out the viscous term of the mean flow, causing the viscous term of the instantaneous flow to tend towards zero. Due to the mismatch between the energy loss and the velocity magnitude at these points, they become physical singularities of the Navier-Stokes equation. The instability of these singularities leads to velocity discontinuities, where the norm of the velocity gradient tends to infinity. This proves that there is no smooth solution to the Navier-Stokes equation in the global domain. This paper takes the plane Poiseuille flow as an example to demonstrate the physical logic and mathematical derivation process of the proof.

Keywords: Navier-Stokes Equation; criterion; singularity; discontinuity; turbulence

MSC: 35A01; 35A02; 35A21; 25A20

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1. Introduction

The smoothness of global solutions to the three-dimensional (3D) Navier-Stokes equation is a foundational open problem in mathematical fluid mechanics (Doering 2009). In 2000, the Clay Mathematics Institute (CMI) selected the existence and smoothness of the Navier-Stokes equation as one of the seven Millennium Prize problems (Fefferman 2006). The difficulty of the problem lies that solutions to the Navier-Stokes equation include turbulence, which is still not fully understood.

In previous years, the existence and smoothness of the Navier-Stokes equation has been extensively studied. Leray (1934) speculated that turbulent flow is an irregular motion with singularity, at which the velocity tends towards to blow up. Up to the time when singularity appears, the solution to the equation is smooth, and the breakdown of solution at the singularity is called the finite-time singularity (FTS) of the Navier-Stoke equation. Although nearly a century has past, Leray's conjecture concerning the appearance of singularities in three-dimensional turbulent flows has neither been proved nor disproved (Foias et al. 2004). Leray (1934) proved the existence of weak solutions to the Navier-Stokes equation for three-dimensional (3D) flows. Ladyzhenskaya (1969) proved the existence and smoothness of the Navier-Stokes equation for two-dimensional (2D) flows. Tao (2016) constructed an averaged three-dimensional Navier-Stokes equation, and proposed a possible approach for the blowup solution to the Navier-Stokes equation. Buckmaster and Vicol (2019) proved the nonuniqueness of weak solutions to the three-dimensional Navier-Stokes equation.

Mathematicians have tried to explore the regularity of solutions to the Navier-Stokes equation from the weak solutions to the Navier-Stokes equation (Scheffer 1976; Caffarelli et al. 1982; Serrin 1962; Berselli and Galdi 2002), but the answer to the question on the existence of global smooth solutions has not been reached.

Experiments and numerical simulations showed that the transition from laminar flow to turbulent flow depend on both the Reynolds number and the disturbance. These effects should be taken into account when the Navier-Stokes equation is analyzed.

Recently, Dou (2025) carried out detailed analysis on the 3D Navier-Stokes equation for plane Poiseuille flow, under the interaction of the base flow and the external disturbance. It has been revealed that the velocity discontinuity induced by the interaction of the nonlinear term and the viscous term in the Navier-Stokes equation indicates the onset of turbulent transition. The relevant theoretical results are compared with the data from numerical simulations and experiments, and good agreement has been obtained (Dou 2022; Dou 2025; Dou 2006; Niu et al. 2024; Niu et al. 2025; Zhou et al. 2025a; Zhou et al. 2025b).

In this study, mathematical analysis is carried out for laminar flow with disturbances following previous studies (Dou 2022; Dou 2025). The onset of turbulence is marked by the appearance of singularities in physics, which results in a finite-time velocity discontinuity. The solutions to the Navier-Stokes equation cannot be continued beyond this critical point of temporal evolution. Based on these results, the existence and smoothness of solutions to the NS equation in global domain are disproved.

2. Governing Equations and Flow Decomposition

2.1. Navier-Stokes Equation and Continuity Equation

The Navier-Stokes equation and the continuity equation as well as the boundary condition and the initial condition are:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \frac{1}{\rho}\nabla p = 0, \quad (x, y, z, t) \in \Omega \times (0, \infty) \tag{1}$$

$$\nabla \cdot u = 0 \quad (x, y, z, t) \in \Omega \times (0, \infty) \tag{2}$$

$$u(x, 0, z, t) = u(x, H, z, t) = 0, \quad u(x, y, z, 0) = u_0(x, y, z) \tag{3}$$

where $\Omega = (0, Lx) \times (0, H) \times (0, Lz)$ ($y = 0, H$ are walls), $u = (u_x, u_y, u_z)$ is the instantaneous velocity field satisfying no-slip boundary conditions, ρ is the fluid density, ν is the kinmatic viscosity, t is the time, p is the static pressure. This group of equations can be solved with the time increase accoring to the given boundary conditions and the initial condition.

2.2. Flow Decomposition

The instantaneous flow velocity is decomposed into the time-averaged velocity U and the disturbed velocity v.

$$U(x, y, z, t) = \frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} u(x, y, z, \tau)d\tau \tag{4}$$

where Δt is the time-averaging scale (satisfying $\Delta t \gg$ disturbance characteristic time and $\Delta t \ll$ time for laminar flow to evolve to singularity);

$$v(x, y, z, t) = u(x, y, z, t) - U(x, y, z, t) \tag{5}$$

satisfying

$$\frac{1}{\Delta t} \int_{t-\Delta t/2}^{t+\Delta t/2} v(x, y, z, \tau)d\tau = 0. \tag{6}$$

where $U = (U_x, U_y, U_z)$, and $v = (v_x, v_y, v_z)$.

It is well known that the disturbance velocity is far less than the time-averaged velocity and the instantaneous velocity is always larger than zero except on the walls, in the range of laminar flows. Thus, we have $u > 0$ and $|U| \gg |v|$ (off solid walls) for the laminar flow in plane Poiseuille flow configurations.

In the mathematical analysis, the time-averaged flow is updated for all the time, while the disturbance flow is obtained by the instantaneous flow minus the time-averaged flow.

3. Preliminaries

3.1. Local Vanishing of Composite Viscous Term

The instantaneous velocity is decomposed as $u = U + v$, where U is the time-averaged velocity and $v = u - U$ is the disturbance. The viscous term in the Navier-Stokes equation represents the loss of the total mechanical energy for pressure-driven flows (Dou 2006; Dou 2022; Dou 2025). The viscous term of the instantaneous flow can be considered as the superposition of those of the averaged flow and the disturbed flow. If the Reynolds number is sufficient large and the disturbance amplitude is sufficient high, the instantaneous velocity, can lead to

$$\Delta u = \Delta U + \Delta v \to 0 \tag{7}$$

with the increase of the disturbance amplitude at a local point, implying zero viscous energy loss there. It is pointed out that the Laplace term Δu represents the loss of the total mechanical energy per unit length for pressure-driven flows (Dou 2022).

3.2. Monotonic Relationship Beween the Energy Loss and the Velocity Magnitude in Viscous Flow

A fundamental property of viscous flows is the monotonic correspondence between the energy loss and the velocity magnitude. This is the key mechanism of singularity generation in viscous flows.

The Axiom (2) in Dou (2025) reads,

Axiom: The velocity of a fluid particle varies monotonically with the energy lost during its motion, and vice versa.

This Axiom is the necessary result of the second law of thermodynamics. Thus, we have,

(1) If $u \neq 0$, then $\Delta s > 0$ and $\|\Delta u(t)\| > 0$, where s is the entropy and Δs represents the entropy increase.

(2) If $\|\Delta u(t)\| = 0$, then $\Delta s = 0$ and $u(t) = 0$.

Therefore, for any point $x \in \Omega$, if the local viscous energy loss vanishes (i.e., $\|\Delta u(x,t)\| \to 0$), the velocity at x must tend to zero to maintain the energy-velocity consistency—formally:

$$\lim_{\|\Delta u(x,t)\| \to 0} u(x,t) = 0 \quad \forall x \in \Omega \setminus \partial \Omega \tag{8}$$

3.3. Key Definitions

These definitions are for singularity (in physics) and velocity discontinuity.

Definition 3.1 (Navier-Stokes Singularity): A point $(x^*, t^*) \in \Omega \times (0, \infty)$ is a singularity in physics if:

The composite viscous term vanishes locally:

$$\lim_{t \to t^*} \Delta u(x^*, t) = 0 \tag{9}$$

The velocity fails to tend to zero:

$$\limsup_{t \to t^*} u(x^*, t) > 0 \tag{10}$$

(violating energy-velocity monotonicity).

Definition 3.2 (Velocity Discontinuity): A singularity (x^*, t^*) induces a velocity discontinuity if:

$$\lim_{t \nearrow t^*} u(x^*, t) > 0 \quad \text{and} \quad \lim_{t \searrow t^*} u(x^*, t) = 0 \tag{11}$$

where $t \nearrow t^*$ denotes approaching t^* from below (pre-singularity) and $t \searrow t^*$ from above (post-singularity).

4. Main Results and Proofs

4.1. Main Theorem

Theorem 4.1: For 3D plane Poiseuille flow with initial laminar velocity $u_0 \in H^3_{\sigma,0}(\Omega)$ (divergence-free, no-slip), there exists a finite time $t^* > 0$ and a point

$$x^* \in \Omega \setminus \partial\Omega \tag{12}$$

such that: (x^*, t^*) is a Navier-Stokes singularity (local viscous term vanishes, velocity remains positive).

Then immidiately, as $t \to t^*$, a velocity discontinuity forms at (x^*, t^*), causing

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \to \infty \tag{13}$$

Thus, no global smooth solution exists for the 3D Navier-Stokes equation.

4.2. Proof Process

Step 1: Existence of Points with Vanishing Composite Viscous Term

(1) For laminar plane Poiseuille flow, the composite viscous term is $\Delta u = \Delta U + \Delta v$.

(2) The initial base flow U has a steady-state Laplacian $\Delta U_0 = \frac{1}{\mu}\frac{dp}{dx}$ (constant, negative for pressure-driven flow), where μ is the dynamic viscosity . Afer the initial base flow is disturbed, the averaged flow is modified by the nonlinear role. Then, the updated new averaged flow can be taken as the new base flow, and the disturbance is $v = u - U$.

Disturbances induce time-dependent $\Delta v(t)$, which can be positive in a period of disturbance.

As the disturbance amplitude grows (initially small but amplified by 3D nonlinearity), there exists a time t^* and point x^* where $\Delta v(x^*, t^*) = -\Delta U(x^*, t^*)$, hence:

$$\Delta u(x^*, t^*) = \Delta U(x^*, t^*) + \Delta v(x^*, t^*) = 0 \tag{14}$$

This implies that the local viscous energy loss at (x^*, t^*) vanishes:

$$\lim_{t \to t^*} \|\Delta u(x^*, t)\| = 0. \tag{15}$$

Step 2: Energy-Velocity Mismatch Induces Singularity

By the monotonic energy-velocity relationship (Section 3.2), $\lim_{t \to t^*} \|\Delta u(x^*, t)\| = 0$ requires $\lim_{t \to t^*} u(x^*, t) = 0$. However, in pre-singularity laminar flow ($t < t^*$):

(1) The averaged flow $U(x^*, t) > 0$ (pressure-driven, non-zero away from walls);

(2) Disturbances $v(x^*, t)$ are small and do not reverse the flow direction, so $u(x^*, t) = U(x't) + v(x't) > 0$.

This contradiction—zero energy loss but positive velocity—satisfies Definition 3.1: (x^*, t^*) is a Navier-Stokes singularity in physics.

Step 3: Singularity Instability and Velocity Discontinuity

Physical singular points are dynamically unstable due to the energy-velocity mismatch. For $t > t^*$ (post-singularity):

(1) The flow cannot sustain non-zero velocity with zero viscous dissipation;

(2) The physical singularity undergoes instantaneous stabilization, forcing $u(x^*, t) \to 0$ to restore energy-velocity consistency.

Thus,

$$\lim_{t \nearrow t^*} u(x^*, t) > 0 \quad \text{and} \quad \lim_{t \searrow t^*} u(x^*, t) = 0$$

satisfying Definition 3.2: a velocity discontinuity forms at (x^*, t^*).

Step 4: Velocity Discontinuity Causes Divergence of Velocity Gradient Norm

A velocity discontinuity at (x^*, t^*) means that the velocity field is discontinuous at x^* for $t \geq t^*$. For any neighborhood $B_\epsilon(x^*)$ around x^*:

(1) For $x \in B_\epsilon(x^*) \setminus x^*$, $u(x,t) \approx u(x^*,t^-)$ (pre-singularity velocity, positive);

(2) At $x = x^*$, $u(x^*,t) = 0$ (post-singularity velocity).

The velocity gradient at x^* is:

$$\nabla u(x^*,t) = \lim_{\epsilon \to 0} \frac{u(x^* + \epsilon, t) - u(x^*,t)}{\epsilon} \approx \lim_{\epsilon \to 0} \frac{u(x^*,t^-)}{\epsilon} \to \infty \tag{16}$$

Taking the L^∞ norm (supremum over Ω):

$$\|\nabla u(t)\|_{L^\infty(\Omega)} \geq |\nabla u(x^*,t)| \to \infty \quad \text{as } t \to t^* \tag{17}$$

This is re-written as

$$\lim_{t \to t^*} \|\nabla u(t)\|_{L^\infty(\Omega)} = \infty \tag{18}$$

Step 5: Contradiction with Global Smoothness

A smooth solution requires $\nabla u \in C(\Omega \times [0,\infty))$ (continuous velocity gradient everywhere for all time). However, $\|\nabla u(t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to t^*$ implies that the gradient is unbounded and discontinuous at (x^*,t^*). This contradicts the definition of a global smooth solution, completing the proof.

5. The BKM Criterion for Solution Breakdown

Vorticity dynamics is important in understanding the flow stability and turbulent generation (Lugt 1983). The BKM criterion based on vorticity dynamics for Euler equation was proposed in Beale et al. (1984). In later studies, it has been proved that the BKM criterion is also valid for the Navier-Stokes equation (Kozono and Taniucj 2000; Zhao 2017; Gibbon et al. 2018). Constantin and Fefferman (1993) focused on the problem of global regularity for the Navier-Stokes equation. They explored the relationship between the direction of vorticity and the breakdown of smooth solutions.

The relation between the velocity gradient and the vorticity can be found from the definition of vorticity, as expressed by $\omega = \nabla \times u$. By utilizing the Biot-Savart Law, the relation of the velocity gradient ∇u and the vorticity ω was obtained as follow (equation (21) in Beale et al., 1984):

$$\|\nabla u(t)\|_{L^\infty} \leq C\{(\gamma^{1/4}\| u(t)\|_3 + (1 - log\ \gamma)\|\omega(t)\|_{L^\infty} + \|\omega(t)\|_{L^2}\} \tag{19}$$

where C is a universal constant and the first term on the right hand side $\gamma^{1/4}\|\mathbf{u}\|_3 \leq 1$, and $\gamma \leq 1$.

The discontinuity in u causes ω to concentrate. By consideration of equations (18) and (19), we obtain,

$$\lim_{t \to t^*} \|\omega(t)\|_{L^\infty} = \infty \tag{20}$$

where t^* is the first time such that the solution cannot be continued. This establishes vorticity growth as the critical indicator of regularity breakdown.

Thus, the following equation can be obtained from equation (20),

$$\int_0^{t^*} \|\omega(t)\|_{L^\infty} dt = \infty \tag{21}$$

By the BKM criterion, this means that the solution cannot be extended beyond t^*, contradicting the assumption of global smoothness. This breakdown of solutions is due to the singularity-induced vorticity blow up. Thusfar, the finite-time velocity discontinuity revealed in this study also satisfies the BKM criterion for solution breakdown of the Navier-Stokes equation.

6. Discussions

6.1. Physical Interpretation of the Singularity Mechanism

The core of this work is linking local viscous energy loss to velocity evolution:

This work identifies the energy-velocity mismatch induced by local viscous term vanishing as the root cause of singularities. The key novelty lies in linking local viscous energy loss to velocity evolution constraints, providing a physically intuitive and mathematically rigorous singularity formation mechanism for plane Poiseuille flow.

In laminar flow, the loss of the total mechanical energy is the source to produce the power for the motion of fluid particle in pressure-driven flows, generating velocity. When the energy loss vanishes at a point, the flow can no longer sustain non-zero velocity without violating energy conservation.

The singularity here is not a result of "velocity blow-up" as in Leray (1934) but of a fundamental inconsistency between local energy consuming and velocity magnitude.

6.2. Robustness of the Energy-Velocity Monotonicity

The energy-velocity relationship $\lim_{\|\Delta u\| \to 0} u = 0$ is robust for laminar Poiseuille flow:

It follows directly from the law of the second thermodynamics: if $u(x, t)$ remained positive while the mechanical energy loss $\|\Delta u(x, t)\| \to 0$, the kinetic energy at x would persist without energy consuming, creating a "perpetual motion" paradox.

Numerical simulations of plane Poiseuille flow at high *Re* confirm that regions of low energy loss correlate with velocity decay, supporting the monotonicity assumption (Dou 2022; Niu et al. 2024; Niu et al. 2025).

6.3. Implications for Turbulence Onset

The velocity discontinuity at the singularity marks the onset of turbulence:

The discontinuous velocity gradient generates large-scale vortices via the "negative velocity spikes" phenomenon. Gibbon (2010) studied the regularity and singularity in solutions of the three-dimensional Navier-Stokes equation and suggested that solutions are intermittent but spikes may be the manifestation of true singularities. The present study confirms Gibbon's suggestion that spikes are produced by velocity discontinuity.

These vortices amplify disturbances and drive velocity fluctuations, leading to the transition from laminar to turbulent flow.

This aligns with experimental observations that turbulence initiates at localized "bursts"—consistent with our singularity model.

6.4. Confirmation from Simulations and Experiments

In our simulations for transitional flows (Niu et al. 2024; Niu et al. 2025; Zhou et al. 2025a; Zhou et al. 2025b), it is found that turbulent vortices are produced via singularities of the Navier-Stokes equation. The birth of these turbulent vortices must be reflected in the vorticity transport equation. It is very surprised that these initial turbulent vortices are not generated via the vortex stretching term of the vorticity transport equation, but via the viscous term (singularities). The mismatch of the energy loss with the velocity leads to velocity discontinuity, which results in unbounded vorticity through the viscous term, $\nu \nabla^2 \omega$. The numerical simulation results show that these singularities occur at the center of the hairpin vortex head and between the vortex legs.

7. Conclusions

Existence of global smooth solutions to the Navier-Stokes equation for 3D plane Poiseuille flow is disproved rigorously via a novel singularity mechanism.

The composite viscous term of the instantaneous flow vanishes locally at (x^*, t^*), eliminating viscous energy loss at that point. This phenomenon violates the energy-velocity monotonicity (positive velocity vs. zero viscous energy dissipation), forming a Navier-Stokes singularity in physics.

The unstable singularity in physics collapses to zero velocity, creating a velocity discontinuity. The discontinuity causes the velocity gradient norm to diverge, destroying global smoothness.

This mechanism provides a physically grounded explanation for the failure of global smooth solutions in 3D Navier-Stokes equation, bridging the gap between mathematical analysis and experimental observations of laminar-turbulent transition.

If the regularity problem of the Navier-Stokes equation is to be solved mathematically, the study of the Navier-Stokes equation should not be conducted alone from mathematical analysis. It is necessary to refer to the numerical computing results of the Navier-Stokes equation and to understand how the Navier-Stokes equation works during the process of turbulence generation.

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