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Article

A Combinatorial Proof of the Existence of a 4-Coloring for Planar Graphs

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Abstract

A combinatorial tree (Pt) is used to record valid colors to be assigned to each of the vertices of a planar graph G of order n . The main process consists of a loop that incrementally builds G vertex by vertex, starting from the most internal triangular face of G , and in $n - 3$ iterations, the paths constructed in Pt will have the valid color labels assigned to the vertices of G . This method ultimately generates all proper 4-colorings of G . In each iteration, a vertex $v_i \in V(G)$ is selected to be aggregated to the current induced subgraph G_i of G . This process, alongside the use of the Pt tree (which results in a binary tree of depth $n - 3$), ensures all proper 4-colorings of G , regardless of the topology of the maximal planar graph G . Additionally, we develop an existential theorem that shows that for any maximal planar graph G , it is always possible to create a proper 4-coloring. Furthermore, we detail a method through which such a 4-coloring can be constructed. Also, we present the extremal topologies of planar graphs, highlighting those with the maximum and minimum number of 4-coloring functions.

Keywords: 4-coloring problem; existence theorem 4-coloring; maximal planar graph; vertex building graph; combinatorial tree

1. Introduction

The graph vertex coloring problem involves assigning colors to the vertices of a graph with the aim of using the fewest possible colors, while ensuring that adjacent vertices have different colors. The minimum number of colors required in any such coloring of a graph G is known as the chromatic number of G , denoted by $\chi(G)$. If it is possible to color G using a set of k colors, then G is said to be k -colorable, and such a coloring is referred to as a k -coloring. When $\chi(G) = k$, then G is described as k -chromatic, and any k -coloring represents a minimum coloring of G .

Since its initial proposal in 1852 by Francis Guthrie [7], a conjecture was proposed to know if it is possible to use a maximum of 4 colors for map coloring, in such a way that no two adjacent regions of the map share the same color (The Four Color Conjecture '4CC'). This conjecture has presented a significant challenge to the scientific community for constructing a mathematical proof to validate it.

The famous Four-Color Theorem (4CT) states that every planar graph is vertex 4-colorable [18]. However, the 4CT could involve the 3-coloring problem, which is a classic NP-complete problem, even for planar graphs [11]. The 4CT settles a conjecture that has been for more than a century, the most famous unsolved problem in graph theory and perhaps in all of mathematics [10]. The 4CC is now, a broad area of research, with hundreds of contributors and thousands of contributions [3]

A proof of the 4CT proposed by Kempe [12] in 1879, turned out to be flawed. The seeming impossibility of proving the 4CT, and the increasing power of computers in the early 1960's, led scientists to use computers for showing that there are no counter-examples for the 4CT. A programming proof of the 4CT was introduced in 1977 by Appel and Haken [1,2], assisted in algorithmic work by Koch [2].

The orientation in resolving the 4CT, based on Appel and Haken's algorithm [1,2], is based on the application of the discharging method plus a large finite case check by computer. Initially, this involved analysing 1,834 configurations in order to identify all possible unavoidable sets of configurations in a

planar graph. Robertson [18] has narrowed down the number of unavoidable sets of configurations to close to 700. However, this reduction requires significant computational support to manage the numerous cases that can be created for any planar graph. As a result, this computer program becomes impractical to present as a mathematical proof of the 4CT.

Initially, the proof did not gain general approval among mathematicians, however, it did fuel a debate [19] on the meaning of a mathematical proof. Nonetheless, a variation of the initial programming proof of the 4CT published in 1997 by Robertson, Sanders, Seymour, and Thomas [18] has been predominantly accepted.

The discharging method has proven to be an effective proof technique, particularly for graph coloring problems. However, its main drawback is that it often necessitates extensive case analysis. To address this challenge, various automatic methods have been developed to search for discharging proofs automatically [4,6].

For example, in 2005, Gonthier [9] applied an automatic theorem software (the Coq proof assistant) to build an automatic proof for the 4CT. The computer-assisted proofs contributed significantly to the development of "computer formal methods as a primary proof technique." For instance, in the 4-coloring problem, these methods enabled the mathematical argument to be verified more thoroughly, with various cases encoded and checked by an automatic proof assistant. For a comprehensive history of the four-color problem and in-depth review of plane graph 3- and 4-colorings, refer to Fritsch and Fritsch [7] and to Borodin [3].

Additionally, there are numerous publications available online focused on the 'proof' of the 4CT theorem. However, to date, none of these have been acknowledged by researchers in the field as a formal, complete, and correct proof of the 4CT.

We address the 4CT problem from a combinatorial perspective, through the vertex incremental construction of a maximal planar input graph G . Our method utilizes a binary tree to list the valid colors of the vertices of G . This article also provides a clear and novel demonstration of the 4CT by using graph theory and combinatorial techniques, all without the assistance of a computer program.

This paper is structured as follows: the introduction section 1 provides a general introduction about the 4 coloring theorem. Basic definitions and the notation to be used are presented in section 2 - Preliminaries. In section 3, we introduce extremal topologies of planar graphs for the 4-coloring problem. In section 4, the main structures and procedures to be used in our method are introduced. In section 5, the existence' theorem for the 4CT is proved. The final section offers conclusions drawn from this work.

2. Preliminaries

Let us consider a planar graph $G = (V, E, F)$, where $n = |V|, m = |E|$ and in the case of planar graphs, let $k = |F|$ be the number of faces in G . We assume that the reader is acquainted with common terms and symbols related to graph theory, specifically concerning planar graphs (see e.g. [10] and [17]). In this section, we will introduce only a few notation that will be utilized.

When two distinct vertices are connected by an edge $\{v, w\} \in E$, we describe v and w as adjacent vertices. A graph where every pair of different vertices is adjacent is termed a complete graph. The complete graph with n vertices is symbolized as K_n .

The set $N_G(x)$, defined as the neighborhood of x in a graph G , is given by $N_G(x) = \{y \in V(G) : \{x, y\} \in E(G)\}$, while its closed neighborhood, denoted $N_G[x]$, is the union of $N_G(x)$ with $\{x\}$, i.e., $N_G[x] = N_G(x) \cup x$. The size or cardinality of a set A is the number of elements of A , denoted by $|A|$. The degree of a vertex $x \in V(G)$, denoted by $\delta_G(x)$, is $|N_G(x)|$. Only when there is no doubt about the graph G to which the vertex's neighborhood refers is it omitted. The degree of the graph is denoted as $\Delta(G) = \max\{\delta(x); x \in V(G)\}$.

A subgraph of a graph $G = (V, E)$, is also a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. When the subgraph G' contains all the vertices from G , G' is a spanning subgraph of G , and G' is termed as induced by V' , if it includes all edges from G connecting the vertices of V' . Such an induced subgraph

is also denoted as G/V' . Then, $G - V'$ would be the induced subgraph from $V - V'$. On the other hand, when $E' \subseteq E$, $G - E' = (V, E - E')$. When $V' = \{v\}$ and $e = \{u, v\}$, previous notation is usually simplified as $G - v$ and $G - e$, respectively.

2.1. Elements of a Planar Graph

A graph drawing of a graph G , denoted as Γ , represents each vertex $v \in V(G)$ as a unique point $\Gamma(v)$ in the plane, and each edge $\{u, v\} \in E(G)$ as a simple Jordan arc $\Gamma(u, v)$ with endpoints $\Gamma(u)$ and $\Gamma(v)$. A drawing is considered planar if it can be embedded in a plane without any two edges intersecting, except possibly at their shared endpoints. When a graph G has a planar drawing, it is called a *planar graph*.

One relevant result in graph theory is Kuratowski's theorem, which provides constraints for determining whether a graph G is planar. According to this theorem, G is planar when it does not contain a subgraph that is a subdivision of K_5 or $K_{3,3}$ [13]. It is worth mentioning that a disconnected graph is planar if and only if each of its connected components is planar. Similarly, the theorem of Wagner states that a graph G is planar if and only if it does not have K_5 or $K_{3,3}$ as minor [15].

Planar graphs are significant in both graph theory and graph drawing. They possess several interesting properties: they are sparse, four-colorable, and their structure is described succinctly and elegantly [5].

A planar graph drawing divides the plane into connected areas known as faces. The face that extends infinitely is referred to as the outer face or the external face of the graph. f_0 denotes the outer face of a given planar graph G . Meanwhile, f_0^i will denote the outer face for the graph G_i .

If all the vertices of a planar graph can be drawn incident to the outer face of the graph, then it is called an outerplanar graph. Similarly, an outerplanar graph may be characterized by the two forbidden minors K_4 and $K_{2,3}$.

Let $F(G) = \{f_1, \dots, f_{k-1}\}$ be the set of closed non-intersected faces of G . The internal area of each face $f \in F(G)$ is delineated by the set of edges that enclose it. Notice that the outer face f_0 of the graph is not included in $F(G)$, because we want to consider only the internal faces related to G . $Int(G)$ denotes the internal vertices of G , which are vertices not incident to f_0 , i.e. $Int(G) = V(G) - V(f_0)$. Let $f \in F(G)$, a *facial walk* w corresponding to f is the shortest closed walk induced by all edges incident with f . If the boundary of f is a cycle, the walk is called a *facial cycle*. The length of a face f is denoted by $|f|$ and equals $|V(f)|$. We say that a face $f \in F(G)$ is a triangle when $|f| = 3$.

Two faces f_i and f_j in $F(G)$ are considered adjacent if they share edges, meaning $(E(f_i) \cap E(f_j)) \neq \emptyset$. If two faces do not share any edges, they are termed independent faces. It is worth noting that independent faces may share vertices but do not share edges. A collection of faces is independent if every pair of faces within it is independent. An edge $\{u, v\}$ in G that is not part of a facial cycle of any internal face in $F(G)$, is called an *acyclic edge*. An acyclic edge is considered adjacent to a face if it shares only one vertex. Similarly, two acyclic edges are adjacent if they have a common endpoint.

Given a planar drawing, the (clockwise) circular order of the edges incident to each vertex is fixed. Two planar drawings are equivalent if they determine the same circular orderings of the edges incident to each vertex (sometimes called rotation scheme). A (planar) embedding is an equivalent class of planar drawings and is described by the clockwise circular order of the edges incident to each vertex. A graph put together with one of its planar embedding is sometimes referred to as a plane graph [5].

3. Determination of Extremal Topologies for the 4-Coloring

Let $Four = \{c_1, c_2, c_3, c_4\}$ be a set containing 4 different colors. Given a planar graph $G = (V, E, F)$, let $c : V \rightarrow Four$ be the assigning 4-colouring function which assigns to each vertex $v \in V$ a unique colour in $Four$. The assigning function c is proper if for each edge $\{u, v\} \in E$, $c(u) \neq c(v)$. Given a planar graph $G = (V, E, F)$, we analyse in this section, the set of all possible 4-coloring functions that can be built on $V(G)$.

We consider that exist an arbitrary ordering on the set vertex from G , i.e. $V = \{v_1, v_2, \dots, v_n\}$. Let H be a subgraph from G , $Col(H)$ denotes a proper 4-coloring on the set vertex from H and inferred

by the ordering on V . For example, given an arbitrary edge $e = \{u_i, u_j\} \in G$, $Col(e) = (a_1, a_2)$, where $a_1, a_2 \in Four, a_1 \neq a_2$, and $c(u_i) = a_1, c(u_j) = a_2$, if $i < j$, otherwise $c(u_j) = a_1, c(u_i) = a_2$. $Col(G) = (a_1, a_2, \dots, a_n)$ denotes a total 4-coloring of G according to the order given on the set vertex from G .

Let $4Col(G) = \{C_1, \dots, C_t : C_i = Col(G), C_i \neq C_j, i \neq j, i, j = 1, \dots, t\}$ be the set of all proper 4-colouring that can be built on a graph G . And let $4c(G) = |4Col(G)|$ be the number of different proper 4-colorings that can be built on the graph G .

3.1. Counting 4-Colouring Process

We introduce a systematic way to count all 4-coloring of any subgraph $H \subseteq G$, induced by the ordering set vertex from V . In order to calculate $4c(H)$, we consider that each vertex and each edge from H is visited only once.

The first visited vertex would consider 4 different colors since it has not any restriction for assigning one element from $Four$. When the following vertices are considered, they could have restriction according to the previous visited vertices from its neighbourhood.

For example, for an isle vertex u , $4c(u) = 4$. Given an aisled unitary edge $e = \{u, v\}$, $4c(e) = 4 * 3 = 12$, since when v is visited, it has only one restricted color, the color of u .

Let us consider the planar graph formed by n vertices, where each vertex is an isle, $Is = (V, E)$, $V = \{v_1, \dots, v_n\}$, and $E = \emptyset$. Then, Is is a completely disconnected graph. In this case, each vertex could be colored with whatever of the four colors, since there is not constraint between colors for any pair of vertices, then $4Col(Is) = \{(a_1, \dots, a_n) : a_i \in Four, i = 1, \dots, n\}$. $4Col(Is)$ consists of all lists of n items formed by the colors in the set $Four$. There are $4c(Is) = |4Col(Is)| = 4^n$ different 4-colorings for Is .

An edge $e = \{v_1, v_2\}$ constraints the combination of colors that can be assigned to its extreme points. In the case of the 4-colorings, and only one edge e , although v_1 can be assigned one of the four colors from $Four$, for each color assigned to v_1 , v_2 can only be assigned 3 possible colors. $4Col(\{v_1, v_2\}) = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4, 3)\}$, and therefore, $4c(e) = 4 * 3 = 12$.

In fact, when a path P_n (with n edges and $n + 1$ vertices) is considered, the previous numeric formula is kept. We are traversing on P_n linearly, beginning from vertex v_1 . To the first vertex visited v_1 , it can be assigned whatever color from $Four$. But for each new vertex visited $v_i \in V(P_n), i = 2, \dots, n + 1$, it has three possible colors to be assigned, avoiding the use of the color from its previous neighbour (See Figure 1). Then, it follows that $4c(P_n) = 4 * 3^n$.

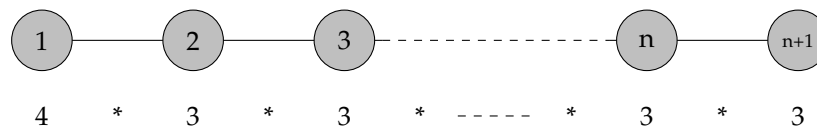


Figure 1. Counting the number of 4-colorings on a path P_n .

When a tree T_n is traversed in post-order, after visiting the root node $v_r \in V(T_n)$, the following vertices to be considered have only one restricted color. Considering a connected planar graph, a similar formula for the number of 4-colorings is shared for any tree graph T_n with n vertices and $n - 1$ edges (see for example, Figure 2). But for other topologies, the vertices can take more than one restricted color.

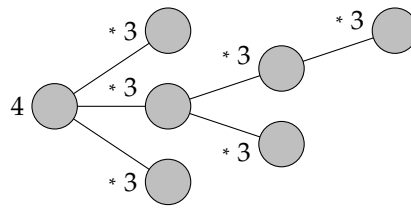


Figure 2. Counting 4-colorings on a tree.

Lemma 1. *The connected planar graph with a maximum number of 4-colorings is a tree.*

Proof. Let T_n be a tree of order n , whose root vertex is v_1 . T_n is traversed in pre-order, then v_1 is the first vertex visited. v_1 can take one of the 4 possible colors from *Four*.

For any node $v_i \in V(T_n)$ where $v_i \neq v_1$, when v_i is visited during the traversal, it has one unique restriction: it cannot be colored with the same color as its already-visited parent node. Therefore, v_i has three possible colors available for use. As a result, the formula for the total number of colorings is given by $4c(T_n) = 4 * 3^{n-1}$.

A tree T_n represents the topology of a graph with the minimum number of edges required to keep it connected to any planar graph G_n containing n vertices. Therefore, the maximum number of distinct 4-colorings for any connected planar graph G_n with n vertices is given by $\max\{4c(G_n)\} = 4 * 3^{n-1}$. \square

When new edges are added to a connected planar graph without adding any new vertices, the number of possible 4-coloring functions decreases. This is because the additional edges place constraints on the colors that can be assigned to the vertices at their endpoints.

When acyclic edges are added to a connected planar graph G , it necessitates the addition of new vertices. As a result, the number of valid 4-coloring functions increases. This is because each new vertex contributes to a threefold increase in the previous value of $4c(G)$.

In a planar graph, the acyclic subgraphs are 4-colorable and, in fact, they are 2-colorable. To simplify the counting process for 4-coloring planar graphs, we will assume from now on that any input planar graph G does not contain any acyclic edges. Consequently, all faces in G will have a facial cycle.

For example, consider simple cycles such as the triangle and the square. In Figures 3 and 4, we show the combinatorial enumeration of the all 4-coloring functions created for both cycles.

It is not complicated to count 4-colorings in the case of simple cycles. For instance, consider a triangular cycle $f_t : v_1 - v_2 - v_3 - v_1$. The first vertex v_1 can be assigned one of the 4 colors. The second vertex v_2 can choose from only 3 colors, as it cannot use the same color as v_1 . Finally, the third vertex v_3 is left with only 2 available colors. Therefore, the total number of 4-colorings for a triangular face f_t can be calculated as $4c(f_t) = 4 * 3 * 2 = 4! = 24$ (see Figure 3).

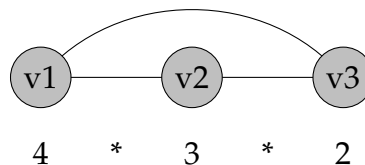


Figure 3. counting 4-colorings on a triangle cycle.

In the case of even cycles, such as a square $f_c : v_1 - v_2 - v_3 - v_4 - v_1$, we can assign colors to the vertices as follows:

- The first vertex, v_1 , can be assigned one of 4 colors.
- The second vertex, v_2 , has only 3 color options since it cannot repeat the color of v_1 .
- The third vertex, v_3 , can also be assigned one of 3 colors, provided it is different from the color of v_2 . However, two of these choices for v_3 limit the available color options for v_4 to 2 colors. This is because v_4 cannot have the same color as v_1 or v_3 if they are different. While one of the colors of v_3 can have 3 colors assigned for v_4 .

Thus, the total number of 4-colorings for a square cycle f_c is $4c(f_c) = 4 * 3 * (2 * 2 + 3) = 12 * 7 = 84$. In Figure 4, only one of the 4 branches derived from assigning the color c_1 (out of the 4 possible colors) to vertex v_1 is shown.

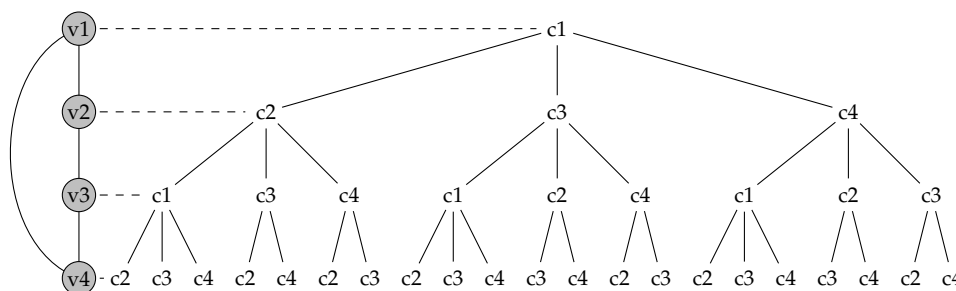


Figure 4. Counting 4-colorings on a square cycle, considering only one of the 4 combinations for v_1 .

Perhaps the most renowned property for planar graphs is the one stated by Euler’s theorem, which states that planar graphs are sparse. Namely, given a plane graph with n vertices, m edges, and f faces, planar graphs hold $n - m + f = 2$.

A direct consequence of Euler’s formula is that in a maximal planar graph (where every face is a triangle) with at least three vertices, we have $2m = 3f$, and then $m \leq 3n - 6$. This number reduces to $m = 2n - 3$ for maximal outerplanar graphs with at least three vertices (and $m \leq 2n - 3$ for general outerplanar graphs). Furthermore, when the planar graph is triangle-free, then $m \leq 2n - 4$. Finally, if the graph is a tree, then $m = n - 1$.

The previous insights enable us to substitute m with n in asymptotic calculations concerning planar graphs, whereas for general graphs, we can only assume that $m \in O(n^2)$. From a practical standpoint, this enables us to determine the non-planarity of denser graphs without examining all edges, which would necessitate a quadratic-time algorithm.

On the other hand, relating to 4-coloring functions on planar graphs, it is known that any general planar graph G can be reduced to a maximal planar graph G' and where all 4-coloring for G' is also a 4-coloring for G .

Considering maximal planar graphs, the bound ($2m = 3f$) can be substituted for relating number of faces with number of vertices, and the Euler’s equality holds $f = 2/3m$, obtaining $f \leq 2n - 4$. Considering maximal planar graphs, and tabulating the maximum value for the number of edges and number of faces while the number of vertices is increased one by one, the Table 1 is obtained.

Table 1. Growth (denoted by Δ) in the number of edges and faces corresponds to the increase in the number of nodes by one.

n	m	f
2	1	1
3	3	2
4	6	4
5	9	6
6	12	8
7	15	10
...
$\Delta 1$	$\Delta 3$	$\Delta 2$

Let $G = (V, E, F)$ be a maximal planar graph of order n . We consider a subgraph $G_0 = f_1$, which is induced from G and formed by f_1 , where f_1 is the most internal triangular face in G .

According to Figure 3, we have $4c(f_1) = 4!$. However, when we fix the first three colors for the vertices of f_1 , the factor $4!$ simplifies to 1. This simplification avoids the need to consider the $4!$ possible permutations of the vertex colors in *Four*.

Table 1 shows that when a planar graph grows incrementally, one vertex at a time, the addition of a new vertex v_i to the current subgraph, i.e. $G_i = (G_{i-1} \cup \{v_i\})$, at this point, the maximum number of edges that can be incident from v_i to G_{i-1} is three, and since G_{i-1} maintains the maximum number of edges for each new vertex added to it.

Since for each G_i , the size of its outer face remains constant at three (denoted as $|f_0^i| = 3$), we can ensure that in each iteration, when a new vertex v_i is incorporated into G_{i-1} , the maximum number of new edges continues to form. This process results in an extreme topology for planar graphs where G has a minimum number of 4-coloring functions.

Lemma 2. *The planar graph with a minimum number of 4-colorings is the graph formed by the nesting of K_4 .*

Proof. Let G_n be a planar graph that has been formed iteratively and incrementally, beginning from the triangle face f_1 , which is the innermost face of G_n . Let $G_1 = f_1$. From example 1, we know that $4c(G_1) = 4! = 24$. Although if we fix the three different colors for the endpoints of f_1 , we reduce the factorial number of permutations $4!$ on *Four*, and simplify the number of colours to be assigned to the following vertices to be aggregated.

As G_n is formed by adding vertex by vertex, starting with G_1 , and in each iteration i , a new vertex $u_i \in (G_n - G_i)$ is added, then $G_{i+1} = (G_i \cup \{u_i\})$. We want that each time that the new vertex $u_i, i = 1, \dots, n - 3$ is aggregated to G_i , the change in the value of $4c(G_i)$ to $4c(G_{i+1})$ has a minimal increase. The minimum growth of the function $4c(G_i)$ occurs when $4c(G_{i+1}) = 1 \times 4c(G_i)$. This situation arises when $N_{G_i}(u_i)$ is incident with exactly three different colors from the outer face of G_i .

In order to use a minimum number of edges for u_i , at least $|N_{G_i}(u_i)| = 3$. And in this case, $c(u_i)$ has just only one possible color to be assigned, then $4c(G_{i+1}) = 1 * 4c(G_i)$.

Observe that K_4^i (see Figure 5) forms a nested pattern created by aggregating each new vertex u_i to the current graph. \square

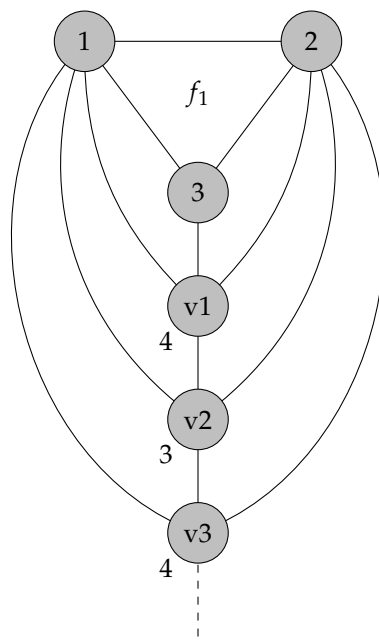


Figure 5. The extremal planar graph minimizing the number of 4-colorings (Illustrating K_4^3).

Figure 5 illustrates an extreme planar graph G_m with the minimum number of 4-coloring functions. By fixing the colors of the first triangular face f_1 , the colors assigned to the vertices alternate between only two possible colors. In this way, G_m has only one option of colours for its vertices. However, the factorial number of permutations $4!$ appears when no color is fixed to the initial triangle f_1 . Thus, considering the general case, $4c(G_m) = 4! = 24$.

Only in the case that $N_{G_i}(u_i)$ is incident to the 4 different colors in *Four*, it follows that for this combination of colors on G_i , it is not possible to assign a color from *Four* to u_1 . However, in the following sections, we will show that there is no planar graph G for which $4c(G) = 0$.

4. Principal Procedures and Structures

In the process of constructing a 4-coloring for an input planar graph G_I , useful preprocessing involves transferring certain elements to a Stack. This includes both vertices of degree less than 3 and acyclic edges, following an iterative order that starts with the objects of the lowest degree. By doing so, we obtain a simplified graph G from G_I . The elements in the stack can then be 4-colored without difficulty after 4-coloring the remaining graph G .

Additionally, it is beneficial to reduce the remaining graph to one of its maximal versions, where all internal face of G are triangles. Although our proposal can work on non-maximal graphs, the analysis of the method is simplified by considering only triangle as internal faces. As it is known, if an input graph G_I is not maximal, this can be polynomially reduced to a maximal graph G , where any 4-coloring of G is also a 4-coloring for G_I . Therefore, in this section, we will focus on processing a maximal connected planar graph G , in which all vertices have a degree greater than 2.

During the counting of 4-colouring functions, it is not necessary traversing the vertices and edges in a specific order. However, all object (vertices and edges) have to be considered during the counting of 4-colourings.

As it is known, an arbitrary ordering on the vertex set does not help to reduce the computational complexity of graph problems. Furthermore, imposing an arbitrary vertex ordering does not change the NP-hard status, because the hardness of a graph problem arises from the combinatorial structure of the graph, not from the labelling of the vertices [8], unless the ordering captures the graph's symmetry and combinatorial structure related among vertices [16].

Some orderings (e.g., tree decomposition) help in parametrized complexity, but these are not arbitrary [14]. In our case, we provide an ordering on the vertex set according to the incremental and iterative construction of the input planar graph $G = (V, E, F)$ starting from the most internal triangular face $f_1 \in F(G)$.

4.1. Decomposition in Layers of a Planar Graph

Let $G = (V, E, F)$ be a planar graph, where $F(G) = \{f_1, f_2, \dots, f_k\}$ is formed by only triangular faces. The goal is to rebuild the graph G starting with the most internal triangular face $f_1 \in F(G)$, which has no vertices in its interior space. The idea, is that the "most internal" face f_1 would be the one that is surrounded by other faces and lies deepest inside the nesting of the different levels of outer faces in G . Notice that the most internal face is different to other topological concepts, such as the most central face of a graph.

To identify f_1 as the most internal triangular face, a decomposition of G is carried out based on layers. Each layer, denoted as L_i , is formed by the outer face of the current subgraph G_i of G .

At the beginning of the layer's decomposition, $G_1 = G$ and $L_1 = f_0$ - the current outer face of G . In each iteration, a new subgraph $G_{i+1} = (G_i - V(L_i))$ is formed. Given the new subgraph G_{i+1} , its current outer face is identified, and it is denoted as L_{i+1} .

In this way, G is decomposed into layers based on the outer face of each subgraph of the decomposition. Each layer L_i corresponds to the outer face of G_i , and the vertices of this layer are removed from G_i to create the subsequent subgraph G_{i+1} .

This iterative process finishes until obtain as last subgraph G_r , or a set of faces forming a unique outer face, or forming a forest. Let $Ly(G) = \{L_1, \dots, L_r\}$ be the set of layers obtained from the decomposition of G by outer faces. If there are triangular faces in the last layer L_r , choose the most central face as f_1 . This selection should help establish a clockwise order of the vertices of the current outer face L_r .

When L_r is a forest, the innermost triangular face f_1 is formed by selecting its endpoints from the dominated vertices of L_r . To complete this triangular face, additional vertices are taken from L_{r-1} . These extra vertices from L_{r-1} must have maximal degree in G .

For example, in Figure 5 there are two layers formed by the vertices $L_1 = \{1, 2, v_3\}$ and $L_2 = \{3, v_1, v_2\}$. f_1 is formed by the vertex 3 since such vertex is dominated by v_1 in L_2 . Meanwhile, the other endpoints for f_1 are the vertices 1 and 2 because they have maximal degree in G , then the most internal face for graph in Figure 5, is $f_1 : 1 - 2 - 3 - 1$.

4.2. Rebuilding the Original Planar Graph

After to recognize the most internal triangular face $f_1 \in F(G)$, to the three endpoints of f_1 are assigned fixed colors, reducing the $4!$ initial permutations from *Four*. A process of building G is beginning by a vertex addition algorithm, forming a family of planar graphs: G_1, G_2, \dots, G_{n-3} of induced subgraphs from G , where $G_1 = f_1$, and $G = G_{n-3}$.

The construction of G is an incremental and iterative process, where in each iteration $i = 1, \dots, n - 3$, a vertex $v_i \in (G - G_i)$ is selected to form $G_{i+1} = G_i \cup \{v_i\}$. Then, in $n - 3$ iterations, G is obtained derived from $G = G_{n-3} = G_{n-4} \cup \{v_{n-4}\}$. At the same time, $j = r, \dots, 1$ is the index pointing to the current layer L_j in which the vertex v_i is selected.

In each iteration, the selection of the vertex $v_i \in (G - G_i)$ holds the following hierarchy of criteria:

1. The remaining vertices $(G - G_{i+1})$ are in the outside face of G_{i+1} .
2. $\delta_{G_i}(v_i)$ is maximum into the set of vertices in $(G - G_i)$.
3. Considering the current layer L_j from the process of decomposition in layers of G , v_i is in a first instance in L_j , or in L_{j-1} of G .
4. When some vertices are holding all previous criteria, v_i is the following vertex according to the clockwise direction from the vertex v_{i-1} .

We denote by f_0^i the outer face of the corresponding subgraph G_i , into the family of graphs formed by the construction of G . $Color(H)$ denotes the set formed by the different colors used in a 4-coloring of H , and $\#C(H) = |Color(H)|$ is the number of different colors in the set $Color(H)$, then $\#C(H) \leq 4$.

To obtain a proper 4-coloring for G while building it, we need to consider when a new vertex v_i is added to G_i . At this point, we identify the set $Color(N_{G_i}(v_i)) \subseteq Four$, which consists of the colors already used by the neighbors of v_i . The potential colors that v_i could take are given by the set $Four - Color(N_{G_i}(v_i))$. These colors are used as the edges' labels of a combinatorial tree.

Thus, it is enough to show that $(c(v_i) \cup Color(N_{G_{i-1}}(v_i))) \subseteq Four$ and that $c(v_i) \notin Color(N_{G_{i-1}}(v_i))$. As v_i is added from the outside face from G_i , and as G_i is planar, then $N_{G_i}(v_i) \subset V(f_0^i)$. When we are considering a maximal planar graph, where all internal face in G is a triangle, then $\delta_{G_i}(v_i)$ is bottom bounded, since if $\delta_{G_i}(v_i) = 1$, then G_i is not maximal, since all new vertex aggregated must form at least one new triangular face in G_{i+1} that it does not exist in G_i . Thus, $\delta_{G_i}(v_i) \geq 2$.

In fact, for our method, it is not necessary that G to be maximal. Instead, the requirement is that when adding each new vertex v_i , it must be incident to at least two colors of G_i , allowing the formation of a binary tree recording the possible colors assigned to v_i . A key point when a vertex v_i is aggregated to G_i is that $N_{G_i}(v_i) \subseteq V(f_0^i)$, since G_i is planar, v_i is in the outer face of G_i and all edge from v_i has endpoint in the outer face (f_0^i).

On the other side, when $\delta_{G_i}(v_i) = 2$, then $|V(f_0^{i+1})|$ is increasing in one with respect to $|V(f_0^i)|$, since v_i is added to (f_0^i) and not new internal vertices are formed in G_{i+1} . When $\delta_{G_i}(v_i) = 3$, v_i is part of the new (f_0^{i+1}), but also a new internal vertex is formed in G_{i+1} , which is one of the adjacent vertices of v_i in (f_0^i). Then, the cardinalities between (f_0^i) and (f_0^{i+1}) do not change.

When $\delta_{G_i}(v_i) > 3$, only v_i and two of the extreme vertices of $N_{G_i}(v_i)$ are kept in f_0^{i+1} , but the other vertices from $N_{G_i}(v_i)$ are now internal vertices in G_{i+1} decreasing the cardinality $|V(f_0^{i+1})|$ with respect to $|V(f_0^i)|$.

4.3. Combinatorial Tree

The tree structure effectively illustrates various algorithmic properties, such as the use of proof trees in Gentzen systems, recursion trees for monitoring recursive processes, and execution trees to observe the execution flow in Prolog.

In our approach, we utilize a *combinatorial tree* (Pt) to systematically label the colors that can be assigned to the vertices of G . In this tree, nodes in Pt are labelled by the vertices $v_i \in (G - G_i)$, which are selected during the graph reconstruction process. Meanwhile, edges in Pt coming from v_i record the possible colors that can be assigned to v_i .

As the three endpoints of f_1 have assigned fixed colors, for each new vertex v_i aggregated to G_i , it could take a maximum of two possible colors, since $\delta_{G_i}(v_i) \geq 2$. Thus, Pt is a binary tree, whose root node is the first selected vertex v_1 .

All nodes at level $i, i = 1, \dots, n - 3$ of Pt have the same label: v_i , which corresponds to the vertex that has been added to G_i in the iteration i . We denote as $Pt(i)$ at level $i, i = 1, \dots, n - 3$ of Pt . Each edge derived from $v_i \in Pt$ is labelled with one of the colors that can be assigned to v_i . Since $\delta_{G_i}(v_i) \geq 2$, there are either one or two edges originating from v_i , because v_i is incident to at least two different colors in G_i . Furthermore, if two edges come out from v_i in Pt , as the labels of the paths coming from v_i are different, this establishes a system of mutually exclusive color-paths within Pt .

A path of size $n - 3 \geq j \geq 0$ of Pt starts at the root v_1 up to level $Pt(j + 1)$. For paths where $1 \leq j < n - 3$, the last vertex is a 'dummy' vertex (note that this dummy vertex is not illustrated in any figure of Pt). When the paths in $Pt(j)$ are extended to one level higher (achieving level $Pt(j + 1)$), a new edge is inserted before the dummy vertex. Such edge connects v_{j+1} with the dummy vertex, and the edge is labelled by the color assigned to v_{j+1} .

Any path from $Pt(j)$ establishes a 4-coloring on the vertices of $G_j, j = 1, \dots, n - 3$. Within the paths of Pt , there can be failure paths, which are denoted by placing the symbol X as a leaf node in the path of Pt . Meanwhile, the parent node v_{j-1} of X has only one unlabelled edge towards the leaf X , indicating that it is impossible to assign a color to v_{j-1} . From a failure node X , the corresponding path is truncated and will not be expanded further.

The paths of success in Pt are paths of length $n - 3$, that is, they are paths that start from v_1 , reach the vertex v_{n-3} , and establish a single proper coloring to be assigned to each vertex $v_i, i = 1, \dots, n - 3$ on the path. The leaf node of a successful path in Pt is labelled by E ('exit').

Each outer face f_0^i is formed by the facial cycle surrounding G_i (the induced graph G_i of G), regardless of the parity of this cycle, it is known that three colors are sufficient to color any simple cycle. Since the paths in Pt represent all possible color combinations that can be assigned to $V(G_i)$, including its outer face f_0^i , there will always be paths in Pt that use only 3 colors to color $V(f_0^i)$. The existence of these paths will be an important property in the proof of our main theorem.

Summarizing, we utilize three support structures in the construction of a 4-coloring for a planar graph G . For an iteration $j < (n - 3)$, the first structure is the graph G_j with $j + 3$ vertices, which is induced from G . In G_j , only the endpoints from the face f_1 have fixed colors, while the remaining vertices do not have any colors assigned. However, the colors for those vertices can be inferred from the paths in $Pt(j)$.

The second structure is the outer face f_0^j corresponding to the graph G_j . Note that since $G_j, j = 1, \dots, n - 3$ is built incrementally, its corresponding outer face f_0^j depends on the last vertex v_{j-1} added to G_{j-1} . Lastly, the third structure is the binary tree Pt . In Pt , there may be failure paths; however, in the next section, we will demonstrate that there will always be non-failure paths in Pt , regardless of the level j considered, even when reaching the maximum depth of Pt , which is $n - 3$.

The following figure (Figure 6) illustrates the three structures used as support in the 4-coloring of the graph G . The order in which the vertices were selected during the reconstruction of G is indicated by the index associated with each vertex label. Notice that G is, in fact, not a maximal planar graph.

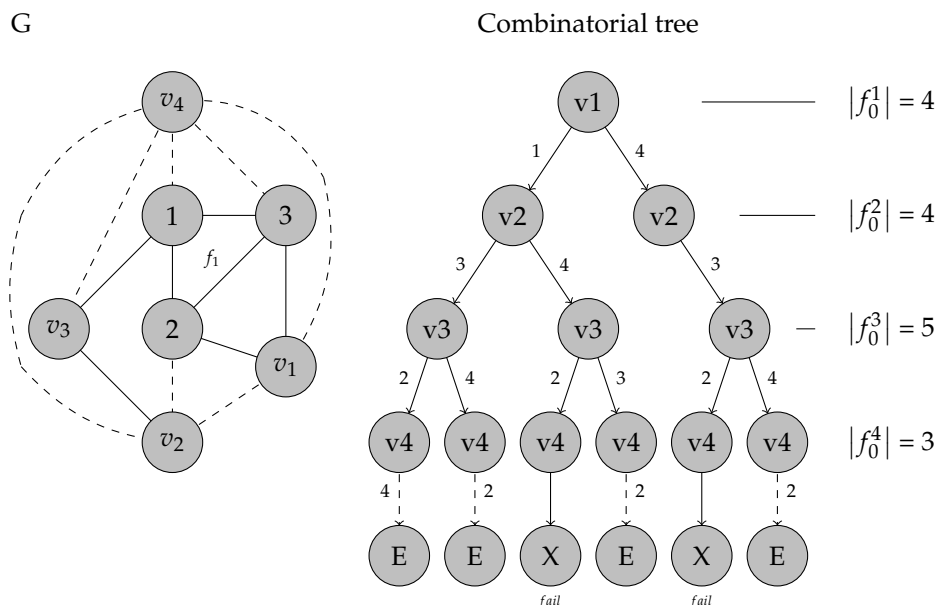


Figure 6. Main structures used in a 4-coloring of a planar graph.

5. Proof of the Existence of a 4-Coloring for Planar Graphs

Theorem 1. Based on the incremental and iterative construction of G by the family G_1, \dots, G_{n-3} , for each iteration $i, i = 1, \dots, n - 3$, there always exists paths in $Pt(i)$ that use fewer than 4 colors for coloring the outer face of G_i , i.e. $\#C(f_0^i) < 4$.

Proof. The proof is by the induction principle applied on the level of the tree: $Pt(i)$.

- 1) For $i = 1$, the first subgraph $G_1 = f_1$ holds that $\#C(f_0^1) = 3 < 4$, since the outer face is a triangular face.
 From G_1 only two cases can be derived to form G_2 . When $\delta_{G_1}(v_1) = 2$ (illustrated in Figure 7(ii), and when $\delta_{G_1}(v_1) = 3$ illustrated in Figure 7(i). Notice that for both cases, there is a path in $Pt(2)$ such that $\#C(f_0^2) < 4$.

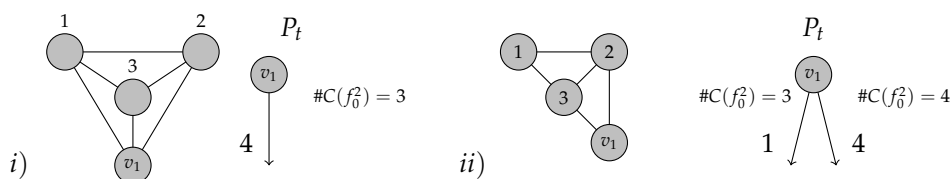


Figure 7. i) $\delta_{G_1}(v_1) = 3$, ii) $\delta_{G_1}(v_1) = 2$.

For example, in the case $\delta_{G_1}(v_1) = 2$. (Figure 7(ii), As one of the possible colours to be assigned to v_1 , it already appears in $Color(f_0^1)$ (in the most left path of Pt), $c(v_1)$ does not add any new colour to the current outer face and then, $\#C(f_0^2) < 4$.

- 2) By induction hypothesis, at level $i - 1$, there are paths in $Pt(i - 1)$ where the current outer face of G_{i-1} has less than 4 colours, i.e. $\#C(f_0^{i-1}) < 4$.
- 3) When a new vertex v_{i-1} is aggregated to G_{i-1} , we form G_i with outer face f_0^i , and the combinatorial tree is extended one level more, this is from $Pt(i - 1)$ to $Pt(i)$. We analyze different cases:

For the paths $Pt(i - 1)$ where $\#C(f_0^i) = 2$, or $\#C(f_0^i) = 3$, if $c(v_{i-1})$ assigns a color that already appears in $Color(f_0^{i-1})$, it will be satisfied that $\#C(f_0^i) < 4$, and therefore, the theorem holds.

Let us consider the case when $\#C(f_0^{i-1}) = 3$, and analyze its development according to the value of $\delta_{G_{i-1}}(v_{i-1})$.

- i) When $\delta_{G_{i-1}}(v_{i-1}) = 2$, then $Pt(i - 1)$ ramifies with two possible colors, say c_1 and c_2 . If $c_1 \notin Color(f_0^{i-1})$, then f_0^i could have 4 colors when the color c_1 is included in $Color(f_0^i)$. However as $\#C(f_0^i) = 3$ and $c_1 \notin Color(f_0^{i-1})$, then c_2 must be in $Color(f_0^{i-1})$ and the branch for c_2 will not increment $\#C(f_0^i)$ with respect to $\#C(f_0^{i-1})$, and therefore for this path $\#C(f_0^i) < 4$ (as it was illustrated in the leftmost path of Figure 7(ii)).
- ii) If $\delta_{G_{i-1}}(v_{i-1}) = 3$. Assuming $|f_0^{i-1}| = 3$, by adding to G_{i-1} a vertex v_{i-1} of degree 3, the cardinality is preserved: $|f_0^i| = |f_0^{i-1}| = 3$ (by results from Table 1 - derived of the Euler's equality), and therefore $\#C(f_0^i) < 4$.
Let us assume $|f_0^{i-1}| > 3$, so now we could take $\delta_{G_{i-1}}(v_{i-1}) \geq 3$. This combination of parameters is analysed in the following case.
- iii) To have $|f_0^{i-1}| > 3$, is only achieved by adding vertices $v_k, k < (i - 1)$, where $\delta_{G_k}(v_k) = 2$, which allows increasing the cardinality of the outer face f_0^{k+1} above 3, and also, it generates binary ramifications at the level $Pt(k)$, using two different colors for v_k , let us say c_a and c_b .
In one of the colors (either c_a or c_b), we have $\#C(f_0^{k+1}) < 4$. This is because, in all ramifications, at least one of the colors used is already present in $Color(f_0^k)$. In fact, for all levels from k to $i - 1$, there are paths where $\#C(f_0^j) < 4$ for $j = k, \dots, n - 1$. Furthermore, according to the inductive hypothesis, there are paths at level $Pt(i - 1)$ where $\#C(f_0^{i-1}) < 4$. Within these set of paths of $Pt(i - 1)$, there is at least one path where the color $c(v_{i-1})$ is already included in $Color(f_0^{i-1})$. This is due to the exhaustiveness in the color assignment to the branches of Pt . For these particular paths, the cardinality of $Color(f_0^{i-1})$ does not increase for the following level; hence, we conclude that $\#C(f_0^i) < 4$, and this confirms that the theorem holds true.
Furthermore, by adding vertices v_{i-1} with $\delta_{G_{i-1}}(v_{i-1}) \geq 3$, new internal vertices are formed in G_i , thus causing some of the colors assigned to those internal vertices to no longer appear in $Color(f_0^i)$.

□

For example, Figure 8(i) shows that two vertices v_1 and v_2 that are added to f_1 , fulfil $\delta_{G_1}(v_1) = \delta_{G_2}(v_2) = 2$, so $|f_0^3| = 5$. The third vertex v_3 that is added to G_3 fulfils that $\delta_{G_3}(v_3) = 5$. The Pt tree (Figure 8(ii)) shows how paths where $\#C(f_0^k) < 4$ will always exist, regardless of the k level of the tree.

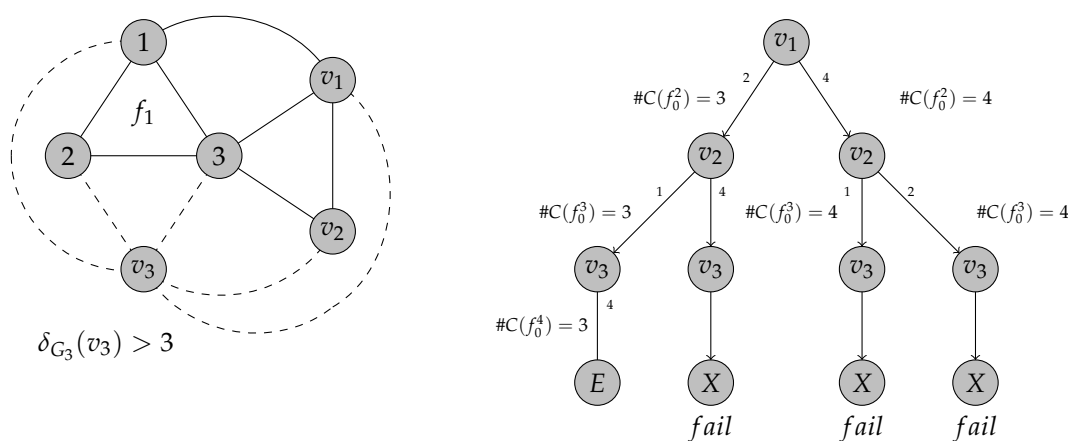


Figure 8. i)Subgraphs with different $|f_0^i|$ ii) $\forall i, Pt(i)$ always has paths where $\#C(f_0^i) < 4$.

It is possible that a path $P' \in Pt(i - 1)$ would have a corresponding value of $\#C(f_0^{i-1}) = 4$ at level $i - 1$. Although, the same path P' could define at level i , a coloring for the new outer face holding $\#C(f_0^i) < 4$. Importantly, regardless of the level i of the tree, there will be paths in $Pt(i)$ where the condition $\#C(f_0^i) < 4$ is satisfied.

The previous theorem is the key in the proof of the existence of a 4-coloring for any planar graph G . By building G vertex by vertex, and forming its combinatorial tree Pt , we acknowledge that there may be failure paths within Pt . However, the previous theorem ensures the existence of paths satisfying the condition $\#C(f_0^i) < 4$, regardless of the level i of the tree and the topology of the induced subgraph G_i of G .

Corollary 1. *For any planar graph G , it always exists a 4-coloring for G .*

Proof. It is known that G can be polynomially reduced to a maximal planar graph G' , where all internal faces are triangular. Based on the vertex construction of G' and the formation of its combinatorial tree Pt , the previous theorem guarantees the existence of paths in $Pt(i)$ where $\#C(f_0^i) < 4$, for all level i of Pt . This ensures that whenever a new vertex v_i is added in the vertex formation of G' , there is always a free color available from the set *Four* that can be assigned to v_i .

Consequently, every vertex added to form the G_i subgraph family, for $i = 1, \dots, n - 3$, one of the four available colors will be assigned to each vertex. This process guarantees that a proper 4-coloring can be achieved for the graph G' and, consequently, for the original graph G . \square

6. Conclusion

A mathematical proof of the Four Color Theorem (4CT) has eluded the efforts of scientists in the area for hundreds of years. Interestingly, the computational verification of the 4-coloring of planar graphs has been known since 1977[1,2], including refinements to the initial algorithm [9,18]. Unfortunately, the vast number and complexity of wanting to mathematically verify the thousands of configurations that can arise during the computational process have led the continued search for a formal and succinct proof of the 4CT.

This article presents a novel method that enables the enumeration of the exhaustive process involved in the 4-coloring of maximal planar graphs. Our method is based on using a combinatorial tree Pt to enumerate all valid color combinations to be assigned to each vertex of the input graph G (except for the vertices of the innermost triangular face of G).

The breadth-first search used in constructing Pt , while incrementally reconstructing G from its innermost triangular face, enables the analysis, construction, and encoding of all possible 4-coloring assignment functions for the vertices of G .

Our theorem confirms that in the combinatorial tree of a planar graph G , there exist paths that ensure, regardless of the size or complexity of the graph G , that each new vertex $v_i \in V(G)$, when added to the induced subgraph G_i of G , will always have at least one color available from the four colors that can be assigned to v_i . Thus, confirming the existence of a 4-coloring for any planar graph. The proof of the theorem is a formal and combinatorial demonstration that is manageable by the mathematical community, due to its extent and the number of cases to be considered in its development.

Although our method guarantees that every planar graph can be colored using four colors, a future task is to propose fast algorithms based on our method, where Pt can be developed via a depth-first search as G is built incrementally, such that upon finding the first proper 4-coloring of G , the process ends.

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Abbreviations

The following abbreviations are used in this manuscript:

4CC	The Four Color Conjecture
4CT	Four-Color Theorem
4Col(H)	the set of all proper 4-coloring of H
4c(H)	the total number of different 4-colorings functions of H
Color(H)	the set of colors used in a 4-coloring of H
#C(H)	the number of different colors used in a 4-coloring of H

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