Article

Not peer-reviewed version

A Riesz-Medvedev Stieltjes Integral Method and Nearness Principle for Solvability of Fractional Integral Equations

Sikiru Sanni , Godspower Emmanuel * , Udeme Johnson , Uduak Edet , Xavier Alexius Udo-utun *

Posted Date: 19 September 2025

doi: 10.20944/preprints202509.1695.v1

Keywords: fractional integral equations; nearness of operators; quadratic integral equation; resolvents; Riesz-Medvedev variation; small perturbations; variation of parameters

Article

A Riesz-Medvedev Stieltjes Integral Method and Nearness Principle for Solvability of Fractional Integral Equations

Sikiru Sanni [1], Godspower Emmanuel [2,*], Udeme Johnson [1], Uduak Edet [1] and Xavier Udo-utun [1,*]

[1] Department of Mathematics, University of Uyo, Nigeria
[2] Department of Mathematics, University of Florida, Gainesville, Florida -USA
* Correspondence: og.emmanuel@ufl.edu; xavierudoutun@uniuyo.edu.ng

Abstract

A Riesz-Medvedev type of total variation method is employed together with the principle of nearness of superposition operators to study solvability of quadratic Voltera-Stieltjes integral equations. This solvability method is then shown to be an invaluable approach to the solvability of a very general class of fractional dynamic problems and other singular integral equations. We have given several illustrative examples to coroborate known solvabilty cases for particular fractional and singular integral equations in literature.

Keywords: fractional integral equations; nearness of operators; quadratic integral equation; resolvents; Riesz-Medvedev variation; small perturbations; variation of parameters

MSC: MSC2020: 45H10, 47H10, 45N20

---

1. Introduction

In this paper we present, below, solvability results for quadratic type Volterra-Stieltjes integral equations with applications to a very general class of fractional dynamic problems and various singular problems taking the general form.

$$u(t) = h(t) + f(t, u(t)) \int_0^t k(t,s) g(s, u(s)) d_s \Phi(t,s); t \in [0,1]. \tag{1}$$

In this presentation, we are particularly motivated by the newer approach reported in a series of publications [10,11,13,14] by Banas and his followers using the properties of the classical total variation functions. We recall, in 2011, Jozef Banas and Tomasz Zajac [10] presented a new approach to the investigation of functional equations of Volterra-Stieltjes type integral equations of fractional integral order. In particular, they initiated a very natural solvability formulation for functional integral equation of Volterra-Stieljes type, subject to existence of solution r_0 to the two auxiliary inequalities below:

$$\|h\| + K(kF_1)\Phi(r) \leq r; \text{ and } kK\Phi(r) < 1. \tag{2}$$

In the same year, Banas and Rzepka [11] published another set of enlightening results for the solvability of the more general formulation below:

$$u(t) = h(t, u(t)), u(a(t)) + (Gu)(t) \int_0^t k(t,s) v(t,s) g(u(s)) ds; \tag{3}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1695.v1

subject to solvabilty requirement on a more complicated auxiliary inequalities. viz the solvability requirement that the following auxiliary inequalities,

$$\eta_1 r + \bar{h} + \bar{k}\bar{v}\varphi(r)\psi(r) \leq r; \tag{4}$$

has a solution r_0 which satisfies

$$\eta_1 + \bar{k}\bar{v}\eta_2\psi(r_0) < 1. \tag{5}$$

The aim of this paper is to employ the more general Riesz-Medvedev type of total variation and the principle of nearness of superposition operators to modify this new approach to obtain a more natural and more relaxed but similar constraint on solvability results which yield invaluable applications to fractional dynamics problems. The quadratic fractional integral equations we study herein are the form below:

$$u(t) = h(t) + \frac{f(t,u(t))}{\Gamma(\alpha)} \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} g(s,u(s))ds;\ t \in [0,1]. \tag{6}$$

Though our method is newer and more natural, equations (6) are special cases of (3) investigated by Banas Rzepka [11] which constitutes generalizations and extensions of problems studied in literature [12,28,33] under the broad class called quadratic integral equations which includes a wide class of singular integral equations as special cases. The class of quadratic integral equations are mostly used to formulate models describing numerous events and problems of real world problems like the theory of radiative transfer [17,18,20], the kinetic theory of gases [29], the theory of neutron transfer [17,20] and the theory of vehicular traffic [6,12]. The Chandrasekhar [20] type quadratic integral equations are special cases of such problems often encountered in models formulations for neutron transport theory and vehicular traffic theory [4,6,12,22,33,37].

It should be noted that solvability requirement for the pair of auxiliary inequalities in (36) is an inevitable constraint (inherent in applications of Darbo fixed point theorem) on the hypotheses for existence of solution of associated equations like (3). It is also important to highlight that the inequalities in (36) are more complicated than the inequalities in (2) because the associated integral equation is less complicated than the integral equation (3). Apart from these complications, the inequalities further impose smallness constraints on the data h, f, k, and g of the integral equations. But the novelty of our presentation includes relaxation of these smallness constraints on these data by allowing for alternations of smallness among the data in our formulation herein. In particular, it is only the Young function $\Phi(t,s)$ that is affected by smallness constraints in our presentation.

Under similar constraints by sets of auxiliary inequalities, various authors have attempted improvements in order to relax the requirement of solvability of complicated auxiliary inequalities (2) and (5). In this direction, we mention the following recent contributions: Benkerouche [15], Darwish [22–24]. El-Sayed et al [26], Zhou et al [38], Cicho [21], Danville [25], Kelley [28] and various others [1,2] and the references therein.

2. Preliminaries

Our methodology requires applications of the following classical results.

Definition 1. *[31,37] Let Ω denote a compact metric space and $Z(\Omega)$ denote a class of functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$. The function $f(t,x)$ is said to generate a superposition operator $F : Z_1(\Omega) \to Z_2(\Omega)$ if for each $x \in Z_1(\Omega)$ there corresponds a function $y \in Z_2(\Omega)$ such that*

$$\begin{gathered}(Fx)(t) = y(t) = f(t,x(t)) \\ i.e.\ F(x) = y \in Z_2(\Omega), x \in Z_1(\Omega)\end{gathered} \tag{7}$$

The theory of superposition operators $(Fx)(t) = f(t,x(t))$ often called Nemytskii operators (or outer composition operators or substitution operators) still has many open problems [3,35] which are extension of the basic basic problem in the theory of superposition operators. The basic problem of superposition operators concerned with formulations of conditions (called defining conditions) to guarantee validity of definition of a superposition operator $F : Z_1(\Omega) \to Z_2(\Omega)$ between any given pair of function spaces $Z_1(\Omega)$ and $Z_2(\Omega)$. But this theory for superposition operators $F : C[0,1] \to C[0,1]$ on the space of continuous functions on a bounded closed interval $[a,b]$ has been well established [3,31].

In the space $C[0,1]$ of continuous functions defined on the unit interval $[0,1]$ we have the following important properties;

Theorem 1. *[3] The superposition operator $F : \Omega \to [0,1]$ generated by a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ maps the space $C[0,1]$ of continuous functions into itself if and only if $f(t,x)$ is continuous on $\Omega' \times \mathbb{R}$ where Ω' denotes the derived set of Ω (i.e. Ω' is the set of all limit points of Ω i.e.*

$$F(\Omega) \subseteq C(\Omega) \text{ iff } f : \Omega' \times \mathbb{R} \to \mathbb{R}$$

is continuous.

Observe that theorem 1 implies continuity of $f(.,u)$ on Ω' $\forall u \in \mathbb{R}$.

Theorem 2. *[3] Suppose that the superposition $F : C(\Omega) \to C(\Omega)$ generated by $f(t,x)$ maps $C(\Omega)$ into itself. Then F is bounded if and only if $f(x,.)$ is bounded $\forall\, t \in \Omega \backslash \Omega'$ where Ω' denotes the derived set of Ω.*

Theorem 3. *Suppose $F : C[0,1] \to C[0,1]$ is a superposition operator $F(u) =$ generated by $f : [0,1] \times \mathbb{R} \to \mathbb{R}$. Then the following condonations are equivalent:*

- *(a) There exists a superposition operator $G : B_r(C[0,1]) \to B_{k(r)}(C(0,1))$ generated by a function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ such that $|f(t,x) - f(t,y)| \leq g(t,z)|x-y|$ where, $z = \max |x|, |y|$.*
- *(b) The superposition operator $F(u) = f(t,u)$ satisfies the Lipschitz condition $|F(x) - F(y)| \leq k(r)|x-y|$*
- *(c) The superposition operator $F(u) = f(t,u)$ satisfies the Darbo condition $H_C(F(\Omega)) \leq k(r) H_C(\Omega)$; for all $\Omega \subseteq B_r(C[0,1])$, where H_C denotes the Hausdorff measure of no compactness in $C[0,1]$.*

Theorem 4 (cf [5]). *A function $\varphi(t,s)$ is a convex function if, and only if, $\Phi(t,s)$ has monotonically increasing one-sided partial derivative $\varphi(t,s) = \frac{\partial \Phi(t,s)}{\partial s}$ with respect to s.*

Corollary 1 (cf [5]). *Let $f(x)$ be a twice differentiable function. Then $f(x)$ is convex if, and only if, $f''(x) \geq 0$ for all x of our interval.*

Definition 2. *A continuous function $\Phi : [0,\infty) \to [0,\infty)$ is called a young function if*

- *(i)* $\Phi(0) = 0$
- *(ii)* $\Phi(t) > 0 \;\forall\, t > 0$
- *(iii)* $\lim_{t\to\infty} \Phi(t) = \infty$

A Young function is often called a gauge function.

Definition 3. *Given a Young function $\Phi : [0,\infty) \to [0,\infty)$, and a partition $P = \{t_0, t_1, \ldots, t_n\} \in \mathcal{P}([\prime,\infty])$ and a function $f : [0,1] \to \mathbb{R}$, the Wiener-Young variation $var^W_\Phi(f,P) = \sum_{j=i}^{\infty} \Phi(|f(t_j) - f(t_{t-1})|)$.*

Definition 4. *Given a Young function Φ, a partition $P = \{t_0, t_1, t_2, \ldots, t_n\}$ of $[0,1]$ and a function $f : [0,1] \to \mathbb{R}$. The Riesz-Medvedev variation of f on $[0,1]$ with respect to a partition P is defined by*

$$var^R_\Phi(f) = \sum_{j=1}^{n} \Phi\left(\frac{|f(t_j) - f(t_{j-1})|}{t_j - t_{j-1}}\right) \cdot (t_j - t_{j-1})$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1695.v1

Observe that the Riesz-Medvedev variation with respect to a partition P is the Wiener-Young variation if $\Phi(|f(t_j) - f(t_{j-1}))$ is replaced with $\left(\frac{|f(t_j)-f(t_{j-1})|}{t_j-t_{j-1}}\right) \cdot (t_j - t_{j-1})$.

Definition 5. *The Riesz-Medvedev total variation of f on $[0,1]$ is given as $\sup\{u_\Phi^R(f,P) : P \in \mathcal{P}([0,1])\}$ where $\mathcal{P}([\prime,\infty])$ denotes the set of all possible finite partitions P of $[0,1]$.*

Example 1. *Let $0 \le s \le t \le 1$. Then the function $\Phi(t,s) = \frac{t^\alpha-(t-s)^\alpha}{\alpha}$ is of bounded Riesz-Medvedev total variation.*

We recall [36] a mapping $A : X \to E$ of a subset X of a Banach space E is said to be near a mapping $B : X \to E$ if there exist two constants $\lambda > 0$ and $\eta_2 \in [0,1)$ such that the following property holds:

$$\|(Bu - Bv) - \lambda(Au - Av)\| \le \eta_2\|Bu - Bv\| \tag{8}$$

It follows from (8) that when B is the identity mapping i.e. $B = I$ then A is said to be near identity mapping I. Our investigation is limited to the property of nearness of A to identity mapping given below:

$$\|(u - v) - \lambda(Au - Av)\| \le \eta_2\|u - v\|$$

The nearness principle [16,34] follows from the following generalization of Neuman's lemma by Campanato

Theorem 5. *[16,19] Let E be a real Banach space. $T : E \to E$ a nonlinear mapping such that there exist $\lambda > 0$ and $\eta_2 \in [0,1)$ and*

$$\|(u - v) - \lambda(Tu - Tv)\| \le \eta_2\|u - v\| \quad \textit{then} \quad Lip(I - \lambda A) \le \eta_2 \quad \textit{and} \quad Lip(A^{-1}) \le \frac{\lambda}{1-c}$$

Where $Lip(T)$ is the Lipchitz norm of T.

Our solvability procedure is based on application of Rothe fixed point theorem below to prove existence and uniqueness of the quadratic integral equations studied herein.

Theorem 6. *[27,32] Let E be a Banach space, X a closed unit ball in E with boundary ∂X. Let T be a countable compact mapping of X into E such that $T(\partial X) \subset X$, then T has a fixed point.*

Main Result

Proposition 1. *Let $\nu(s)$ be right differentiable increasing function such that $\nu'(s) > 0 \,\forall\, s > 0$. Then any composition function which is convex in s is of bounded Riesz-Medvedev variation.*

Proof. Given a differentiable increasing functions $\varphi(s)$ with $\varphi(s) > 0 \,\forall\, s > 0$, a composition function $(\Phi \circ \varphi)(s)$ which is convex in s. We are to prove that the Riesz-Medvedev variation

$$\begin{aligned} V_\Phi^R(\nu,P) &= \sum_{j=1}^{n} \Phi\left(\frac{\nu(s_j) - \nu(s_{j-1})}{s_j - s_{j-1}}\right)(s_j - s_{j-1}) \\ &= \sum_{j=1}^{n} \Phi(\nu'(c_j))(s_j - s_{j-1}) \end{aligned}$$

$$
\begin{aligned}
\implies \sup V_{\Phi}^{\mathbb{R}}(\nu, P) &= \sup_{P\in\mathcal{P}([0,1])} \sum_{j=1}^{n} \Phi(\nu'(c_j))(s_j - s_{j-1}) \\
&= \sup_{c_j\in\nu'(0,1)} \Phi(\nu'(c_j)) \sum_{j=1}^{n}(s_j - s_{j-1}) \\
&= \sup_{c_j\in\nu'(0,1)} \nu(c_j)(t_0 - 0) < \infty
\end{aligned}
$$

□

Corollary 2. *Let $\Phi(t,s) = t^{\alpha} - (t-s)^{\alpha}$ Then the total Riesz-Medvedev variation $v_s^{t_0}ar_{\Phi}^{\mathbb{R}}(\nu)$ of the function $\nu(t-s) = -(t-s)^{\alpha}$ satisfies*

$$\lim_{t_0\to 0} v_s^{t_0}ar_{\Phi}^{\mathbb{R}}(\nu(t,s)) = 0$$

Example 2. *For all $t_0 \in (0,1)$, the function $\nu(s) = -(t-s)^{\alpha}$ is of bounded Riesz-Medvedev bounded variation.*

PROOF

The function $\nu(s) = -(t-s)^{\alpha}$ is an increasing function of s with $\frac{\partial \nu(t,s)}{\partial s} > 0 \; \forall \; s \in (0,1)$. Let $(\Phi \circ \nu)(s)$ be a positive increasing function of s. Then we have

$$
\begin{aligned}
\Phi\left[\frac{-(t-s_j)^{\alpha} + (t-s_{j-1})^{\alpha}}{s_j - s_{j-1}}\right] &= \Phi\left[\frac{(t-s_{j-1})^{\alpha} - (t-s_j)^{\alpha}}{s_j - s_{j-1}}\right] \\
&= \Phi\left[\frac{1}{(t-c_j)^{1-\alpha}} \cdot \frac{(t-s_{j-1}) - (t-s_j)}{s_j - s_{j-1}}\right] \\
&= \Phi\left(\frac{1}{(t-c_j)^{1-\alpha}}\right)
\end{aligned}
$$

for some $c_j \in (t - s_{j-1}, t - s_j)$

$$\implies \sum_{j=1}^{n} \Phi\left[\frac{\nu(s_j) - \nu(s_{j-1})}{s_j - s_{j-1}}\right](s_j - s_{j-1}) = \sum_{j=1}^{n}\left(\frac{s_j - s_{j-1}}{(t-c_j)^{1-\alpha}}\right)$$

$$\implies \sup_{P\in\mathcal{P}([t,\infty])} \left[\frac{\nu(s_j) - \nu(s_{j-1})}{s_j - s_{j-1}}\right](s_j - s_{j-1}) = \sup_{c_j} \frac{t_o}{(t-c_j)^{1-\alpha}} < 0$$

In what follows, we shall investigate solvability of the quadratic integral equation (**??**) shown below

$$u(t) = h(t) + f(t,u(t)) \int_0^t k(t,s)g(s,u(s))d_s\Phi(t,s);$$

subject to under-listed hypotheses on its data; viz - the functions $h(t)$, $k(s,t)$, $\Phi(t,s)$ and the superposition operators $(Fu)(t) = f(t,u)$ and $(Gu)(t) = g(t,u)$:

H1 $h \in C[0,1]$

H2 $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the Lipschitz condition in its second variable with Lipschitz constant η_1 ie;

$$|f(t,u_1) - f(t,u_2)| \leq \eta_1 |u_1 - u_2|$$

H3 $k : \Delta \to \mathbb{R}$ is continuous on the triangle $\Delta = \{(t,s) \in [0,1] : 0 \leq s \leq t \leq 1\}$

H4 The nonlinearities $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ generates the superposition operators $F, G : C[0,1] \to C[0,1]$ respectively; i.e $(Fu)(t) = f(t,u(t))$ and $(Gu)(t) = g(t,u(t))$

H5 The superposition operator $G(u)$ satisfies the following nearness condition with nearness constant η_2, with respect to the identity operator I i.e; there exists $\eta_2 \in (0,1)$ s.t

$$|(u_1(t) - u_2(t)) - [(Gu_1)(t) - (Gu_2)(t)]| \leq \eta_2 |u_1(t) - u_2(t)|$$

H6 The function $\Phi(t,s) = \Phi : \Delta \to \mathbb{R}$ is a Young's function with respect to the second variable s and its one-sided partial derivative with respect to s is denoted by

$$\frac{d\Phi(t,s)}{ds} = \varphi(t,s) = \frac{\partial\Phi(t,s)}{\partial s} \tag{9}$$

To resolve existence issues for the quadratic integral equation (**??**), we need to solve the corresponding linear equation (10) below. After which, we will employ variation of parameter method involving the linear part and the nonlinearity to resolve the nonlinear problem.

Theorem 7. *Let the data $h(t)$, $f(u,t)$ and $\Phi(t,s)$ of the quadratic integral equation*

$$u(t) = h(t) + f(t,u) \int_o^t k(t,s) g(u(s)) d_s\Phi(t,s) \tag{10}$$

satisfy the properties $(H1), (H2), (H4), (H4)$ and $(H6)$. Let $t_0 \in (0,1)$ be such that the following condition holds:

$$\begin{aligned}
\eta_1 \bar{k} \Phi_0 \|h\| &< (1 - \overline{fhk}\Phi_0)^2 \text{ where :} && (11)\\
\bar{f} &= \sup_{(t,s)\in[0,t_0]\times[0,t_0]} f(t,s); && (12)\\
\bar{k} &= \sup_{(t,s)\in\Delta_0} k(t,s). && (13)\\
\Phi_0 &= \sup_{(t,s)\in\Delta_0} \Phi(t,s);; && (14)\\
\Delta_0 &= \{(t,s) \in [0,t_0] \times [0,t_0] : 0 \leq s \leq t_0 < 1\}. && (15)
\end{aligned}$$

Then the integral equation (10) has a unique solution $z \in C[0,t_0]$.

PROOF
Let B_r be a ball centered at the origin with radius r, We are to determine a value r_0 for r such that a mapping $T : B_{r_0} \to C([0,1])$ satisfies the requirements of Theorem 6 of Rothe; where the mapping T is defined by

$$Tz = h(t) + f(t,z) \int_0^t z(s) d_s\Phi(t,s).$$

Clearly,

$$|(Tz)(t)| \leq |h(t)| + |f(t,z(t))| \int_0^t |k(t,s)||z(s)| d_s\Phi(t,s)$$

Hence,

$$\|Tz\| \leq \|h\| + \overline{fk}\Phi_0 \|z\|.$$

This leads to an auxiliary inequality below:

$$\|h\| + \overline{fk}\Phi_0 \|z\| \leq r. \tag{16}$$

For $z \in \partial B_r(h)$ the inequality (16) becomes the following:

$$\bar{h} + \overline{fk}\Phi_0 r \leq r. \tag{17}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1695.v1

But (17) yields the first estimate for r_0 as below:

$$r \geq \frac{\|h\|}{1-\bar{k}\Phi_0\bar{f}}. \tag{18}$$

Next, is to show that T is Lipschitz

$$\begin{aligned}
|(Tz_1)(t)-(Tz_2)(t)| = & \left|\int_0^t k(t,s)f(t,z_1(t)))z_1(s) - f(t,z_2(t)))z_2(s)d\Phi(t,s)\right| \\
& \leq \int_0^t |k(t,s)|[|f(t,z_1(t))||z_1(s)-z_2(s)| + z_2(t)|f(t,z_1(t)) - f(t,z_2(t))|]d_s\Phi(t,s) \\
& \leq \int_0^t |k(t,s)|[|f(t)||z_1(s)-z_2(s)| + |z_2(t)|\eta_1|z_1(t)-z_2(t)|]d_s\Phi(t,s) \\
& \leq \bar{k}(\bar{f}+\eta_1\|z_2\|)\|z_1-z_2\| \times \int_0^t d_s\Phi(t,s) \\
& \leq \bar{k}\Phi_0(\bar{f}+\eta_2\|z_2\|)\|z_1-z_2\|.
\end{aligned}$$

On assuming $z \in \partial(B_r)$ and in view of contraction principle, this yields:

$$\bar{k}\Phi_0(\bar{f}+\eta_2 r)\bar{k}\Phi_0 < 1. \tag{19}$$

From (19) we obtain a complementary estimate for r_0 below:

$$r < \frac{1-\bar{k}\Phi_0\bar{f}}{\eta_2\bar{k}\Phi_0}. \tag{20}$$

Combining (18) and (20) and in view of (11) we obtain the desired valid estimates for r_0 as follows:

$$\frac{\|h\|}{1-\bar{k}\Phi_0\bar{f}} \leq r_0 < \frac{1-\bar{k}\Phi_0\bar{f}}{\eta_2\bar{k}\Phi_0}. \tag{21}$$

So, choosing $t_0 \in (0,1)$ such that (19) holds we obtain a contraction constant η_3 for T as below:

$$\eta_3 = \bar{k}\Phi_0(\bar{f}+\eta_2 r_0)\bar{k}\Phi_0. \tag{22}$$

Based on above we apply Rothe's fixed point theorem (Theorem6) to conclude that there exists a unique fixed point $z \in C([0,t_0])$ of T which is the desired unique solution of (10). □

We will employ solvabity procedure of Theorem **??** fo obtain solution of the linear part of the problem below to enable us apply method of variation of parameters for its sovability.

Theorem 8. *Let the data $h(t)$, $f(u,t)$, $\Phi(t,s)$, and $g(t,u)$, and of the quadratic integral equation*

$$u(t) = h(t) + f(t,u)\int_o^t k(t,s)g(s,u(s))d_s\Phi(t,s) \tag{23}$$

 doi:10.20944/preprints202509.1695.v1

satisfy the properties $(H1), (H2), (H3)(H4), (H4)$ *and* $(H6)$*. Let* $t_0 \in (0,1)$ *be such that the following condition holds:*

$$\begin{aligned}
\Phi_0 \bar{k}\eta_1(\eta_2+1)\|h\| &< (1-\Phi_0\bar{k}\eta_1\bar{f})(1-\Phi_0\bar{k}\eta_1\bar{f}(\eta_2+1))\text{where}: && (24)\\
\bar{f} &= \sup_{(t,s)\in[0,t_0]\times[0,t_0]} f(t,s); && (25)\\
\bar{k} &= \sup_{(t,s)\in\Delta_0} k(t,s). && (26)\\
\Phi_0 &= \sup_{(t,s)\in\Delta_0} \Phi(t,s);; && (27)\\
\Delta_0 &= \{(t,s)\in[0,t_0]\times[0,t_0] : 0\le s\le t_0<1\}. && (28)
\end{aligned}$$

Then the integral equation (23) has a unique solution $u \in C[0,t_0]$*.*

PROOF
Without loss of generality, we shall assume that $g(t,0)=0$ for all $t \in (0,t_0)$ and as in the proof of Theorem **??** we are to estimate values r_0 for $r>0$ to enable application of Rothe fixed point theorem on an operator $T : B_r \to C[0,t_0]$.

Again, B_r denotes a ball centered at the origin with radius $r>0$, we are to determine a value r_0 for r such that a mapping $T : B_{r_0} \to C([0,1])$ satisfies the requirements of Rothe's Theorem 6. Here, we define the mapping T by

$$(Tu)(t) = h(t) + f(t,u(t))\int_0^t k(t,s)g(u(s))d_s\Phi(t,s).$$

To apply nearness of the superposition operator $G(u) = g(t,u)$ to identity map I; the last equation is modified into:

$$(Tu)(t) = h(t) + f(t,u(t))\int_0^t k(t,s)u(s)d_s\Phi(t,s) + f(t,u(t))\int_0^t k(t,s)[g(u(s)) - u(s))]d_s\Phi(t,s).$$

Clearly,

$$\begin{aligned}
|Tu(t)| &\le |h(t) + |f(t,u(t)|\int_0^t |k(t,s)||u(s)|\,d_s\Phi(t,s) + |f(t,u(t))|\int_0^t |k(t,s)[g(s,u(s)) - u(s)]\,d_s\Phi(t,s)\\
&\le |h(t)| + |f(t,s)|\int_0^t |k(t,s)||u(s)|\,d_s\Phi(t,s) + |f(t,u(t))|\int_0^t |k(t,s)|[(u(s)-0) - g(s,u(s)) - g(s.0)]\,d_s\Phi(t,s)\\
&= |h(t)| + |f(t,u(t))|\int_0^t |k(t,s)|((u(s)-0) - [(G(u))(s) - G(0)]\,d_s\Phi(t,s)\\
&\le |h(t)| + \eta_1|f(t,u(t))|\int_0^t |k(t,s)||u(s)|d_s\Phi(t,s)\\
\implies \|Tu\| &\le \|h\| + \overline{fk}\Phi_0 r + \Phi_0\overline{fk}\eta_2 r \le r\\
&\implies \|h\| + [\Phi_0\overline{fk} + \Phi_0\bar{f}\eta_2\bar{k}]r \le r\\
&= \|h\| + [\Phi_0\overline{fk}[\eta_2+1]]r \le r
\end{aligned} \tag{29}$$

yielding:

$$r \ge \frac{\|h\|}{1-\bar{f}\Phi_0\bar{k}[\eta_2+1]} \tag{30}$$

To show that the mapping T is a contraction, we proceed as follows to find its contraction constant η_T.

But

$$\begin{aligned}
|(Tu_1)(t) - (Tu_2)(t)\| &\le |f(t,u(t)|\int_0^t k(t,s)[u_1(s) - u_2(s)]\,d_s\Phi(t,s) + \left|\int_0^t k(t,s)[f(t,u_1(t)(g(u_1) - u_1) - f(t,u(t))g(u_2) - u_2]\,d_s\Phi(t-s)\right|\\
&\le |f(t,u(t)|\int_0^t |k(t,s)||u_1(s) - u_2(s)|\,d_s\Phi(t,s) + \int_0^t |k(t,s)|[f(t,u_1(t)||(g(u_1) - u_1 - (g(u_2) - u_2)| + |g(u_2)||f(t,u_1) - f(t,u_2)|]\,d_s\Phi(t-s)\\
&= |f(t,u(t)|\int_0^t |k(t,s)||u_1(s) - u_2(s)|\,d_s\Phi(t,s) + \int_0^t |k(t,s)|[f(t,u_1(s)||(u_1 - u_2)(g(u_1(s)) - g(u_2(s))| + |g(u_2)||f(u_1) - f(u_2)|]\,d_s\Phi(t,s)\\
&= |f(t,u(t)|\int_0^t |k(t,s)||u_1(s) - u_2(s)|\,d_s\Phi(t,s) + \int_0^t |k(t,s)|[\eta_2 f(t,u_1(s)||(u_1(s) - u_2(s)| + |g(u_2)|\eta_1|u_1(s) - u_2(s)|\,d_s\Phi(t,s)\\
\implies \|T\| &\le \bar{f}\Phi_0\bar{k}\|u_1 - u_2\| + \bar{k}\Phi_0[\eta_2\bar{f}\|u_1 - u_2\| + r_0\eta_1\|u_1 - u_2\|]\\
&= \bar{f}\Phi_0\bar{k} + \bar{k}\Phi_0[\eta_2\bar{f} + r_0\eta_1]\|u_1 - u_2\|
\end{aligned} \tag{31}$$

To obtain a contraction constant $\eta_3 < 1$, we set

$$\eta_3 = \overline{f}\Phi_0\overline{k} + \overline{k}\Phi_0\eta_2\overline{f} + \overline{k}\Phi_0\eta_1 r < 1 \tag{32}$$

But 32 can yield a complementary estimate for r_0 as follows:

$$\overline{k}\Phi_0\eta_1 r < 1 - \overline{f}\Phi_0\overline{k}(\eta_2 + 1)$$

$$\implies r < \frac{1 - \overline{f}\Phi_0\overline{k}(\eta_2 + 1)}{\overline{k}\Phi_0\eta_1} \tag{33}$$

we, combining estimate 30 and estimate 33, we obtain the desired valid estimates for r_0 using

$$\frac{\|h\|}{1 - \overline{f}\Phi_0\overline{k}(\eta_2 + 1)} \leq r \leq \frac{1 - \overline{f}\Phi_0\overline{k}(\eta_2 + 1)}{\overline{k}\Phi_0\eta_1}$$

In view of the requirement (24) we conclude that for $T(\partial B_{r_0}) \subset B_{r_0}$ we must have

$$r_0 \in \left(\frac{\|h\|}{1 - \overline{f}\Phi_0\overline{k}(\eta_2 + 1)}, \frac{1 - \overline{f}\Phi_0\overline{k}(\eta_2 + 1)}{\overline{k}\Phi_0\eta_1} \right)$$

From above, we conclude that the operator T has a unique fixed point $u \in C[0, t_0])$.

3. Application to Fractional Dynamics

The results in this section are Corollaries following from our main results above and applications to problems of fractional dynamics and other singular problems.

Corollary 3. *Let the hypotheses H1, H2, H3, and H5 hold, Suppose $\phi(t)$ is the right derivative of the Young's function $\Psi(t,s)$ then the singular integral equation*

$$u(t) = h(t) + f(t, u(t)) \int_0^t k(s,t)\phi(t,s)g(t,s)\, ds$$

has a unique solution.

The next theorem illustrates applications of nearness principle to resolution of hitherto difficult solvability of fractional dynamic problems with degenerate perturbation of identity by nonlinearities which are neither small nor contractive.

Theorem 9. *Let h, f, k, and g satisfy the hypotheses H1 - H6 and $\int_0^s \varphi(t,\tau)\, d\tau$ is a Young function in variable s, then the quadratic integral equation*

$$u(t) = f(t, u(t)) \int_0^t k(t,s)\varphi(t,s)g(s, u(s))\, ds$$

has a unique solution.

Theorem 10. *Let the hypotheses (H1-H6) be satisfied for h, f, k and g in the equation*

$$u(t) = h(t) + f(t, u(t)) \int_0^t \frac{k(t,s)}{(t-s)^{1-\alpha}} g(s, u(s))\, ds$$

if the condition below is satisfied

$$\frac{\tau_0}{\alpha}\overline{k}\eta_1(\eta_2 + 1)\|h\| < (1 - \Phi_0\overline{k}\eta_1\overline{f})(1 - \overline{f})$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1695.v1

then the quadratic equation $u(t)$ has a unique solution.

4. Illustrative Examples

Example 3. *Investigate the solvability of the following quadratic integral equation*

$$u(t) = h(t) + f(t, u(t)) \int_0^t \frac{t^2 - s^2}{1 + s} \sin u(s)\, ds,\ 0 \le t \le 1 \tag{34}$$

where $f(t)$
Solution
But $\frac{ds}{1+s} = d \ln(1+s)$ So we set $\Phi(t,s) = \ln(1+s)$ Since $\ln(1+s)$ is a convex increasing function. In fact $\ln(1+s)$ is convex since by theorem **??**

$$\frac{d}{ds} \ln(1+s) = \frac{1}{1+s} > 0\ \forall\ s > 0.$$

The function $\sin u$ is near identity with nearness constant i.e.

$$\|u_1 - u_2 - \lambda(\sin u_1 - \sin u_2)\| \le k\|u_1 - u_2\|$$

with $\lambda > 0$ and $k \in [0,1)$ Hence, by theorem 8 Also $\overline{f1} = \sup f(t, u(t)), \bar{k} = 1,\ \Phi_0 = \ln(1+t_0),\ \Phi_0 \le \eta_1,$ is the Lipchitz constant of f η_1,$\eta_2 = 1 - cos\gamma$ $r \in (0,1)$

$$\ln(1+\tau_0)\eta_1(2-\cos\gamma)\|h\| < (1 - ln(1-\tau_0)\eta\bar{f})$$

Hence there exists solution $(u(t)$ on $C[0,t_0]$ for the equation 34.

Example 4.

$$u(t) = h(t) + f(t, u(t)) \int_0^t \frac{t^2 - s^2}{(t-s)^{1-\alpha}} \sin u(s)\, ds,\ 0 \le t \le 1 \tag{35}$$

Conclusion

It is important to remark that these auxiliary inequalities are bound to become more complicated as the integral equation (**??**) is made more complex. So, the novelty of the formulation in this present paper is in circumventing (or relaxing on) solvability requirements on equivalent auxiliary inequalities. In due course we will illustrate the complexities of these auxiliary inequalities with respect to different level of complexity of the corresponding integral equations. For instance, the next equation which is more complicated than (**??**) requires solvabilty of a more complicated pair of inequalities below:

$$\eta_1 r + \bar{h} + \bar{k}\bar{v}\varphi(r)\psi(r) \le r \text{ and } \eta_1 + \bar{k}\bar{v}\eta_2\psi(r) < 1. \tag{36}$$

The versatility of the nearness principle of operators is that, given any two functions A and B in a given Banach space, we have the following alternatives either:

(a) A is near B
(b) Or $-A$ is near B
(c) Or A is orthogonal to B.

So, this informs the motivation of various researches into diverse connections between the nearness property and orthogonality property. These connections yield many positive possibilities leading to various generalizations of the concept of orthogonality in very general spaces like locally convex spaces and arbitrary metric spaces.

References

1. Rasim Abood, Saleh Redhwan, Omar Bazighifan and Kamsing Nonlaopon, Investigating a generalized fractional quadratic integral equation. Fractal Fract. 2022, 6, 251.
2. Alahmadi, J., Abdou, M. and Abdel-Aty, M., Analytical and numerical applications via quadratic integral equations, Axioms, 13 (2024), 621.
3.
4. I.K. Argyros, Quadratic equations and application to chambers. *Bull. Austral. Math. Soc.* equations 275–289.
5. Emil Artin. *The Gamma Function*. Translated by Michael Butler. Holt, Rinehart and Winston, New York, 1964. (Translated from the German original: "Einführung in die Theorie der Gammafunktion" (1931)).
6. N. Bellomo and M. Delitala, On the mathematical theory of vehicular traffic flow: Fluid Dynamics and kinetic modelling. Mathematical Models and Methods in Applied Sciences, Vol 12, No. 12(2002), 1801 - 1843.
7. J. Banas, M. Lecko and W. G. El-Sayed. Existence theorems of some quadratic integral equations. *J. Math. Anal. Appl.* 222 (1998) 276–285.
8. J. Banas, B. Rzepka, Nondecreasing solutions of a quadratic singular Volterra integral equation, *Math. Comput. Modelling* **49** (2009) 488–496.
9. J. Banas, A. Chlebowicz, On integrable solutions of a nonlinear Volterra integral equation under Carathéodory conditions, *Bull. Lond. Math. Soc.* **41** (2009) 1073–1084.
10. J. Banas, Z. Zajac, A new approach to the theory of functional integral equations of fractional order, *J. Math. Anal. Appl.* **375** (2011) 375–387.
11. Jozef Banas and Beata Zepka, The technique of Volterra-Stieltjes integral equations in the application to infinite systems of nonlinear integral equations of Fractional orders. Computers and Mathematics with Applications, 64 (2012), 3208 - 3116.
12. Jozef Banas, J. Martin and K. Sadarangani, On solutions of a quadratic integral equation of Hammerstein type. Mathematical and Computer modelling, 43(2006), 97 - 104.
13. Jozef Banas and Agnieska Dubiel, Solvability of a Volterra-Stieltjes integral equation in the class of functions having limits at infinity. Electronic Journal of Qualitative Theory of Differential Equations, No 53 (2017) 1 - 17.
14. Jozef Banas and Agnieszka Dubiel, Solutions of a quadratic Volterra-Stieltjes integral equation in the class of functions converging at infinity. Electronic Journal of Qualitative Theory of Differential Equations, No 80 (2018)
15. Amar Benkerouche, Mohammed Souid, Gani Stamov and Ivanko Stamov On the solution of quadratic integral equation of Urysohn type of Fractional order. Entropy, 2022, 24, 886.
16. Barbagallo, Annamaria, Ernst, Emil and Thera, Michel, Orthogonality in locally convex spaces: Two nonlinear generalizations of Neumann's Lemma, J. Math Anal Appl.
17. L. W. Busbridge. The Mathematics of Radioactive Transfer. Cambridge Uni. Press 1960
18. J. Caballero, A.B. Mingarelli and K. Sadarangani. Existence of solutions of an integral equation of Chandrasekhar type in the theory of radiative transfer. *Electron. J. Differential Equ.* 2006, no. 57, 1–11.
19. Campanato, Sergio (1994) On the Condition of Nearness Between Operators. Annali di Matematica pura ed applicata (IV), Vol. CLXVII, pp. 243-256
20. S. Chandrasekhar. *Radiative Transfer*. Dover Publication, New York, 1960.
21. M. Cichoń and M. A. M. Metwali. Monotonic solutions for quadratic integral equations. *Discuss. Math. Differ. Incl. Control Optim.* 31 (2011) 157–171.
22. M. A. Darwish. On monotonic solutions of an integral equation of Abel type.
23. M.A. Darwish, S.K. Ntouyas, Monotonic solutions of a perturbed quadratic fractional integral equation, *Nonlinear Anal.* **71** (2009) 5513–5521.
24. M.A. Darwish, On monotonic solutions of a quadratic integral equation with supremum, *Dynam. Systems Appl.* **17** (2008) 539–550.
25. M.A. Danville, J. Henderson, Existence and asymptotic stability of solutions of a perturbed quadratic fractional integral equation, *Fract. Calc. Appl. Anal.* **12** (2009) 71–86.
26. A. El-Sayed and H. Hasham, Integrable and continuous solutions of a nonlinear quadratic integral equation. Electronic Journal of Quadratic Theory of Differential Equations, No. 25 (2008), 1 - 10.
27. Granas, A. and J. Dugundj (2003), Fixed point theory. Springer, New York.
28. C.T. Kelley. Approximation of solutions of some quadratic integral equations in transport theory. *J. Integral Eq.* 4 (1982) 221–237.

 doi:10.20944/preprints202509.1695.v1

29. S. Ho, M. F. Khavani and W. Zhuang. Integral equations arising in the kinetic theory of gases. *Appl. Anal.* 34 (1989) 261–266.
30. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science B. V., Amsterdam, 2006.
31. Krasnoselskii, M. (1958). Some problems of nonlinear analysis. American Mathematical Society Translation, 2(0):345–409.
32. Smart, D. R. (1974), Fixed Point Theorems, Cambridge University Press, London.
33. Hussein salem, On the quadratic integral equations and their applications wish applications. 62(2011), 2931 - 2943.
34. Fattorusso, F and Tarsia, A., Applications of near operators Campanato's Theory. Boll. Accademia Gioenica di Scienze Naturali - Catania, BOLLOG Vol 46, No 376 (2012), pp. 77 - 90.
35. Xavier Udo-utun, Zakawat Siddiqi, M. Balla, A. U. Afuwape and M. Mshelia, On integral representation of causal operators, Far East Journal of Mathematical Sciences (FJMS) 59(1) 57 - 63
36. Xavier Udo-utun, Uduak Edet and Udeme Johnson, Nearness of fractional operators for solvability of natural perturbations of fractional dynamic equations. World Journal of Applied Sciences and Technology, 15(2) (2023), 275 - 278.
37. P. P. Zabrejko, A. I. Koslev, M. A. Krasnoseł'skiĭ, G. G. Mikhlin, L. S. Rakovskchik and V. J. Stecenko. *Integral Equations*. Noordhoff, Leyden, 1975.
38. Zhou,M., Wan, H., Xu, Z. and Zhang, I., The study of fractional quadratic integral equations involves generalized fractional integrals. Fractional Fract., 2025, 9, 249