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Arya Pathak *

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Self-Healing Numbers: A New Class of Integers with Positional Divisibility Properties

Arya Pathak

Independent Researcher, India; write9524@gmail.com

Abstract

We introduce a new class of positive integers called Self-Healing Numbers, defined by a unique positional divisibility property. For each digit position i, the number obtained by removing the digit at position i must be divisible by i. We present complete computational enumerations up to seven digits, establish fundamental structural properties, prove non-hereditary behavior, and analyze the growth of the sequence. Our findings suggest a regular, exponential growth pattern, supporting the conjecture that this sequence is infinite. We also provide a more rigorous mathematical framework by expressing the divisibility conditions as a system of congruences, which allows for deeper theoretical analysis of the number's structure.

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1. Introduction

The study of integers with special divisibility properties has been a recurring theme in number theory. Classical examples include the divisibility tests for small primes, Harshad numbers (integers divisible by the sum of their digits), and self-divisive numbers (integers divisible by each of their individual digits).

We introduce a variant based on positional digit removal and divisibility.

Definition 1.1. *Let n be a positive integer with decimal representation $d_1d_2 \dots d_k$ where $d_1 \neq 0$. For each $i \in \{1, 2, \dots, k\}$, let $n^{(i)}$ denote the $(k-1)$-digit integer obtained by removing the digit d_i from position i. We say that n is a Self-Healing Number (SHN) if $i \mid n^{(i)}$ for all $i \in \{1, 2, \dots, k\}$.*

Example 1.2. *The integer $n = 20$ is self-healing*:

$n^{(1)} = 0,$	$1 \mid 0$	(1)
$n^{(2)} = 2,$	$2 \mid 2$	(2)

Example 1.3. *The integer $n = 111$ is not self-healing*:

$n^{(1)} = 11,$	$1 \mid 11$	(3)
$n^{(2)} = 11,$	$2 \nmid 11$	(4)

Since $11 \not\equiv 0 \pmod 2$, the number 111 fails the condition for $i = 2$.

2. Basic Properties and Congruence Relations

Theorem 2.1. *Every single-digit positive integer is self-healing.*

Proof. For any $n \in \{1, 2, \dots, 9\}$, removing the digit at position 1 yields $n^{(1)} = 0$. Since $1 \mid 0$, the divisibility condition is satisfied. □

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Theorem 2.2. *A two-digit integer $n = 10a + b$ with $a \in \{1, 2, \ldots, 9\}$ and $b \in \{0, 1, \ldots, 9\}$ is self-healing if and only if a is even.*

Proof. For $n = 10a + b$, we have:

$$n^{(1)} = b \tag{5}$$

$$n^{(2)} = a \tag{6}$$

The condition $1 \mid n^{(1)} = b$ is always satisfied. The condition $2 \mid n^{(2)} = a$ requires a to be even. Therefore, n is self-healing if and only if $a \in \{2, 4, 6, 8\}$. □

Corollary 2.3. *There are exactly 40 two-digit self-healing numbers, namely*:

$\{20, 21, \ldots, 29, 40, 41, \ldots, 49, 60, 61, \ldots, 69, 80, 81, \ldots, 89\}$

Proposition 2.4. Let $n = d_1 d_2 \ldots d_k$ be a k-digit integer. The integer $n^{(i)}$ can be expressed as:

$$n^{(i)} = \sum_{j=1}^{i-1} d_j 10^{k-1-j} + \sum_{j=i+1}^{k} d_j 10^{k-j}$$

The divisibility condition $i \mid n^{(i)}$ is equivalent to the congruence relation $n^{(i)} \equiv 0 \pmod{i}$. For $i > 1$, this can be simplified. For example, for a 3-digit integer $n = 100d_1 + 10d_2 + d_3$, the conditions are:

- $1 \mid 10d_2 + d_3$ (always true)
- $2 \mid 100d_1 + d_3 \implies d_3 \equiv 0 \pmod{2}$
- $3 \mid 100d_1 + 10d_2 \implies d_1 + d_2 \equiv 0 \pmod{3}$

Proof. The formula for $n^{(i)}$ is a direct consequence of the positional representation of integers. The simplification of the congruences uses standard divisibility rules and the fact that $10 \equiv 1 \pmod{3}$ and $100 \equiv 1 \pmod{3}$. For $i = 2$, $100 \equiv 0 \pmod{2}$, so $2 \mid 100d_1 + d_3$ simplifies to $2 \mid d_3$. For $i = 3$,

$100 \equiv 1 \pmod{3}$ and $10 \equiv 1 \pmod{3}$, so $3 \mid 100d_1 + 10d_2$ simplifies to $3 \mid d_1 + d_2$. □

3. Computational Results

We have performed exhaustive computational enumeration of self-healing numbers through seven digits. Each integer in the specified ranges was tested algorithmically according to Definition 1.1.

Table 1. Counts of Self-Healing Numbers by digit length.

Digit length k	Count $a(k)$	Cumulative
1	9	9
2	40	49
3	150	199
4	858	1,057
5	4,146	5,203
6	19,908	25,111
7	95,526	120,637

The sequence of self-healing numbers begins:
1, 2, 3, 4, 5, 6, 7, 8, 9, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 40, . . .

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The counting function $a(k)$ exhibits substantial growth.

4. Structural Properties

Theorem 4.1. Every self-healing number with two or more digits must have an even last digit.

Proof. Let n be an SHN with $k \geq 2$ digits. By definition, n must satisfy the divisibility condition for $i = 2$, which means $2 \mid n^{(2)}$. The number $n^{(2)}$ is obtained by removing the second digit of n. The parity (even or odd) of an integer is determined solely by its last digit. Since $n^{(2)}$ has the same last digit as n, and $n^{(2)}$ must be even, the last digit of n must also be even. □

Theorem 4.2. The self-healing property is not hereditary. That is, if n is self-healing, then prefixes of n need not be self-healing.

Proof. We exhibit a counterexample. Consider $n = 152$, and we verify that it is an SHN:

$$n^{(1)} = 52, \quad 1 \mid 52 \tag{7}$$
$$n^{(2)} = 12, \quad 2 \mid 12 \tag{8}$$
$$n^{(3)} = 15, \quad 3 \mid 15 \tag{9}$$

Since all conditions are met, 152 is a self-healing number.

However, the two-digit prefix 15 is not self-healing:

$$15^{(1)} = 5, \quad 1 \mid 5 \tag{10}$$
$$15^{(2)} = 1, \quad 2 \nmid 1 \tag{11}$$

Since $1 \not\equiv 0 \pmod 2$, the number 15 is not self-healing. This completes the proof. □

5. Growth Analysis

The data in Table 1 reveals significant growth in the counting function. To analyze this growth, we examine the ratios $a(k+1)/a(k)$:

Table 2. Growth ratios for consecutive digit lengths.

k	$a(k+1)/a(k)$
1	4.44
2	3.75
3	5.72
4	4.83
5	4.80
6	4.80

The growth ratios suggest exponential behavior, with ratios stabilizing around 4.8 for lengths 4-6.

Let $\rho(k) = a(k)/(9 \times 10^{k-1})$ denote the density of self-healing numbers among k-digit integers.

We observe:

$\rho(1) = 1.000, \rho(2) = 0.444, \quad \rho(3) = 0.167, \quad \rho(4) = 0.095$

While the density decreases with k, it decreases slowly enough that the absolute count continues to grow substantially.

Conjecture 5.1. The sequence of self-healing numbers is infinite.

The sustained growth through seven digits and the regular pattern in growth ratios provide empirical support for this conjecture, though a proof remains elusive. A naive probabilistic model would suggest that the number of k-digit SHNs, $a(k)$, is proportional to $\frac{10^{k-1}}{k!}$, leading to a growth ratio of $\frac{10}{k+1}$. The discrepancy between this simple model and our empirical data (which shows a ratio stabilizing around 4.8) suggests that the divisibility conditions are not independent and require a more sophisticated analysis.

6. Algorithmic Considerations

The verification of whether an n-digit integer is self-healing requires checking n divisibility conditions. For each position i, we must:

1. Remove the digit at position i
2. Convert the resulting $(n-1)$-digit string to an integer
3. Check divisibility by i

The time complexity for testing a single n-digit number is $O(n^2)$, accounting for the digit manipulations and divisibility tests. Exhaustive enumeration of all k-digit self-healing numbers requires $O(k^2 \cdot 10^k)$ operations.

Theorem 4.1 provides a crucial computational optimization: we need only examine integers with even terminal digits, reducing the search space by a factor of 2.

7. Related Work

Self-healing numbers are related to several existing classes of integers:

- **Harshad numbers**: integers divisible by the sum of their digits
- **Self-divisive numbers**: integers divisible by each of their nonzero digits
- **Polydivisible numbers**: k-digit numbers where the first j digits form a number divisible by j for $j = 1, 2, \ldots, k$

The key distinction is that self-healing numbers involve *removal* of digits rather than manipulation or concatenation, leading to different mathematical behavior.

8. Open Problems

Several questions remain open:

1. Prove or disprove Conjecture 5.1.
2. Determine the asymptotic growth rate of $a(k)$ as $k \to \infty$.
3. Characterize the distribution of self-healing numbers within each digit length.
4. Investigate analogous sequences in bases other than 10.
5. Find connections to deeper number-theoretic structures.

9. Conclusion

We have introduced self-healing numbers as a new class of integers with interesting positional divisibility properties. Complete enumeration through seven digits reveals substantial growth and several structural theorems. The non-hereditary nature and terminal digit constraints distinguish these numbers from related sequences in the literature.

The substantial growth observed computationally, combined with the structural properties we have established, suggests that self-healing numbers form a rich class worthy of further mathemat- ical investigation.

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