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Spanning Hypertrees and Spanning Superhypertrees

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Abstract

Graph theory provides a rigorous foundation for representing relationships and connectivity through vertices and edges. Hypergraphs extend this framework by introducing *hyperedges* that connect more than two vertices. Superhypergraphs further enhance the model via iterated powerset constructions, capturing hierarchical and self-referential structures among hyperedges. A spanning tree is a connected, acyclic subgraph that covers all vertices of a graph with exactly $|V| - 1$ edges. A spanning hypertree is a connected, Berge-acyclic subhypergraph of a uniform hypergraph that spans all vertices with hypertree structure. In this paper, we study the notion of a *spanning superhypertree* as the natural spanning tree concept within superhypergraphs. We also discuss several concrete real-world examples of spanning superhypertrees and analyze their structural properties.

Keywords: SuperHyperGraph; hypergraph; spanning superhypertree; spanning tree; spanning hypertree

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1. Preliminaries

We collect the basic terminology and notation used in what follows. Unless explicitly stated otherwise, all graphs considered are finite.

1.1. SuperHyperGraphs

Graph theory provides a rigorous foundation for representing relationships and connectivity through vertices and edges [1,2]. A classical hypergraph generalizes an ordinary graph by permitting an edge to connect an arbitrary (finite) number of vertices, which makes it suitable for representing multiway relationships [3–5]. A *SuperHyperGraph* carries this idea further by forming vertices and edges from iterated powersets of a base set; this viewpoint has appeared in several recent contexts [6–8]. Reported applications include, among others, molecular structure modeling, complex network analysis, and signal processing [9–12]. Throughout, the *level n* is a fixed nonnegative integer.

Definition 1 (Base set)**.** A *base (ground) set* is a fixed finite set S from which higher-level objects are generated:

$$S = \{ x \mid x \text{ belongs to the chosen domain} \}.$$

All structures introduced below ultimately draw their elements from S.

Definition 2 (Powerset)**.** [13,14] Given a set X, its powerset is

$$\mathcal{P}(X) = \{ A \subseteq X \}.$$

We also use the *nonempty* powerset $\mathcal{P}^*(X) := \mathcal{P}(X) \setminus \{\varnothing\}$.

Definition 3 (Iterated powerset)**.** [15–18] For $k \in \mathbb{N}_0$ define

$$\mathcal{P}^0(X) := X, \qquad \mathcal{P}^{k+1}(X) := \mathcal{P}(\mathcal{P}^k(X)).$$

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For the nonempty version set

$$(\mathcal{P}^*)^0(X) := X, \qquad (\mathcal{P}^*)^{k+1}(X) := \mathcal{P}^*((\mathcal{P}^*)^k(X)).$$

Example 4 (Iterated powerset — menu planning with courses (real life))**.** Let the base set of available dishes be

$$X = \{\text{rice}, \text{ fish}\}.$$

Then

$$\mathcal{P}^0(X) = X = \{\text{rice}, \text{fish}\}, \qquad \mathcal{P}^1(X) = \mathcal{P}(X) = \{\varnothing, \{\text{rice}\}, \{\text{fish}\}, \{\text{rice}, \text{fish}\}\}.$$

Elements of $\mathcal{P}^1(X)$ are *menus* (sets of dishes). The second iterated powerset $\mathcal{P}^2(X) = \mathcal{P}(\mathcal{P}(X))$ consists of *families of menus* (e.g., a weekly plan). Two concrete members are

$$F_1 = \{\{\text{rice}\}, \{\text{rice}, \text{fish}\}\}, \qquad F_2 = \{\{\text{fish}\}\}.$$

F_1 collects two acceptable menus (rice only; rice+fish), while F_2 is a single-menu family (fish only). Using the nonempty tiers, $(\mathcal{P}^*)^1(X) = \mathcal{P}(X) \setminus \{\varnothing\}$ excludes the empty menu, and $(\mathcal{P}^*)^2(X)$ excludes empty families.

Definition 5 (Hypergraph [4,19])**.** A *hypergraph* is a pair $H = (V(H), E(H))$ with $V(H) \neq \varnothing$ and $E(H) \subseteq \mathcal{P}^*(V(H))$. Throughout this paper both $V(H)$ and $E(H)$ are finite.

Example 6 (Hypergraph — task forces in an emergency response (real life))**.** Let the responders be the vertex set

$$V = \{\text{Alice (nurse)}, \text{ Bob (paramedic)}, \text{ Chen (driver)}, \text{ Dana (firefighter)}, \text{ Eli (logistics)}\}.$$

Define the family of hyperedges $E \subseteq \mathcal{P}^*(V)$ by

$$E = \{\{\text{Alice}, \text{Bob}, \text{Chen}\}, \{\text{Bob}, \text{Dana}\}, \{\text{Chen}, \text{Dana}, \text{Eli}\}\}.$$

Each hyperedge is a team required for a specific incident type: {nurse, paramedic, driver} for patient transport; {paramedic, firefighter} for a rescue; {driver, firefighter, logistics} for debris removal and supply. Thus $H = (V, E)$ is a finite hypergraph modeling multi-person tasks beyond pairwise interactions.

Definition 7 (n-SuperHyperGraph)**.** [20,21] Fix a finite base set V_0 and a level $n \in \mathbb{N}_0$. An *n-SuperHyperGraph over V_0* is a triple

$$\mathrm{SHG}^{(n)} = (V, E, \partial),$$

where

- $V \subseteq \mathcal{P}^n(V_0)$ is a finite set of *n-supervertices;*
- E is a finite set of *(super)edge identifiers;*
- $\partial : E \to \mathcal{P}^*(V)$ is an *incidence map* sending each edge to a nonempty finite subset of V.

For $e \in E$, the set $\partial(e) \subseteq V$ is called the *(super)edge incidence set*.

Remark 8 (Simple, uniform, and nonempty-tier options)**.** (i) *Simple*: ∂ is injective (no parallel superedges). (ii) *k-uniform*: $|\partial(e)| = k$ for all $e \in E$. (iii) To exclude empties at every tier, one may require $V \subseteq (\mathcal{P}^*)^n(V_0)$.

Remark 9 (Subset presentation)**.** If parallel superedges are unnecessary, one may identify each edge with its incidence set and work with a pair $(V, \mathcal{E})$ where $\mathcal{E} \subseteq \mathcal{P}^*(V)$. This is equivalent to Definition 7 by taking $E := \mathcal{E}$ and $\partial := \mathrm{id}$.

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Example 10 (n-SuperHyperGraph — program playlists of reading lists (real life))**.** Let the base set of short papers be $V_0 = \{\text{L1}, \text{L2}, \text{L3}\}$. Then $\mathcal{P}^1(V_0)$ are *reading lists* (subsets of papers) and $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$ are *playlists of reading lists*. Choose level $n = 2$ and define the set of 2-supervertices

$$V = \Big\{ v_A = \{\{\text{L1}\}, \{\text{L1}, \text{L2}\}\},\ v_B = \{\{\text{L2}, \text{L3}\}\},\ v_C = \{\{\text{L1}, \text{L3}\}, \{\text{L2}\}\} \Big\} \subseteq \mathcal{P}^2(V_0).$$

Introduce two (super)edges $E = \{\alpha, \beta\}$ and the incidence map $\partial : E \to \mathcal{P}^*(V)$ by

$$\partial(\alpha) = \{v_A, v_B\}, \qquad \partial(\beta) = \{v_B, v_C\}.$$

v_A, v_B, v_C are *program playlists* (families of reading lists). Edge α links playlists used in the "Foundations" module; edge β links those used in the "Capstone" module. Then

$$\text{SHG}^{(2)} = (V, E, \partial)$$

is a finite 2-SuperHyperGraph: vertices live at level 2 (families of reading lists), and each (super)edge connects a nonempty set of such vertices via ∂.

Example 11 (3–SuperHyperGraph: University Programs and Shared Resources)**.** Let the base set of atomic items (foundational courses) be

$$V_0 = \{\text{Data}, \text{ Algebra}, \text{ ML}, \text{ Systems}\}.$$

Level 1 (tracks; subsets of courses) in $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$:

$$T_1 = \{\text{Data}, \text{Algebra}\}, \quad T_2 = \{\text{Algebra}, \text{ML}\}, \quad T_3 = \{\text{ML}, \text{Systems}\}.$$

Level 2 (program bundles; sets of tracks) in $\mathcal{P}^2(V_0)$:

$$B_A = \{T_1, T_2\}, \qquad B_B = \{T_2, T_3\}.$$

Level 3 (degree plans; sets of bundles) in $\mathcal{P}^3(V_0)$:

$$L_1 = \{B_A\}, \qquad L_2 = \{B_A, B_B\}, \qquad L_3 = \{B_B\}.$$

Define the 3–supervertex set and the edge set by

$$V = \{L_1, L_2, L_3\} \subseteq \mathcal{P}^3(V_0), \qquad E = \{\, e_{\text{capstone}},\ e_{\text{lab}},\ e_{\text{admin}} \,\}.$$

Define the incidence map $\partial : E \to \mathcal{P}^*(V)$ by

$$\partial(e_{\text{capstone}}) = \{L_1, L_2\}, \qquad \partial(e_{\text{lab}}) = \{L_2, L_3\}, \qquad \partial(e_{\text{admin}}) = \{L_1, L_2, L_3\}.$$

Then

$$\text{SHG}^{(3)} = (V, E, \partial)$$

is a 3–SuperHyperGraph over V_0 in the sense of Definition 7. Here, 3–supervertices encode degree plans (level 3 objects), while superedges encode shared structures: a joint capstone (e_{capstone}), shared laboratories (e_{lab}), and university-wide administration (e_{admin}).

1.2. Spanning Tree and Hypertree

A spanning tree is a connected, acyclic subgraph that includes all vertices of the graph with exactly $|V| - 1$ edges[22–25]. A spanning hypertree is a connected, Berge-acyclic subhypergraph of a uniform hypergraph that spans all vertices with hypertree structure[26–31].

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Definition 12 (Spanning tree of a graph)**.** [22–25] Let $G = (V, E)$ be a finite (simple, undirected) graph. A subgraph $T = (V, E_T)$ of G is a *spanning tree* of G if T is connected and acyclic. Equivalently, $|E_T| = |V| - 1$ and T is connected.

Example 13 (Spanning tree in a simple graph)**.** Let $G = (V, E)$ with

$$V = \{1,2,3,4\}, \qquad E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}, \{1,3\}\}.$$

Consider the subgraph $T = (V, E_T)$ where

$$E_T = \{\{1,2\}, \{2,3\}, \{3,4\}\}.$$

Then $|E_T| = 3 = |V| - 1$ and T is connected (the unique simple path $1-2-3-4$ joins any two vertices). Since a connected simple graph on $|V|$ vertices with $|V| - 1$ edges is acyclic, T is a spanning tree of G. Equivalently, T is a tree (no cycles) and uses all vertices of G.

Definition 14 (Hypertrees in uniform hypergraphs: recursive h-hypertrees)**.** (cf.[26–31]) Fix $h \geq 2$. An *h-hypertree* is an h-uniform hypergraph $T = (X, \mathcal{E})$ defined recursively as follows:

1. If $|X| = h$, then T has the unique edge X.
2. If $|X| \geq h + 1$, there exists a vertex $x \in X$ such that, writing $\mathcal{E}(x) = \{E_1, \ldots, E_q\}$ for the edges of T containing x, the family $\{E_1 \backslash \{x\}, \ldots, E_q \backslash \{x\}\}$ induces an $(h-1)$-hypertree on $X \backslash \{x\}$ and the remaining edges (those not containing x) induce an h-hypertree on $X \backslash \{x\}$.

(For $h = 2$, this coincides with the usual notion of a tree.) *Spanning h-hypertree in H.* If $H = (V, E)$ is an h-uniform hypergraph, a subhypergraph T of H is a *spanning h-hypertree of* H if T is an h-hypertree and $V(T) = V(H)$ (i.e., T spans all vertices of H and uses only edges of H).

Example 15 (A 3-hypertree and a spanning 3-hypertree)**.** Fix $h = 3$. Let $X = \{a, b, c, d\}$ and define the 3-uniform hypergraph

$$T = (X, \mathcal{E}_T), \qquad \mathcal{E}_T = \{\{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}.$$

We verify that T is a 3-hypertree by the given recursion. Choose $x = a$. The edges of T containing a are $\{a, b, c\}$ and $\{a, c, d\}$. Removing a from these gives the 2-edges

$$\{b,c\}, \{c,d\} \subseteq X \setminus \{a\} = \{b,c,d\}.$$

These two 2-edges form a (connected, acyclic) graph on $\{b, c, d\}$, hence an $(h - 1) = 2$-hypertree on $X \setminus \{a\}$. The remaining edges of T that do not contain a form the family

$$\{\{b,c,d\}\},$$

which, on the vertex set $X \setminus \{a\} = \{b, c, d\}$, is exactly the base case of a 3-hypertree (a single 3-edge on 3 vertices). Thus the recursive conditions are satisfied, and T is a 3-hypertree.

To exhibit a *spanning* 3-hypertree inside a larger 3-uniform hypergraph, enlarge T to

$$H = (X, \mathcal{E}_H), \qquad \mathcal{E}_H = \mathcal{E}_T \cup \{\{a,b,d\}\}.$$

Then T is a subhypergraph of H with $V(T) = V(H) = X$, hence T is a spanning 3-hypertree of H.

Example 16 (Spanning 3-hypertree — handoff teams in a project (real life))**. Formal instance.** Let $h = 3$. Take the vertex set

$$X = \{w_1, w_2, w_3, w_4, w_5, w_6, w_7\}$$

and the 3-uniform hyperedge family

$$\mathcal{E} = \{\ E_1 = \{w_1, w_2, w_3\},\ \ E_2 = \{w_3, w_4, w_5\},\ \ E_3 = \{w_5, w_6, w_7\}\ \}.$$

Let the host hypergraph be $H = (X, \mathcal{E})$. Then $T = (X, \mathcal{E})$ is a spanning 3-hypertree of H:

- *Spanning:* $V(T) = X = V(H)$.
- *Connectivity:* E_1 meets E_2 at w_3, and E_2 meets E_3 at w_5, so the incidence graph is a path E_1—E_2—E_3.
- *Berge-acyclicity:* Any Berge cycle would require $E_1 \cap E_3 \neq \varnothing$, but $E_1 \cap E_3 = \varnothing$; hence no cycle exists.

Thus T is a connected, Berge-acyclic 3-uniform subhypergraph spanning all vertices, i.e., a spanning 3-hypertree.

Each hyperedge models a *handoff team* of three people working a project stage. Consecutive stages share exactly one member (w_3 then w_5) to transfer know-how. All seven workers are covered (spanned), and the pipeline has no loops.

Example 17 (Spanning 4-hypertree — co-requisite course blocks across terms (real life))**. Formal instance.** Let $h = 4$. Take

$$X = \{a, b, c, d, e, f, g, h, i, j\}, \qquad \mathcal{E} = \{\ E_1 = \{a, b, c, d\},\ E_2 = \{d, e, f, g\},\ E_3 = \{g, h, i, j\}\ \}.$$

Let $H = (X, \mathcal{E})$. Then $T = (X, \mathcal{E})$ is a spanning 4-hypertree of H:

- *Spanning:* $V(T) = X = V(H)$.
- *Connectivity:* $E_1 \cap E_2 = \{d\}$ and $E_2 \cap E_3 = \{g\}$, so the incidence graph is the path E_1—E_2—E_3.
- *Berge-acyclicity:* A Berge cycle would force $E_1 \cap E_3 \neq \varnothing$, but $E_1 \cap E_3 = \varnothing$; contradiction.

Hence T is a connected, Berge-acyclic 4-uniform subhypergraph that spans all vertices—i.e., a spanning 4-hypertree.

Each hyperedge represents a *term block* of four co-requisite courses. Successive terms share exactly one course (d then g) to ensure curricular continuity. All courses in the program are covered without cycles.

2. Main Results: Spanning SuperHyperTree

A spanning superhypertree is a connected, Berge-acyclic substructure of a superhypergraph, covering all supervertices, generalizing spanning trees and hypertrees.

Definition 18 (Berge-style superpaths and supercycles)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be an n–SuperHyperGraph (so $\partial : E \to \mathcal{P}^*(V)$).

- A *superpath of length* $\ell \geq 1$ is a sequence

$$v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell$$

  with $v_i \in V$, $e_j \in E$, such that $v_{j-1}, v_j \in \partial(e_j)$ for all j, the vertices $v_0, \ldots, v_\ell$ are pairwise distinct, and the edges $e_1, \ldots, e_\ell$ are pairwise distinct.
- A *supercycle of length* $\ell \geq 2$ is a sequence

$$v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell$$

  with $v_\ell = v_0$, the intermediate vertices $v_1, \ldots, v_{\ell-1}$ pairwise distinct, and the edges $e_1, \ldots, e_\ell$ pairwise distinct, and $v_{j-1}, v_j \in \partial(e_j)$ for all j (indices modulo ℓ).

We say $\mathrm{SHG}^{(n)}$ is *connected* if every two vertices can be joined by a superpath, and it is *Berge-acyclic* if it contains no supercycle.

Example 19 (Real-World Illustration: Multi-Tier Supply Contracts)**.** Consider a 3–SuperHyperGraph modeling a supply chain with atomic parts

$$V_0 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}.$$

Level 1 (kits) in $\mathcal{P}^1(V_0)$:

$$K_1 = \{A, B\}, \quad K_2 = \{B, C\}, \quad K_3 = \{C, D\}.$$

Level 2 (bundles) in $\mathcal{P}^2(V_0)$:

$$B_1 = \{K_1, K_2\}, \qquad B_2 = \{K_2, K_3\}.$$

Level 3 (contracts) in $\mathcal{P}^3(V_0)$:

$$C_1 = \{B_1\}, \qquad C_2 = \{B_1, B_2\}, \qquad C_3 = \{B_2\}.$$

Define the 3–supervertex set and edges

$$V = \{C_1, C_2, C_3\}, \qquad E = \{e_{12}, e_{23}, e_{31}\}.$$

The incidence map $\partial : E \to \mathcal{P}^*(V)$ encodes cooperative agreements:

$$\partial(e_{12}) = \{C_1, C_2\}, \qquad \partial(e_{23}) = \{C_2, C_3\}, \qquad \partial(e_{31}) = \{C_3, C_1\}.$$

Then $\mathrm{SHG}^{(3)} = (V, E, \partial)$ models three contract clusters (C_1, C_2, C_3) with superedges expressing shared logistics or financing channels between them.

Berge–style superpath. The sequence

$$C_1,\ e_{12},\ C_2,\ e_{23},\ C_3$$

is a superpath of length 2 since $C_1, C_2 \in \partial(e_{12})$ and $C_2, C_3 \in \partial(e_{23})$, with distinct vertices and edges.

Berge–style supercycle. The sequence

$$C_1,\ e_{12},\ C_2,\ e_{23},\ C_3,\ e_{31},\ C_1$$

is a supercycle of length 3 because each consecutive pair lies in the incidence of the corresponding edge and all edges are distinct. Hence the structure is connected (every two vertices are joined by a superpath) but not Berge–acyclic (it contains a supercycle).

Definition 20 (Spanning superhypertree)**.** Let $\mathrm{SHG}^{(n)} = (V, E, \partial)$ be an n–SuperHyperGraph. A substructure

$$T := (V,\ E_T,\ \partial\restriction_{E_T}), \qquad \varnothing \neq E_T \subseteq E,$$

is called a *spanning superhypertree* of $\mathrm{SHG}^{(n)}$ if T is connected and Berge-acyclic. Equivalently, T spans all vertices of $\mathrm{SHG}^{(n)}$, uses only edges of $\mathrm{SHG}^{(n)}$, has no supercycle, and connects every vertex pair via a superpath.

Remark 21 (Minimality by edge deletion)**.** If T is a spanning superhypertree and $e \in E_T$, then $T - e$ is disconnected. Indeed, removing any edge on some superpath between two vertices breaks all superpaths between them; if it did not, e would lie on a supercycle, contradicting acyclicity.

Example 22 (Graph case $n = 0$ (reduces to a spanning tree))**.** Let $V = \{1, 2, 3, 4\}$ and consider the 2-uniform $\mathrm{SHG}^{(0)} = (V, E, \partial)$ with edges

$$\partial(e_{12}) = \{1, 2\},\ \partial(e_{23}) = \{2, 3\},\ \partial(e_{34}) = \{3, 4\},\ \partial(e_{14}) = \{1, 4\}.$$

Take

$$E_T = \{e_{12}, e_{23}, e_{34}\}.$$

Then $T = (V, E_T, \partial|_{E_T})$ is connected (paths $1-2-3-4$) and has no supercycle (three edges form a simple path), hence a spanning superhypertree. By Theorem 28, G_T is the usual spanning tree on V with edge set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$.

Example 23 (Proper super case $n = 1$ (supervertices are subsets of a base set))**.** Let the base set be $V_0 = \{a, b, c, d\}$ and set

$$V = \{v_1, v_2, v_3\} = \{\{a, b\}, \{b, c\}, \{c, d\}\} \subseteq \mathcal{P}^1(V_0).$$

Define the 2-uniform 1–SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E, \partial)$ by

$$\partial(e_{12}) = \{v_1, v_2\}, \qquad \partial(e_{23}) = \{v_2, v_3\}, \qquad \partial(e_{13}) = \{v_1, v_3\}.$$

Take the edge subset $E_T = \{e_{12}, e_{23}\}$. Then $T = (V, E_T, \partial|_{E_T})$ is connected (superpath $v_1, e_{12}, v_2, e_{23}, v_3$) and Berge-acyclic (no supercycle since only two edges are used), hence a spanning superhypertree in the level-1 setting. Note that the "vertices" here are *subsets* of V_0, so this example is genuinely beyond ordinary graphs/hypergraphs.

Example 24 (Spanning superhypertree (n=1) — overlapping project teams (real life))**. Formal instance.** Let the ground set of people be

$$V_0 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}.$$

Work at level $n = 1$ so vertices are nonempty subsets of V_0. Define the supervertex set

$$V = \{v_1 = \{\mathrm{A}, \mathrm{B}\},\ v_2 = \{\mathrm{B}, \mathrm{C}, \mathrm{D}\},\ v_3 = \{\mathrm{D}, \mathrm{E}\},\ v_4 = \{\mathrm{E}, \mathrm{F}\}\} \subseteq \mathcal{P}^*(V_0).$$

Let the edge set be $E = \{e_{12}, e_{23}, e_{34}\}$ with incidence map

$$\partial(e_{12}) = \{v_1, v_2\}, \qquad \partial(e_{23}) = \{v_2, v_3\}, \qquad \partial(e_{34}) = \{v_3, v_4\}.$$

Consider the substructure $T = (V, E_T, \partial|_{E_T})$ with $E_T = E$. Then:

- *Spanning:* $V(T) = V$ (all supervertices are included).
- *Connectedness:* for any v_i, v_j there is a superpath along the chain

$$v_1, e_{12}, v_2, e_{23}, v_3, e_{34}, v_4.$$

- *Berge-acyclicity:* the incidence graph on $V \cup E$ is a path, hence contains no supercycle.

Thus T is a spanning superhypertree of the 1–SuperHyperGraph $\mathrm{SHG}^{(1)} = (V, E, \partial)$.

Each supervertex v_i is a *team pod* (subset of people). Edges record handoffs between pods that share a member (e.g., $v_1 \cap v_2 = \{\mathrm{B}\}$). The chain covers all pods (spanning) and has no loop, modeling a linear, non-cyclic delivery pipeline.

Example 25 (Spanning superhypertree (n=2) — compliance dossier bundles (real life))**. Formal instance.** Let the base set of atomic documents be

$$V_0 = \{\mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}\}.$$

Work at level $n = 2$, so supervertices are *families of document sets*, i.e. elements of $\mathcal{P}(\mathcal{P}(V_0))$. Define

$$\begin{aligned} v_1 &= \{\{\mathrm{A}\}, \{\mathrm{A}, \mathrm{B}\}\}, \\ v_2 &= \{\{\mathrm{B}\}, \{\mathrm{B}, \mathrm{C}\}\}, \\ v_3 &= \{\{\mathrm{C}\}, \{\mathrm{C}, \mathrm{D}\}\}, \\ v_4 &= \{\{\mathrm{D}\}\}, \end{aligned} \qquad V = \{v_1, v_2, v_3, v_4\} \subseteq \mathcal{P}_2(V_0).$$

Let $E = \{e_{12}, e_{23}, e_{34}\}$ and define the incidence map

$$\partial(e_{12}) = \{v_1, v_2\}, \qquad \partial(e_{23}) = \{v_2, v_3\}, \qquad \partial(e_{34}) = \{v_3, v_4\}.$$

Set $E_T = E$ and $T = (V, E_T, \partial|_{E_T})$. Then:

- *Spanning:* $V(T) = V$.
- *Connectedness:* there is a superpath $v_1, e_{12}, v_2, e_{23}, v_3, e_{34}, v_4$ joining any endpoints.
- *Berge-acyclicity:* with only the three edges arranged in a chain, no supercycle can occur.

Therefore T is a spanning superhypertree of $\mathrm{SHG}^{(2)} = (V, E, \partial)$.

Each supervertex v_i is a *dossier bundle*: a collection of related document-sets (policies, reports). Consecutive bundles share thematic subsets (e.g., $\{B\}$ or $\{B, C\}$), so review proceeds linearly across quarters. All bundles are covered (spanned) without circular dependencies.

Example 26 (Spanning SuperHypertree (n=3): Multi-Agency Disaster Response)**.** Let the atomic resources be

$$V_0 = \{\text{Med, Log, Comms, Safety}\}.$$

Level 1 (taskforces; subsets of resources) in $\mathcal{P}^1(V_0)$:

$$T_1 = \{\text{Med}, \text{Comms}\}, \quad T_2 = \{\text{Med}, \text{Log}\}, \quad T_3 = \{\text{Log}, \text{Safety}\}, \quad T_4 = \{\text{Comms}, \text{Safety}\}.$$

Level 2 (operations; sets of taskforces) in $\mathcal{P}^2(V_0)$:

$$O_A = \{T_1, T_2\}, \qquad O_B = \{T_2, T_3\}, \qquad O_C = \{T_1, T_4\}.$$

Level 3 (incident plans; sets of operations) in $\mathcal{P}^3(V_0)$:

$$L_1 = \{O_A\}, \qquad L_2 = \{O_A, O_B\}, \qquad L_3 = \{O_B\}, \qquad L_4 = \{O_C\}.$$

Define the 3–supervertex set and an edge set by

$$V = \{L_1, L_2, L_3, L_4\} \subseteq \mathcal{P}^3(V_0), \qquad E = \{e_{12}, e_{23}, e_{34}, e_{14}\}.$$

Define the incidence map $\partial : E \to \mathcal{P}^*(V)$ as

$$\partial(e_{12}) = \{L_1, L_2\}, \quad \partial(e_{23}) = \{L_2, L_3\}, \quad \partial(e_{34}) = \{L_3, L_4\}, \quad \partial(e_{14}) = \{L_1, L_4\}.$$

Consider the substructure

$$T \;:=\; (V,\; E_T,\; \partial\restriction_{E_T}) \quad \text{with} \quad E_T = \{e_{12}, e_{23}, e_{34}\} \;\subseteq\; E.$$

Then T spans all vertices V and is connected via the superpaths

$$L_1 \xleftrightarrow{e_{12}} L_2 \xleftrightarrow{e_{23}} L_3 \xleftrightarrow{e_{34}} L_4.$$

Moreover, T is Berge–acyclic: the edges in E_T form a simple chain and there is no sequence $v_0, e_1, v_1, e_2, \ldots, e_\ell, v_\ell = v_0$ with pairwise distinct edges e_i witnessing a supercycle. Hence T is a *spanning superhypertree* of the 3–SuperHyperGraph $\mathrm{SHG}^{(3)} = (V, E, \partial)$.

Theorem 27 (Restriction is an n–SuperHyperGraph)**.** *If* $\mathrm{SHG}^{(n)} = (V, E, \partial)$ *is an n–SuperHyperGraph and $E_T \subseteq E$ is nonempty, then*

$$T = (V, E_T, \partial\restriction_{E_T})$$

is an n–SuperHyperGraph. In particular, every spanning superhypertree is (by definition) an n–SuperHyperGraph.

Proof. By hypothesis, $\partial : E \to \mathcal{P}^*(V)$ takes edges to nonempty subsets of V. Hence its restriction $\partial\restriction_{E_T}: E_T \to \mathcal{P}^*(V)$ does the same. The vertex set remains V, which is finite, so T satisfies the axioms of an n–SuperHyperGraph. □

Theorem 28 (Generalization of spanning trees in graphs)**.** *Let $n = 0$ and suppose* $\mathrm{SHG}^{(0)} = (V, E, \partial)$ *is* 2-uniform*, i.e.* $|\partial(e)| = 2$ *for all* $e \in E$*. Identify each* e *with the (unordered) pair* $\partial(e) \subseteq V$ *to obtain a simple graph* $G = (V, E')$ *with* $E' = \{\partial(e) : e \in E\}$*. Then:*

> *A substructure* $T = (V, E_T, \partial|_{E_T})$ *is a spanning superhypertree of* $\mathrm{SHG}^{(0)}$ *if and only if the simple graph* $G_T = (V, \{\partial(e) : e \in E_T\})$ *is a spanning tree of* G*.*

Proof. In the 2-uniform, $n = 0$ case, a superpath is exactly a usual graph path (each hyperedge joins precisely two vertices), and a supercycle is exactly a graph cycle. Thus "connected and Berge-acyclic" coincides with "connected and acyclic" in graph theory. Spanning means the vertex set is V in both settings. Hence the equivalence. □

Theorem 29 (Generalization of spanning hypertrees)**.** *Let $n = 0$ and assume* $\mathrm{SHG}^{(0)} = (V, E, \partial)$ *is an* h*–uniform hypergraph (i.e.* $|\partial(e)| = h \geq 2$ *for all* e*). Identify each* e *with its incidence set* $\partial(e) \subseteq V$ *to obtain the* h*–uniform hypergraph* $H = (V, \mathcal{E})$*,* $\mathcal{E} = \{\partial(e) : e \in E\}$*. Then:*

> *A substructure* $T = (V, E_T, \partial|_{E_T})$ *is a spanning superhypertree of* $\mathrm{SHG}^{(0)}$ *if and only if the hypergraph* $H_T = (V, \{\partial(e) : e \in E_T\})$ *is a spanning Berge-acyclic hypertree of* H*.*

Proof. For $n = 0$, our superpaths and supercycles are exactly the classical Berge paths and Berge cycles in hypergraphs (alternating vertex–edge sequences with the prescribed incidences). Therefore, "connected and Berge-acyclic" in T is equivalent to "connected and Berge-acyclic" in H_T. Spanning again means V is the full vertex set. Hence the equivalence. □

3. Conclusions

In this paper, we studied the notion of a *spanning superhypertree* as the natural spanning tree concept within superhypergraphs. In future work, we aim to explore possible extensions based on Fuzzy Sets [32,33], Intuitionistic Fuzzy Sets [34,35], Neutrosophic Sets [36,37], Hesitant Fuzzy Sets [38,39], and Plithogenic Sets [40–42]. Such directions may provide richer generalizations and further applications of the theoretical framework developed in this paper.

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Use of Artificial Intelligence: I use generative AI and AI-assisted tools for tasks such as English grammar checking, and I do not employ them in any way that violates ethical standards.

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Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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