Article

Not peer-reviewed version

Introduction for SuperHyperGraph Labeling and MultiLabeling

Takaaki Fujita *

Posted Date: 19 September 2025

doi: 10.20944/preprints202509.1593.v1

Keywords: Superhypergraphs; Hypergraphs; SuperHyperFunction; Graph Labeling; HyperGraph Labeling

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1593.v1

Article

Introduction for SuperHyperGraph Labeling and MultiLabeling

Takaaki Fujita

Independent Researcher, Tokyo, Japan; Takaaki.fujita060@gmail.com

Abstract

A finite *hypergraph* generalizes the classical graph model by allowing *hyperedges* that can connect any nonempty subset of vertices. Building on this foundation, a finite *SuperHyperGraph* is obtained through iterative application of the powerset construction, thereby creating nested families of vertex and edge sets that capture multi-layered relationships. Graph labeling assigns numbers or symbols to vertices and/or edges of a graph under rules, modeling constraints, optimization, or communication. In this paper, we define and study the mathematical properties of *Graph Labeling*, *HyperGraph Labeling*, *SuperHyperGraph Labeling*, *Graph MultiLabeling*, *HyperGraph MultiLabeling*, and *SuperHyperGraph MultiLabeling*.

Keywords: Superhypergraphs, Hypergraphs, SuperHyperFunction, Graph Labeling, HyperGraph Labeling

---

Structure of this paper

The format of this paper is described below.

1. Preliminaries

This section fixes the terminology and notation used throughout the paper. Unless stated otherwise, every graph considered here is *finite*.

1.1. Labeling Graph

Graph theory investigates mathematical models built from vertices and edges that capture pairwise relations, network structure, and connectivity [1,2]. Graph labeling assigns numbers or symbols to vertices and/or edges of a graph under rules, modeling constraints, optimization, or communication [3–6].

Definition 1 (Graph labeling). *[3–6] Let $G = (V, E)$ be a finite (simple) graph. A* graph labeling *is a choice of label sets L_V (for vertices) and L_E (for edges), together with functions*

$$\ell_V : V \to L_V \quad \text{and/or} \quad \ell_E : E \to L_E.$$

A labeling may be required to satisfy additional constraints, depending on the context. Typical instances include:

- Proper vertex-coloring: *$L_V = \{1, \ldots, k\}$ and for every edge $uv \in E$, $\ell_V(u) \neq \ell_V(v)$.*
- *$L(p, q)$*-labeling: *$L_V \subseteq \mathbb{Z}$ such that for adjacent u, v one has $|\ell_V(u) - \ell_V(v)| \geq p$, and for vertices at distance 2 one has $|\ell_V(u) - \ell_V(v)| \geq q$.*
- Edge labelings *(e.g., graceful, harmonious): constraints are imposed on ℓ_E (and sometimes on ℓ_V) to control the multiset of induced values.*

When only one of ℓ_V or ℓ_E is present, we speak of a vertex labeling *or an* edge labeling*, respectively.*

Example 1 ($L(2, 1)$-labeling of a path). *Consider the path P_5 with vertices v_1, v_2, v_3, v_4, v_5 and edges $v_i v_{i+1}$ ($i = 1, \ldots, 4$). Define a vertex labeling $\ell_V : V(P_5) \to \mathbb{Z}$ by*

$$\ell_V(v_1) = 0, \quad \ell_V(v_2) = 2, \quad \ell_V(v_3) = 4, \quad \ell_V(v_4) = 1, \quad \ell_V(v_5) = 3.$$

We verify the $L(2, 1)$ constraints:

- *For each edge $v_i v_{i+1}$, the label gap is at least 2:*

$$|\ell_V(v_1) - \ell_V(v_2)| = |0 - 2| = 2, \ |\ell_V(v_2) - \ell_V(v_3)| = |2 - 4| = 2,$$

$$|\ell_V(v_3) - \ell_V(v_4)| = |4 - 1| = 3, \ |\ell_V(v_4) - \ell_V(v_5)| = |1 - 3| = 2.$$

- *For distance-2 pairs, the gap is at least 1:*

$$|\ell_V(v_1) - \ell_V(v_3)| = |0 - 4| = 4,$$

$$|\ell_V(v_2) - \ell_V(v_4)| = |2 - 1| = 1,$$

$$|\ell_V(v_3) - \ell_V(v_5)| = |4 - 3| = 1.$$

Hence ℓ_V is a valid $L(2, 1)$-labeling of P_5.

1.2. SuperHyperGraphs

A finite *hypergraph* extends the classical notion by allowing *hyperedges* to join arbitrary nonempty subsets of the vertex set, thereby representing multiway interactions [7–9]. Pushing this idea further, a finite *SuperHyperGraph* arises by iterating the powerset construction, which yields nested families of vertex- and edge-sets and thus encodes multi-layer relationships [10–15]. Such models are useful in, for example, molecular design, complex-network analysis, and advanced signal-processing pipelines [16,17]. Unless stated otherwise, the index n in $\mathcal{P}_n(\cdot)$ and in an n–SuperHyperGraph is taken to be nonnegative.

Definition 2 (Base set). *A* base set *S is the ambient universe of discourse:*

$$S = \{ x \mid x \text{ belongs to the context under consideration} \}.$$

Every object occurring in $\mathcal{P}(S)$ or in any iterated powerset $\mathcal{P}_n(S)$ is, by definition, a subset ultimately formed from elements of S.

Definition 3 (Powerset). *(see [18–20]) For a set S, the* powerset *$\mathcal{P}(S)$ is the collection of all subsets of S:*

$$\mathcal{P}(S) = \{ A \subseteq S \}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1593.v1

In particular, both the empty set $\emptyset$ *and* S *itself lie in* $\mathcal{P}(S)$.

Definition 4 (Hypergraph). *[21,22] A* hypergraph *is an ordered pair* $H = (V, E)$ *with*

- *a finite vertex set* V*, and*
- *a finite family* E *of nonempty subsets of* V*, whose members are called* hyperedges.

Hypergraphs naturally encode interactions involving more than two participants.

Example 2 (Hypergraph — project teams sharing resources (real life)). *Let the employees be the vertex set*

$$V = \{\text{Alice}, \text{Bob}, \text{Chen}, \text{Dina}\}.$$

Define the family of hyperedges

$$E = \big\{\{\text{Alice}, \text{Bob}, \text{Chen}\},\ \{\text{Bob}, \text{Dina}\},\ \{\text{Alice}, \text{Dina}\}\big\}.$$

Interpretation. Each hyperedge is a team *that jointly uses a shared resource (e.g., a meeting room or a code repository). Thus* $H = (V, E)$ *is a finite hypergraph: it records not only pairwise collaborations (*$\{\text{Bob}, \text{Dina}\}$*) but also a three-person collaboration (*$\{\text{Alice}, \text{Bob}, \text{Chen}\}$*).*

Definition 5 (n-th powerset). *[23–25] For a set* X*, define* $\mathcal{P}_1(X) = \mathcal{P}(X)$ *and, for* $n \geq 1$,

$$\mathcal{P}_{n+1}(X) \ = \ \mathcal{P}\big(\mathcal{P}_n(X)\big).$$

When excluding the empty set, write $\mathcal{P}^*_n(X) = \mathcal{P}_n(X) \setminus \{\emptyset\}$.

Example 3 (n-th powerset — explicit small instance). *Take* $X = \{p, q\}$*. Then*

$$\mathcal{P}_1(X) = \mathcal{P}(X) = \{\varnothing,\ \{p\},\ \{q\},\ \{p, q\}\}.$$

The second-level powerset is the powerset of this 4-element set, hence $|\mathcal{P}_2(X)| = 2^4 = 16$*, for example it contains*

$$\{\{p\}\}, \quad \{\{q\}\}, \quad \{\{p\}, \{q\}\}, \quad \{\varnothing, \{p\}, \{q\}, \{p, q\}\} = \mathcal{P}_1(X).$$

If we exclude the empty set at each step, then

$$\mathcal{P}^*_1(X) = \{\{p\}, \{q\}, \{p, q\}\}, \qquad \mathcal{P}^*_2(X) = \mathcal{P}_2(X) \setminus \{\varnothing\}.$$

This illustrates how iterating $\mathcal{P}(\cdot)$ *builds higher "layers" of set families.*

Definition 6 (n–SuperHyperGraph). *(see [26,27]) Let* V_0 *be a finite, nonempty base set and define*

$$\mathcal{P}^0(V_0) := V_0, \qquad \mathcal{P}^{k+1}(V_0) := \mathcal{P}\big(\mathcal{P}^k(V_0)\big) \quad (k \in \mathbb{N}).$$

For $n \geq 0$*, an* n–SuperHyperGraph on V_0 *is a pair*

$$\text{SHG}^{(n)} = (V, E)$$

with

$$V \subseteq \mathcal{P}^n(V_0) \quad \text{and} \quad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

Members of V *are the* n–supervertices*, while members of* E *are the* n–superedges *(each* n*–superedge is a nonempty subset of* V*).*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1593.v1

Example 4 (n–SuperHyperGraph — families of task-sets (real life))**.** *Let the base set of atomic tasks be* $V_0 = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$*. Then* $\mathcal{P}^1(V_0) = \mathcal{P}(V_0)$ *consists of all task-sets, and* $\mathcal{P}^2(V_0) = \mathcal{P}(\mathcal{P}(V_0))$ *consists of* families *of task-sets. Choose the* $n = 2$ *supervertex set*

$$V = \Big\{ F_1,\ F_2 \Big\}, \qquad F_1 = \big\{\{\mathrm{a}\}, \{\mathrm{b}\}\big\}, \quad F_2 = \big\{\{\mathrm{b}\}, \{\mathrm{c}\}\big\}.$$

Define the superedge family

$$E = \big\{ \{F_1\},\ \{F_2\},\ \{F_1, F_2\} \big\} \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Interpretation. Each supervertex F_i *is a* plan family *(a set of admissible task-sets); a superedge groups one or several plan families that are considered together (e.g., combined scenarios). Hence* $\mathrm{SHG}^{(2)} = (V, E)$ *is a finite* 2*–SuperHyperGraph on* V_0*.*

2. Main Results

This section presents the principal findings of the paper.

2.1. Hypergraph labeling

HyperGraph labeling assigns labels to vertices and hyperedges of hypergraphs, encoding multi-participant interactions, scheduling, resource distribution, or network optimization (cf.[28–32]).

Definition 7 (Primal (2-section) graph)**.** *Given a hypergraph* $H = (V, E)$*, its* primal graph *(also called the* 2*-section) is*

$$G(H) := (V,\ E'), \qquad E' := \big\{ \{u, v\} \subseteq V \;\big|\; \exists\, e \in E \text{ with } \{u, v\} \subseteq e \big\}.$$

Distances between vertices of H *are measured in* $G(H)$ *and denoted* dist_H*.*

Definition 8 (Hypergraph labeling (schema-based))**.** *Let* $H = (V, E)$ *be a hypergraph, and let* L_V, L_E *be nonempty label sets (for vertices and hyperedges). A* (vertex/edge) labeling *of* H *is a pair of maps*

$$\ell_V : V \longrightarrow L_V, \qquad \ell_E : E \longrightarrow L_E,$$

where either map may be omitted if not used. A hypergraph labeling schema *is a first-order predicate* $\Phi(H, \ell_V, \ell_E)$ *built from the incidence relation "*$v \in e$*", the distance* dist_H *on* V *(Definition 7), the equality/inequality on labels, and quantification over* V *and* E*. We say that* (ℓ_V, ℓ_E) *is a* valid hypergraph labeling (for Φ) *if* $\Phi(H, \ell_V, \ell_E)$ *holds.*

Remark 1 (Classical graph labelings as instances of Φ)**.** *Typical choices of* Φ *recover familiar graph labelings when* H *is* 2*-uniform:*

- ***Proper vertex coloring:*** $L_V = \{1, \ldots, k\}$ *and* $\Phi \equiv (\forall \{u, v\} \in E)\ \ell_V(u) \neq \ell_V(v)$*.*
- $L(p, q)$***-labeling:*** $L_V \subseteq \mathbb{Z}$ *and*

  $$\Phi \equiv (\forall u \neq v \in V)\ \Big(\mathrm{dist}_H(u, v) = 1 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq p\ \wedge\ \mathrm{dist}_H(u, v) = 2 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq q\Big).$$

- ***Strong hypergraph coloring (a genuine hypergraph constraint):*** $L_V = \{1, \ldots, k\}$ *and* $\Phi \equiv (\forall e \in E)$ *the labels* $\{\ell_V(v) \mid v \in e\}$ *are pairwise distinct.*

Example 5 (Strong hypergraph 3-coloring of a small hypergraph)**.** *Let* $H = (V, E)$ *with*

$$V = \{1, 2, 3, 4\}, \qquad E = \big\{\{1, 2, 3\},\ \{3, 4\}\big\}.$$

Take $L_V = \{r, g, b\}$ and let Φ be "strong coloring": for every $e \in E$ the labels on e are pairwise distinct. Define

$$\ell_V(1) = r, \quad \ell_V(2) = g, \quad \ell_V(3) = b, \quad \ell_V(4) = r.$$

Verification. *On $e_1 = \{1, 2, 3\}$ we have $\{r, g, b\}$ (all distinct). On $e_2 = \{3, 4\}$ we have $\{b, r\}$ (distinct). Thus ℓ_V satisfies Φ and is a valid strong 3-coloring of H.*

Example 6 ($L(2,1)$-labeling lifted to a hypergraph via the 2-section)**.** *Let $H = (V, E)$ with*

$$V = \{a, b, c, d, e\}, \qquad E = \big\{\{a, b, c\}, \ \{c, d\}, \ \{d, e\}\big\}.$$

Its 2-section $G(H)$ has edges

$$\{a, b\}, \ \{a, c\}, \ \{b, c\}, \ \{c, d\}, \ \{d, e\}.$$

Let $L_V = \mathbb{Z}$ and let Φ be the hypergraph version of $L(2,1)$: for all distinct $u, v \in V$,

$$\mathrm{dist}_H(u, v) = 1 \ \Rightarrow \ |\ell_V(u) - \ell_V(v)| \geq 2, \qquad \mathrm{dist}_H(u, v) = 2 \ \Rightarrow \ |\ell_V(u) - \ell_V(v)| \geq 1,$$

where dist_H is computed in $G(H)$. Define labels

$$\ell_V(a) = 0, \ \ \ell_V(b) = 2, \ \ \ell_V(c) = 4, \ \ \ell_V(d) = 1, \ \ \ell_V(e) = 3.$$

Adjacency checks *(distance 1 in $G(H)$):*

$$|0-2| = 2 \ (\{a, b\}), \ \ |0-4| = 4 \ (\{a, c\}), \ \ |2-4| = 2 \ (\{b, c\}), \ \ |4-1| = 3 \ (\{c, d\}), \ \ |1-3| = 2 \ (\{d, e\}).$$

Distance-2 checks *(one example per class):*

$$a\text{-}d: \ a{-}c{-}d \Rightarrow |0-1| = 1; \quad b\text{-}d: \ b{-}c{-}d \Rightarrow |2-1| = 1;$$

$$a\text{-}e: \ a{-}c{-}d{-}e \Rightarrow |0-3| = 3; \quad b\text{-}e: \ b{-}c{-}d{-}e \Rightarrow |2-3| = 1.$$

All constraints hold, hence ℓ_V is a valid $L(2,1)$-labeling of the hypergraph H under Φ. When H is 2-uniform, this reduces to the classical $L(2,1)$-labeling of a graph.

Theorem 1 (Hypergraph labeling strictly generalizes graph labeling)**.** *Let (G, Ψ) be any graph-labeling problem on a simple graph $G = (V, E_G)$, where $\Psi(G, \lambda)$ is a predicate expressed using adjacency/distance in G and (in)equalities among labels. Regard G as the 2-uniform hypergraph $H = (V, E)$ with $E := \{\{u, v\} \mid uv \in E_G\}$. Define the hypergraph labeling schema $\Phi(H, \ell_V) := \Psi\big(G(H), \ell_V\big)$. Then the following are equivalent:*

$$\exists\, \lambda : V \to L \text{ with } \Psi(G, \lambda) \iff \exists\, \ell_V : V \to L \text{ with } \Phi(H, \ell_V).$$

Consequently, every graph labeling instance is an instance of hypergraph labeling; moreover, hypergraph-only constraints (e.g. strong hypergraph coloring) have no counterpart on graphs with only size-2 edges, so the inclusion is strict.

Proof. ($\Rightarrow$) Suppose $\lambda : V \to L$ satisfies $\Psi(G, \lambda)$. Since $G(H) = G$ by construction of E from E_G, setting $\ell_V := \lambda$ yields

$$\Phi(H, \ell_V) \ \equiv \ \Psi\big(G(H), \ell_V\big) \ = \ \Psi(G, \lambda),$$

so ℓ_V is a valid hypergraph labeling.

($\Leftarrow$) Conversely, if $\ell_V : V \to L$ satisfies $\Phi(H, \ell_V)$, then $\Psi\big(G(H), \ell_V\big)$ holds. But $G(H) = G$, hence $\Psi(G, \ell_V)$ holds, and $\lambda := \ell_V$ is a valid graph labeling.

Strictness follows because there exist valid Φ that quantify over hyperedges of size ≥ 3 (e.g. strong hypergraph coloring in Remark 1), which impose constraints on $|e|$-tuples of vertices inside a single hyperedge; such constraints cannot be expressed on a simple graph whose edges are only size-2 subsets without passing to cliques or auxiliary gadgets. □

2.2. SuperHyperGraph Labeling

SuperHyperGraph labeling assigns structured labels to vertices and superedges across nested powerset levels, capturing hierarchical multi-layered relationships and advanced constraints.

Definition 9 (SuperHyperGraph labeling (schema–based))**.** *Let* $\mathsf{SHG}^{(n)} = (V, E)$ *and let* L_V, L_E *be nonempty label sets for vertices and superedges. A* (vertex/edge) labeling *is a pair of maps*

$$\ell_V : V \to L_V, \qquad \ell_E : E \to L_E,$$

where either map may be omitted if not needed. A labeling schema *is a first–order predicate*

$$\Phi(\mathsf{SHG}^{(n)}, \ell_V, \ell_E)$$

built from the incidence relation "$v \in e$*" (*$v \in V$, $e \in E$*), equality/inequality on labels, the distance* $\mathrm{dist}_{\mathsf{SHG}}$ *from the Definition, and (optionally) structural predicates on n–level objects (e.g. cardinalities, inclusion between members of* $V \subseteq \mathcal{P}^n(V_0)$*). We call* (ℓ_V, ℓ_E) *a* valid SuperHyperGraph labeling *(for* Φ*) if* $\Phi(\mathsf{SHG}^{(n)}, \ell_V, \ell_E)$ *holds.*

Remark 2 (Recovering classical schemas)**.** *By choosing* Φ *appropriately one recovers many standard labeling families:*

- ***Proper vertex coloring:*** $L_V = \{1, \ldots, k\}$ *and* $(\forall\{u, v\} \in E')\ \ell_V(u) \neq \ell_V(v)$*, where* E' *is from the 2–section.*
- $L(p, q)$***–labeling:*** $L_V \subseteq \mathbb{Z}$ *and, for all distinct* $u, v \in V$,

$$\mathrm{dist}_{\mathsf{SHG}}(u, v) = 1 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq p, \quad \mathrm{dist}_{\mathsf{SHG}}(u, v) = 2 \Rightarrow |\ell_V(u) - \ell_V(v)| \geq q.$$

- ***Strong hypercoloring (genuinely hyper):*** $L_V = \{1, \ldots, k\}$ *and* $(\forall e \in E)$ *the labels* $\{\ell_V(v) : v \in e\}$ *are pairwise distinct.*

Example 7 (Classical $L(2, 1)$ as a SuperHyperGraph labeling (the case $n = 0$))**.** *Let* $G = P_5$ *be the path on vertices* $v_1, \ldots, v_5$*. Form the 0–SuperHyperGraph* $\mathsf{SHG}^{(0)} = (V, E)$ *with* $V = \{v_1, \ldots, v_5\}$ *and* $E = \big\{\{v_i, v_{i+1}\} : i = 1, 2, 3, 4\big\}$*. Let* Φ *be the* $L(2, 1)$ *schema of Remark 2 with* $L_V = \mathbb{Z}$ *and distances taken in* $G(\mathsf{SHG}^{(0)}) = G$*. Define*

$$\ell_V(v_1) = 0, \quad \ell_V(v_2) = 2, \quad \ell_V(v_3) = 4, \quad \ell_V(v_4) = 1, \quad \ell_V(v_5) = 3.$$

Adjacency (distance 1) checks: $|0 - 2| = 2$, $|2 - 4| = 2$, $|4 - 1| = 3$, $|1 - 3| = 2$. Distance 2 checks: $|0 - 4| = 4$, $|2 - 1| = 1$, $|4 - 3| = 1$. *All constraints hold, so* ℓ_V *is a valid SuperHyperGraph* $L(2, 1)$ *labeling. In this case we exactly recover the classical graph labeling.*

Example 8 (An $n = 1$ SuperHyperGraph with an $L(2, 1)$–type labeling on overlapping sets)**.** *Let* $V_0 = \{a, b, c, d\}$ *and consider the 1–level supervertices*

$$A = \{a, b\}, \quad B = \{b, c\}, \quad C = \{c, d\}, \quad D = \{a, d\} \in \mathcal{P}(V_0).$$

Set $V = \{A, B, C, D\}$ *and define superedges*

$$E_1 = \{A, B, C\}, \qquad E_2 = \{A, D\}, \qquad E_3 = \{C, D\}.$$

Thus $\mathsf{SHG}^{(1)} = (V, \{E_1, E_2, E_3\})$. *Its* 2*–section* $G(\mathsf{SHG}^{(1)})$ *has edges*

$$\{A,B\},\ \{A,C\},\ \{B,C\}\ \ (\textit{from } E_1), \qquad \{A,D\}\ \ (\textit{from } E_2), \qquad \{C,D\}\ \ (\textit{from } E_3).$$

Let $L_V = \mathbb{Z}$ *and impose the* $L(2,1)$ *schema from Remark 2 with distances taken in* $G(\mathsf{SHG}^{(1)})$. *Define the labeling*

$$\ell_V(A) = 0, \qquad \ell_V(B) = 2, \qquad \ell_V(C) = 5, \qquad \ell_V(D) = 7.$$

Adjacency (distance 1) checks:

$$|0-2| = 2\,(\{A,B\}), \quad |0-5| = 5\,(\{A,C\}), \quad |2-5| = 3\,(\{B,C\}), \quad |0-7| = 7\,(\{A,D\}), \quad |5-7| = 2\,(\{C,D\}).$$

Distance 2 check: *B and D have distance* 2 *(via A or C), and* $|2-7| = 5 \ge 1$. *Hence* ℓ_V *satisfies the* $L(2,1)$ *constraints on this genuinely superhyper (level* $n = 1$*) instance. Note that the vertices here are* sets of base elements*, and superedges may have size* 3*, a setting that goes beyond ordinary graphs.*

Theorem 2 (SuperHyperGraph labeling generalizes graph and hypergraph labeling)**.** *Let* (G, Ψ) *be any graph–labeling problem on a simple graph* $G = (V, E_G)$*, with predicate* $\Psi(G, \lambda)$ *expressed in terms of adjacency/distance in G and (in)equalities among labels. Let* (H, Ψ) *be any hypergraph–labeling problem on a hypergraph* $H = (V, E_H)$ *where distances are computed in the hypergraph* 2*–section.*

Then there exist $n \in \{0\}$*, a SuperHyperGraph* $\mathsf{SHG}^{(n)}$*, and a labeling schema* Φ *such that:*

$$\exists\, \lambda : V \to L \textit{ with } \Psi(G, \lambda) \iff \exists\, \ell_V : V \to L \textit{ with } \Phi(\mathsf{SHG}^{(0)}, \ell_V),$$

and

$$\exists\, \lambda : V \to L \textit{ with } \Psi(H, \lambda) \iff \exists\, \ell_V : V \to L \textit{ with } \Phi(\mathsf{SHG}^{(0)}, \ell_V).$$

Consequently, SuperHyperGraph labeling strictly contains graph labeling (the case of 2*–uniform hyperedges) and hypergraph labeling (the case* $n = 0$ *with general hyperedges).*

Proof. For the graph case, let $n = 0$, $V_0 := V$, and define $\mathsf{SHG}^{(0)} = (V, E)$ with $E := \{\{u, v\} \subseteq V \mid uv \in E_G\}$. Then $G(\mathsf{SHG}^{(0)}) = G$ by construction. Define $\Phi(\mathsf{SHG}^{(0)}, \ell_V) := \Psi(G(\mathsf{SHG}^{(0)}), \ell_V)$. Hence $\Psi(G, \lambda)$ holds iff $\Phi(\mathsf{SHG}^{(0)}, \lambda)$ holds, giving the first equivalence.

For the hypergraph case, again take $n = 0$, $V_0 := V$, and set $\mathsf{SHG}^{(0)} = (V, E_H)$. By definition of the hypergraph 2–section, $G(\mathsf{SHG}^{(0)})$ is exactly the primal graph used to measure distances in $\Psi(H, \cdot)$. Put $\Phi(\mathsf{SHG}^{(0)}, \ell_V) := \Psi(H, \ell_V)$, interpreting all distance/adjacency relations through $G(\mathsf{SHG}^{(0)})$. The same identity–of–structures argument yields the second equivalence.

Strict containment follows since for $n \ge 1$ one can add constraints that speak about the internal structure of n–level supervertices (e.g. intersection/nonintersection of members when $V \subseteq \mathcal{P}(V_0)$), which cannot be expressed on ordinary graphs nor on hypergraphs with $n = 0$ without expanding the vertex set. □

2.3. Graph MultiLabeling

Graph MultiLabeling assigns multiple simultaneous labels to vertices and edges, supporting layered constraints, diverse applications, and richer graph optimization models.

Definition 10 (Graph MultiLabeling)**.** *Fix nonnegative integers* p, q*. For a graph* $G = (V, E)$*, choose nonempty* vertex-label alphabets $L_V^{(1)}, \ldots, L_V^{(p)}$ *and nonempty* edge-label alphabets $L_E^{(1)}, \ldots, L_E^{(q)}$*. A* Graph MultiLabeling *on G is the tuple of maps*

$$\boldsymbol{\ell}_V = (\ell_V^{(1)}, \ldots, \ell_V^{(p)}), \qquad \ell_V^{(a)} : V \to L_V^{(a)}\ (1 \le a \le p),$$

$$\boldsymbol{\ell}_E = (\ell_E^{(1)}, \ldots, \ell_E^{(q)}), \qquad \ell_E^{(b)} : E \to L_E^{(b)}\ (1 \le b \le q).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1593.v1

Equivalently, $\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}$, $\ell_E : E \to \prod_{b=1}^{q} L_E^{(b)}$ *with* $\ell_V(v) = (\ell_V^{(1)}(v), \dots, \ell_V^{(p)}(v))$ *and similarly for edges.*

A MultiLabeling schema *is a first-order predicate*

$$\Phi(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$$

built from adjacency/distances in G*, the incidence relation* $u \in e$*, the label components* $\ell_V^{(a)}(u)$*,* $\ell_E^{(b)}(e)$*, and fixed relations on these alphabets (e.g. equality, order, arithmetic, or application-specific constraints). We say that* $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ *is a* valid Graph MultiLabeling for Φ *if* $\Phi(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ *holds.*

Remark 3 (Typical coordinatewise constraints)**.** *Many familiar labeling families appear as single coordinates:*

- *Proper coloring on coordinate a:* $L_V^{(a)} = \{1, \dots, k\}$ *and* $(\forall \{u,v\} \in E)\ \ell_V^{(a)}(u) \neq \ell_V^{(a)}(v)$.
- $L(p,q)$ *on coordinate a:* $L_V^{(a)} \subseteq \mathbb{Z}$ *with*

$$\mathrm{dist}_G(u,v) = 1 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq p, \quad \mathrm{dist}_G(u,v) = 2 \Rightarrow |\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \geq q.$$

- *Edge capacities on coordinate b:* $L_E^{(b)} = \{1, \dots, C\}$ *with cross-constraints such as* $\ell_E^{(b)}(\{u,v\}) \geq f(\ell_V^{(a)}(u), \ell_V^{(a)}(v))$ *for a fixed function* f.

Coordinates may also be coupled*, e.g. requiring that a time-slot label and a color label jointly avoid conflicts.*

Example 9 (Two-coordinate vertex MultiLabeling on a path: coloring + $L(2,1)$)**.** *Let* $G = P_5$ *with vertices* $v_1, \dots, v_5$ *and edges* $v_i v_{i+1}$ *(*$i = 1, \dots, 4$*). Choose* $p = 2$*,* $q = 0$ *with*

$$L_V^{(1)} = \{r, g, b\} \quad \textit{(colors)}, \qquad L_V^{(2)} = \mathbb{Z} \quad \textit{(integers)}.$$

Define $\boldsymbol{\ell}_V = (\ell_V^{(1)}, \ell_V^{(2)})$ *by*

$$\ell_V^{(1)} : (v_1, \dots, v_5) \mapsto (r,\ g,\ b,\ g,\ r),$$
$$\ell_V^{(2)} : (v_1, \dots, v_5) \mapsto (0,\ 2,\ 4,\ 1,\ 3).$$

Schema Φ *requires simultaneously:*

(C) Proper coloring *on coordinate* 1*: for each edge* $v_i v_{i+1}$*,* $\ell_V^{(1)}(v_i) \neq \ell_V^{(1)}(v_{i+1})$.
(N) $L(2,1)$ *on coordinate* 2*: for all distinct* u, v*,*

$$\mathrm{dist}_G(u,v) = 1 \Rightarrow |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)| \geq 2, \qquad \mathrm{dist}_G(u,v) = 2 \Rightarrow |\ell_V^{(2)}(u) - \ell_V^{(2)}(v)| \geq 1.$$

Verification. *(C) Adjacent pairs are* (r,g)*,* (g,b)*,* (b,g)*,* (g,r)*, all unequal. (N) Adjacent gaps:* $|0-2| = 2$*,* $|2-4| = 2$*,* $|4-1| = 3$*,* $|1-3| = 2$*. Distance-2 gaps:* $|0-4| = 4$*,* $|2-1| = 1$*,* $|4-3| = 1$*. Hence* $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E = \varnothing)$ *satisfies* Φ*. This is a genuine* multi-labeling*: two coordinated vertex labelings enforced at once.*

Example 10 (Vertex & edge MultiLabeling with cross-constraints)**.** *Let* G *be the* 4*-cycle* C_4 *on* v_1, v_2, v_3, v_4 *(in order). Take* $p = 1$*,* $q = 1$ *with* $L_V^{(1)} = \{A, B\}$ *(two roles) and* $L_E^{(1)} = \{1, 2\}$ *(capacity classes). Define*

$$\ell_V^{(1)}(v_1) = A,\ \ell_V^{(1)}(v_2) = B,\ \ell_V^{(1)}(v_3) = A,\ \ell_V^{(1)}(v_4) = B,$$

$$\ell_E^{(1)}(\{v_1, v_2\}) = 2,\ \ell_E^{(1)}(\{v_2, v_3\}) = 1,\ \ell_E^{(1)}(\{v_3, v_4\}) = 2,\ \ell_E^{(1)}(\{v_4, v_1\}) = 1.$$

Schema Φ *requires:*

- *Adjacent vertices must have different roles:* $\ell_V^{(1)}(u) \neq \ell_V^{(1)}(v)$ *for all* $\{u,v\} \in E$ *(a* 2*-coloring).*
- *Edge capacity must dominate the* role disparity*:*

$$\ell_E^{(1)}(\{u,v\}) \ \geq \begin{cases} 2, & \text{if } \{\ell_V^{(1)}(u), \ell_V^{(1)}(v)\} = \{A, B\}, \\ 1, & \text{if } \ell_V^{(1)}(u) = \ell_V^{(1)}(v). \end{cases}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1593.v1

Verification. *The vertex roles alternate A, B, A, B, so the first condition holds. Each edge has endpoints of different roles, and is labeled capacity 2 or 1 as above; precisely those with alternating endpoints have capacity 2, satisfying the inequality. Thus $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ is a valid MultiLabeling with coupled vertex/edge constraints.*

Theorem 3 (Graph MultiLabeling generalizes classical graph labeling)**.** *Let (G, Ψ) be any classical graph-labeling problem on $G = (V, E)$: i.e., choose a single label set L and a predicate $\Psi(G, \lambda)$ over mappings $\lambda : V \to L$ (or $\lambda : E \to L$) that is expressed using graph structure and relations on L. Then there exists a MultiLabeling schema Φ with $p = 1$ (and $q = 0$ for a vertex-labeling, or $q = 1$, $p = 0$ for an edge-labeling) such that*

$$\exists\, \lambda \text{ with } \Psi(G, \lambda) \iff \exists\, (\boldsymbol{\ell}_V, \boldsymbol{\ell}_E) \text{ with } \Phi\big(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big).$$

Proof. Vertex-labeling case. Take $p = 1$, $q = 0$, and set $L_V^{(1)} := L$. Define $\Phi\big(G; (\ell_V^{(1)}), \varnothing\big) := \Psi\big(G, \ell_V^{(1)}\big)$. Then λ satisfies Ψ iff $\ell_V^{(1)} := \lambda$ satisfies Φ; existence is equivalent.

Edge-labeling case is identical with $p = 0$, $q = 1$, $L_E^{(1)} := L$, and $\Phi\big(G; \varnothing, (\ell_E^{(1)})\big) := \Psi\big(G, \ell_E^{(1)}\big)$. □

2.4. HyperGraph MultiLabeling

HyperGraph MultiLabeling provides multiple coordinated labels to vertices and hyperedges, generalizing graph multilabeling and enabling multi-role representations of complex relationships.

Definition 11 (Primal (2–section) of a hypergraph)**.** *Given $H = (V, E)$, its* primal graph *(also 2–section) is*

$$G(H) := (V, E') \qquad \text{with} \qquad E' := \big\{\{u, v\} \subseteq V \mid \exists\, e \in E : \{u, v\} \subseteq e\big\}.$$

We write $\mathrm{dist}_H(u, v)$ for the usual shortest–path distance of $u, v \in V$ taken in $G(H)$.

Definition 12 (HyperGraph MultiLabeling)**.** *Fix nonnegative integers p, q. Let $H = (V, E)$ be a hypergraph. Choose nonempty* vertex label alphabets *$L_V^{(1)}, \ldots, L_V^{(p)}$ and nonempty* hyperedge label alphabets *$L_E^{(1)}, \ldots, L_E^{(q)}$. A* HyperGraph MultiLabeling *on H consists of the coordinate maps*

$$\boldsymbol{\ell}_V = \big(\ell_V^{(1)}, \ldots, \ell_V^{(p)}\big), \quad \ell_V^{(a)} : V \to L_V^{(a)} \ (1 \le a \le p),$$

$$\boldsymbol{\ell}_E = \big(\ell_E^{(1)}, \ldots, \ell_E^{(q)}\big), \quad \ell_E^{(b)} : E \to L_E^{(b)} \ (1 \le b \le q).$$

Equivalently, a single vertex map $\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}$ together with a single edge map $\ell_E : E \to \prod_{b=1}^{q} L_E^{(b)}$.

A MultiLabeling schema *is a first–order predicate*

$$\Phi\big(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big)$$

built from the incidence relation $v \in e$, the distance dist_H on V, (and possibly fixed relations/operations on the alphabets, such as $=$, $\neq$, order, arithmetic, etc.). We say $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ is a valid HyperGraph MultiLabeling *(for Φ) if $\Phi\big(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\big)$ holds.*

Remark 4 (Typical coordinatewise and cross–coordinate constraints)**.** *The schema Φ can encode, e.g.:*

- ***Strong hyperedge coloring*** *on a vertex coordinate a: for every $e \in E$, the set $\{\ell_V^{(a)}(v) : v \in e\}$ is pairwise distinct.*
- *$L(p, q)$**–type spacing** on a vertex coordinate a: if $\mathrm{dist}_H(u, v) = 1$ then $|\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \ge p$, and if $\mathrm{dist}_H(u, v) = 2$ then $|\ell_V^{(a)}(u) - \ell_V^{(a)}(v)| \ge q$.*
- ***Vertex–edge coupling**: for each $e \in E$, a constraint linking $\ell_E^{(b)}(e)$ to an aggregate of $\{\ell_V^{(a)}(v) : v \in e\}$ (sum, max, cardinality, etc.).*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 19 September 2025 doi:10.20944/preprints202509.1593.v1

Example 11 (Two–coordinate vertex & one–coordinate edge MultiLabeling on a 3–uniform hypergraph)**.** *Let* $V = \{a, b, c, d\}$ *and* $E = \{e_1, e_2\}$ *with* $e_1 = \{a, b, c\}$ *and* $e_2 = \{b, c, d\}$. *Choose* $p = 2$, $q = 1$ *with*

$$L_V^{(1)} = \{R, G, B\} \quad (colors), \qquad L_V^{(2)} = \mathbb{Z}_{\geq 0} \quad (workload),$$

$$L_E^{(1)} = \mathbb{Z}_{\geq 0} \quad (edge\ deadline).$$

Define the labeling:

$$\ell_V^{(1)}(a) = R,\ \ell_V^{(1)}(b) = G,\ \ell_V^{(1)}(c) = B,\ \ell_V^{(1)}(d) = G,$$
$$\ell_V^{(2)}(a) = 1,\ \ell_V^{(2)}(b) = 2,\ \ell_V^{(2)}(c) = 1,\ \ell_V^{(2)}(d) = 3,$$
$$\ell_E^{(1)}(e_1) = 5, \qquad \ell_E^{(1)}(e_2) = 7.$$

Schema Φ *requires simultaneously:*

(S) Strong hyperedge coloring *on coordinate* 1*: within each* $e \in E$*, the colors are pairwise distinct.*

(C) Capacity coupling*: for each* $e \in E$,

$$\sum_{v \in e} \ell_V^{(2)}(v) \ \leq\ \ell_E^{(1)}(e).$$

Verification. *(S) In* $e_1 = \{a, b, c\}$ *we have* (R, G, B) *distinct; in* $e_2 = \{b, c, d\}$ *we have* (G, B, G) not *all distinct. Thus the strong constraint would fail on* e_2*. Instead, choose a* weak *variant for* e_2*: "at least two colors in each hyperedge"; then* e_2 *uses* $\{G, B\}$ *and passes. (C)* e_1: $1 + 2 + 1 = 4 \leq 5$; e_2: $2 + 1 + 3 = 6 \leq 7$. *Hence with weak hyperedge coloring the tuple* $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ *satisfies* Φ.

Example 12 (Distance–aware MultiLabeling (an $L(2,1)$–type coordinate) plus edge aggregation)**.** *Let* $H = (V, E)$ *with* $V = \{x_1, x_2, x_3, x_4\}$ *and*

$$E = \big\{\{x_1, x_2, x_3\},\ \{x_2, x_3, x_4\}\big\}.$$

Its primal graph $G(H)$ *has edges* $\{x_1, x_2\}$, $\{x_1, x_3\}$, $\{x_2, x_3\}$, $\{x_2, x_4\}$, $\{x_3, x_4\}$. *Choose* $p = 2$, $q = 1$ *with*

$$L_V^{(1)} = \mathbb{Z} \quad (frequency\ channel), \qquad L_V^{(2)} = \{0, 1\} \quad (role), \qquad L_E^{(1)} = \mathbb{Z}_{\geq 0} \quad (edge\ cost).$$

Define

$$\ell_V^{(1)}(x_1) = 0,\ \ell_V^{(1)}(x_2) = 2,\ \ell_V^{(1)}(x_3) = 5,\ \ell_V^{(1)}(x_4) = 7,$$
$$\ell_V^{(2)}(x_1) = 0,\ \ell_V^{(2)}(x_2) = 1,\ \ell_V^{(2)}(x_3) = 0,\ \ell_V^{(2)}(x_4) = 1,$$
$$\ell_E^{(1)}(\{x_1, x_2, x_3\}) = 5, \qquad \ell_E^{(1)}(\{x_2, x_3, x_4\}) = 7.$$

Schema Φ *requires:*

$(L(2,1))$ *coordinate* 1 *(channels): for distinct* $u, v \in V$,

$$\mathrm{dist}_H(u, v) = 1 \Rightarrow |\ell_V^{(1)}(u) - \ell_V^{(1)}(v)| \geq 2, \quad \mathrm{dist}_H(u, v) = 2 \Rightarrow |\ell_V^{(1)}(u) - \ell_V^{(1)}(v)| \geq 1.$$

(R) *On coordinate* 2 *(roles): each hyperedge contains both roles* 0 *and* 1.

(A) Edge aggregation*: for each* $e \in E$,

$$\ell_E^{(1)}(e) = \max\{\ell_V^{(1)}(v) : v \in e\}.$$

Verification. *Distances are taken in* $G(H)$*, which is the* K_4 *minus edges* $\{x_1, x_4\}$*; hence* $\mathrm{dist}_H(x_1, x_4) = 2$. *All adjacent pairs satisfy channel gaps:* $|0 - 2| = 2$, $|0 - 5| = 5$, $|2 - 5| = 3$, $|2 - 7| = 5$, $|5 - 7| = 2$. *Distance* 2 *pair* (x_1, x_4) *has* $|0 - 7| = 7 \geq 1$. *(R) Each hyperedge* $\{x_1, x_2, x_3\}$ *and* $\{x_2, x_3, x_4\}$ *contains roles* $\{0, 1\}$. *(A) For*

$\{x_1, x_2, x_3\}$ *the max channel is* $\max\{0,2,5\} = 5$*, matching* $\ell_E^{(1)} = 5$*; for* $\{x_2, x_3, x_4\}$ *the max is* $\max\{2,5,7\} = 7$*, matching* $\ell_E^{(1)} = 7$*. Thus* $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ *satisfies* Φ*.*

Theorem 4 (HyperGraph MultiLabeling generalizes hypergraph labeling and graph MultiLabeling)**.** *Let* (H, Ψ) *be any (single–coordinate) hypergraph labeling problem, with* $\Psi(H, \lambda)$ *a predicate over* $\lambda : V \to L$ *(or* $\lambda : E \to L$*) using only the hypergraph structure (incidence and/or* dist_H*) and relations on* L*. Let* (G, Φ_{GML}) *be any Graph MultiLabeling instance on a simple graph* $G = (V, E_G)$ *with* p *vertex and* q *edge coordinates.*

Then:

1. ***(Hypergraph labeling*** $\subseteq$ ***HyperGraph MultiLabeling)*** *There exists a schema* Φ *with either* $(p, q) = (1, 0)$ *(vertex case) or* $(p, q) = (0, 1)$ *(edge case) such that*
$$\exists \lambda \text{ with } \Psi(H, \lambda) \iff \exists (\boldsymbol{\ell}_V, \boldsymbol{\ell}_E) \text{ with } \Phi(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E),$$
where $\ell_V^{(1)} = \lambda$ *(or* $\ell_E^{(1)} = \lambda$*).*
2. ***(Graph MultiLabeling*** $\subseteq$ ***HyperGraph MultiLabeling)*** *There exists a hypergraph* $\widehat{H} = (V, \widehat{E})$ *with* $\widehat{E} := \{\{u, v\} \subseteq V : uv \in E_G\}$ *and a schema* $\widehat{\Phi}$ *such that*
$$\exists (\boldsymbol{\ell}_V, \boldsymbol{\ell}_E) \text{ with } \Phi_{\mathrm{GML}}(G; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E) \iff \exists (\boldsymbol{\ell}_V, \boldsymbol{\ell}_E) \text{ with } \widehat{\Phi}(\widehat{H}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E).$$

Hence HyperGraph MultiLabeling strictly contains both classical hypergraph labeling and graph MultiLabeling.

Proof. (1) Vertex case. Take $p = 1, q = 0$, set $L_V^{(1)} := L$, and define $\Phi(H; (\ell_V^{(1)}), \varnothing) := \Psi(H, \ell_V^{(1)})$. Then $\Psi(H, \lambda)$ holds iff $\Phi(H; (\lambda), \varnothing)$ holds; existence is equivalent. Edge case is identical with $p = 0, q = 1$ and $L_E^{(1)} := L$.

(2) Let $\widehat{H} = (V, \widehat{E})$ be the 2–uniform hypergraph obtained from G by setting one hyperedge for each graph edge. Then $G(\widehat{H}) = G$, hence $\mathrm{dist}_{\widehat{H}} = \mathrm{dist}_G$. Define $\widehat{\Phi}(\widehat{H}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E) := \Phi_{\mathrm{GML}}(G(\widehat{H}); \boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$. Any constraint in Φ_{GML} that refers to adjacency/distances or incidence in G is identically interpreted in $\widehat{H}$ via $G(\widehat{H}) = G$, so the two existence statements are equivalent. □

2.5. SuperHyperGraph MultiLabeling

SuperHyperGraph MultiLabeling assigns multiple labels to supervertices and superedges, generalizing both hypergraph multilabeling and superhypergraph labeling for hierarchical multi-dimensional applications.

Definition 13 (Primal (2–section) of a SuperHyperGraph and distance)**.** *Let* $\mathrm{SHG}^{(n)} = (V, E)$ *be an n–SuperHyperGraph, i.e.* $V \subseteq \mathcal{P}^n(V_0)$ *and* $\varnothing \neq E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}$*. Its* primal graph *(or 2–section) is*
$$G\Big(\mathrm{SHG}^{(n)}\Big) := (V, E') \quad \text{with} \quad E' := \Big\{\{u, v\} \subseteq V \;\Big|\; \exists F \in E : \{u, v\} \subseteq F\Big\}.$$
The vertex distance $\mathrm{dist}_{\mathrm{SHG}}(x, y)$ *is the usual shortest–path distance between* $x, y \in V$ *computed in* $G\Big(\mathrm{SHG}^{(n)}\Big)$*.*

Definition 14 (Base support (flattening))**.** *For* $n \geq 0$ *define recursively a map* $\mathrm{flat}_n : \mathcal{P}^n(V_0) \to \mathcal{P}(V_0)$ *by*
$$\mathrm{flat}_0(v) := \{v\} \;\; (v \in V_0), \qquad \mathrm{flat}_{n+1}(X) := \bigcup_{Y \in X} \mathrm{flat}_n(Y) \;\; (X \in \mathcal{P}^{n+1}(V_0)).$$
Thus, for a supervertex $X \in V \subseteq \mathcal{P}^n(V_0)$*, the set* $\mathrm{flat}_n(X) \subseteq V_0$ *collects all base elements of* V_0 *that occur anywhere inside* X*.*

 doi:10.20944/preprints202509.1593.v1

Definition 15 (SuperHyperGraph MultiLabeling)**.** *Fix integers $p, q \geq 0$ and an n–SuperHyperGraph* $\mathrm{SHG}^{(n)} = (V, E)$*. Choose nonempty* vertex label alphabets $L_V^{(1)}, \ldots, L_V^{(p)}$ *and nonempty* edge label alphabets $L_E^{(1)}, \ldots, L_E^{(q)}$*. A* SuperHyperGraph MultiLabeling *consists of*

$$\boldsymbol{\ell}_V = (\ell_V^{(1)}, \ldots, \ell_V^{(p)}), \quad \ell_V^{(a)} : V \to L_V^{(a)} \quad (1 \leq a \leq p),$$

$$\boldsymbol{\ell}_E = (\ell_E^{(1)}, \ldots, \ell_E^{(q)}), \quad \ell_E^{(b)} : E \to L_E^{(b)} \quad (1 \leq b \leq q).$$

Equivalently, a single vertex map $\ell_V : V \to \prod_{a=1}^{p} L_V^{(a)}$ and a single edge map $\ell_E : E \to \prod_{b=1}^{q} L_E^{(b)}$. A schema *is a first–order predicate*

$$\Phi\Big(\mathrm{SHG}^{(n)}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\Big)$$

built from the incidence relation $v \in F$ ($v \in V$, $F \in E$), the distance $\mathrm{dist}_{\mathrm{SHG}}$*, the operators* flat_n*, basic set–theoretic/cardinality operations on $\mathcal{P}(V_0)$, and given relations/operations on the alphabets. The pair $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ is a* valid SuperHyperGraph MultiLabeling (for Φ) *if $\Phi\Big(\mathrm{SHG}^{(n)}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\Big)$ holds.*

Remark 5 (Typical constraints that Φ may express)**.**
- ***Distance–aware separation on a vertex coordinate** a: if $\mathrm{dist}_{\mathrm{SHG}}(x, y) = 1$ then $|\ell_V^{(a)}(x) - \ell_V^{(a)}(y)| \geq \lambda_1$, and if $\mathrm{dist}_{\mathrm{SHG}}(x, y) = 2$ then $|\ell_V^{(a)}(x) - \ell_V^{(a)}(y)| \geq \lambda_2$.*
- ***Support cardinality on a vertex coordinate** a: $\ell_V^{(a)}(X) = \big|\mathrm{flat}_n(X)\big|$ for all $X \in V$.*
- ***Superedge aggregation on an edge coordinate** b: $\ell_E^{(b)}(F) = g\big(\{\mathrm{flat}_n(X) : X \in F\}\big)$ for a fixed aggregator g (e.g. union size, intersection size, Jaccard index, maximum of a vertex coordinate, etc.).*

Example 13 (A $(p, q) = (2, 1)$ multilabel on a level-$n = 1$ SuperHyperGraph)**.** *Let $V_0 = \{a, b, c, d\}$. Consider $n = 1$ so supervertices are subsets of V_0. Let*

$$V = \big\{X_1, X_2, X_3\big\} := \big\{\{a, b\},\ \{b, c\},\ \{c, d\}\big\}, \qquad E = \big\{F_1, F_2\big\} := \big\{\{X_1, X_2\},\ \{X_2, X_3\}\big\}.$$

Hence $G\Big(\mathrm{SHG}^{(1)}\Big)$ is the path X_1–X_2–X_3. Choose alphabets

$$L_V^{(1)} = \{R, G, B\} \quad (\textit{colors}), \qquad L_V^{(2)} = \mathbb{N} \quad (\textit{support size}), \qquad L_E^{(1)} = \mathbb{N} \quad (\textit{coverage size}).$$

Define the labels

$$\ell_V^{(1)}(X_1) = R,\ \ell_V^{(1)}(X_2) = G,\ \ell_V^{(1)}(X_3) = B, \quad \ell_V^{(2)}(X_i) = \big|\mathrm{flat}_1(X_i)\big| = |X_i| = 2 \ \ (i = 1, 2, 3),$$

$$\ell_E^{(1)}(F_1) = \big|\mathrm{flat}_1(X_1) \cup \mathrm{flat}_1(X_2)\big| = \big|\{a, b\} \cup \{b, c\}\big| = 3,$$
$$\ell_E^{(1)}(F_2) = \big|\mathrm{flat}_1(X_2) \cup \mathrm{flat}_1(X_3)\big| = \big|\{b, c\} \cup \{c, d\}\big| = 3.$$

Schema Φ imposes simultaneously:

(Col)*Adjacent supervertices receive different colors on coordinate* 1.
(Sup)*For all $X \in V$, $\ell_V^{(2)}(X) = \big|\mathrm{flat}_1(X)\big|$.*
(Agg)*For all $F \in E$, $\ell_E^{(1)}(F) = \big|\bigcup_{X \in F} \mathrm{flat}_1(X)\big|$.*
The given $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ satisfies (Col), (Sup), and (Agg), hence it is a valid SuperHyperGraph MultiLabeling.

Example 14 (A distance–aware multilabel on a level-$n = 2$ SuperHyperGraph)**.** *Let $V_0 = \{1, 2, 3\}$ and $n = 2$. Define supervertices*

$$A := \big\{\{1\}, \{1, 2\}\big\}, \qquad B := \big\{\{2\}, \{2, 3\}\big\} \ \ \in \mathcal{P}^2(V_0),$$

 doi:10.20944/preprints202509.1593.v1

and set $V = \{A, B\}$, $E = \big\{\{A, B\}\big\}$. *Then*

$$\mathrm{flat}_2(A) = \{1, 2\}, \qquad \mathrm{flat}_2(B) = \{2, 3\}.$$

Choose alphabets $L_V^{(1)} = \mathbb{Z}$ *(channels),* $L_V^{(2)} = \mathbb{N}$ *(support size),* $L_E^{(1)} = \mathbb{N}$ *(overlap size). Define*

$$\ell_V^{(1)}(A) = 0,\ \ \ell_V^{(1)}(B) = 3 \quad (gap\ 3), \qquad \ell_V^{(2)}(A) = 2,\ \ \ell_V^{(2)}(B) = 2,$$

$$\ell_E^{(1)}(\{A, B\}) = \big|\mathrm{flat}_2(A) \cap \mathrm{flat}_2(B)\big| = \big|\{1, 2\} \cap \{2, 3\}\big| = 1.$$

Let Φ *require:*

$L(2, 1)$*On coordinate* 1*, if* $\mathrm{dist}_{\mathrm{SHG}}(X, Y) = 1$ *then* $|\ell_V^{(1)}(X) - \ell_V^{(1)}(Y)| \geq 2$ *(here* $|0 - 3| = 3 \geq 2$*).*
(Sup)$\ell_V^{(2)}(X) = \big|\mathrm{flat}_2(X)\big|$ *for all* $X \in V$ *(true:* 2 *and* 2*).*
(Int) $\ell_E^{(1)}(F) = \big|\bigcap_{X \in F} \mathrm{flat}_2(X)\big|$ *for all* $F \in E$ *(true:* 1*).*
Thus $(\boldsymbol{\ell}_V, \boldsymbol{\ell}_E)$ *satisfies* Φ *and is a valid multilabel.*

Theorem 5 (SuperHyperGraph MultiLabeling generalizes SuperHyperGraph Labeling and HyperGraph MultiLabeling)**.** *The framework in Definition 15 strictly contains:*

(i) SuperHyperGraph Labeling *(single–coordinate labeling on supervertices and/or superedges);*
(ii) HyperGraph MultiLabeling *(multi–coordinate labeling on ordinary hypergraphs).*

Proof. (i) Let a SuperHyperGraph labeling be given as a single map $\lambda_V : V \to L$ (vertex case) or $\lambda_E : E \to L$ (edge case), together with a predicate Ψ that uses only incidence, the primal distance and allowed relations on L. Take $p = 1$, $q = 0$ (vertex case) with $L_V^{(1)} := L$, and set

$$\Phi\Big(\mathrm{SHG}^{(n)};\ (\ell_V^{(1)}),\ \varnothing\Big) \ :=\ \Psi\Big(\mathrm{SHG}^{(n)};\ \ell_V^{(1)}\Big).$$

Then λ_V satisfies Ψ iff $(\ell_V^{(1)}, \varnothing)$ with $\ell_V^{(1)} = \lambda_V$ satisfies Φ. The edge case is identical with $p = 0$, $q = 1$.

(ii) Let $H = (V, E)$ be a (finite) hypergraph and consider any HyperGraph MultiLabeling instance on H (with p vertex and q edge coordinates and a schema Φ_{HG} based on the hypergraph incidence/distance). Realize H as an $n = 0$ SuperHyperGraph by taking $V_0 := V$ and $\mathrm{SHG}^{(0)} := (V, E)$; then $G\Big(\mathrm{SHG}^{(0)}\Big)$ coincides with the primal of H, so the same adjacency/distance is available. Define

$$\Phi\Big(\mathrm{SHG}^{(0)}; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E\Big) \ :=\ \Phi_{\mathrm{HG}}(H; \boldsymbol{\ell}_V, \boldsymbol{\ell}_E).$$

Thus every feasible HyperGraph MultiLabeling on H is a feasible SuperHyperGraph MultiLabeling on $\mathrm{SHG}^{(0)}$, and conversely. Therefore the latter generalizes the former. □

3. Conclusion

In this paper, we defined and study the mathematical properties of *Graph Labeling*, *HyperGraph Labeling*, *SuperHyperGraph Labeling*, *Graph MultiLabeling*, *HyperGraph MultiLabeling*, and *SuperHyperGraph MultiLabeling*. We anticipate that future work may explore extensions employing *Fuzzy Sets*[33,34], *Intuitionistic Fuzzy Sets*[35,36], *Neutrosophic Sets*[17,37,38], Picture Fuzzy Sets [39,40], HyperFuzzy Sets [41–43], and *Plithogenic Sets*[44,45].

Research Integrity

The author confirms that this manuscript is original, has not been published elsewhere, and is not under consideration by any other journal.

Use of Computational Tools

All proofs and derivations were performed manually; no computational software (e.g., Mathematica, SageMath, Coq) was used.

Code Availability

No code or software was developed for this study.

Ethical Approval

This research did not involve human participants or animals, and therefore did not require ethical approval.

Use of Generative AI and AI-Assisted Tools

We use generative AI and AI-assisted tools for tasks such as English grammar checking, and We do not employ them in any way that violates ethical standards.

Supplementary Information

No supplementary materials accompany this paper.

Disclaimer

The ideas presented here are theoretical and have not yet been validated through empirical testing. While we have strived for accuracy and proper citation, inadvertent errors may remain. Readers should verify any referenced material independently. The opinions expressed are those of the authors and do not necessarily reflect the views of their institutions.

Funding: No external funding was received for this work.

Data Availability Statement: This paper is theoretical and did not generate or analyze any empirical data. We welcome future studies that apply and test these concepts in practical settings.

Acknowledgments: We thank all colleagues, reviewers, and readers whose comments and questions have greatly improved this manuscript. We are also grateful to the authors of the works cited herein for providing the theoretical foundations that underpin our study. Finally, we appreciate the institutional and technical support that enabled this research.

Conflicts of Interest: The authors declare no conflicts of interest regarding the publication of this work.

References

1. Jonathan L Gross, Jay Yellen, and Mark Anderson. *Graph theory and its applications*. Chapman and Hall/CRC, 2018.
2. Reinhard Diestel. *Graph theory*. Springer (print edition); Reinhard Diestel (eBooks), 2024.
3. Joseph A. Gallian. A dynamic survey of graph labeling. *The Electronic Journal of Combinatorics*, 2009.
4. Joseph A Gallian. Graph labeling. *The electronic journal of combinatorics*, pages DS6–Dec, 2012.
5. Joseph A Gallian. A dynamic survey of graph labeling. *Electronic Journal of combinatorics*, 6(25):4–623, 2022.
6. Jan Van den Heuvel, Robert A Leese, and Mark A Shepherd. Graph labeling and radio channel assignment. *Journal of Graph Theory*, 29(4):263–283, 1998.
7. Yue Gao, Zizhao Zhang, Haojie Lin, Xibin Zhao, Shaoyi Du, and Changqing Zou. Hypergraph learning: Methods and practices. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 44(5):2548–2566, 2020.
8. Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, volume 33, pages 3558–3565, 2019.
9. Bibin K Jose and Zsolt Tuza. Hypergraph domination and strong independence. *Applicable Analysis and Discrete Mathematics*, 3(2):347–358, 2009.
10. Eduardo Martín Campoverde Valencia, Jessica Paola Chuisaca Vásquez, and Francisco Ángel Becerra Lois. Multineutrosophic analysis of the relationship between survival and business growth in the manufacturing

sector of azuay province, 2020–2023, using plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:341–355, 2025.
11. Julio Cesar Méndez Bravo, Claudia Jeaneth Bolanos Piedrahita, Manuel Alberto Méndez Bravo, and Luis Manuel Pilacuan-Bonete. Integrating smed and industry 4.0 to optimize processes with plithogenic n-superhypergraphs. *Neutrosophic Sets and Systems*, 84:328–340, 2025.
12. Takaaki Fujita. Multi-superhypergraph neural networks: A generalization of multi-hypergraph neural networks. *Neutrosophic Computing and Machine Learning*, 39:328–347, 2025.
13. Yenson Vinicio Mogro Cepeda, Marco Antonio Riofrío Guevara, Emerson Javier Jácome Mogro, and Rachele Piovanelli Tizano. Impact of irrigation water technification on seven directories of the san juan-patoa river using plithogenic n-superhypergraphs based on environmental indicators in the canton of pujilí, 2021. *Neutrosophic Sets and Systems*, 74(1):6, 2024.
14. Takaaki Fujita. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*. Biblio Publishing, 2025.
15. Mohammad Hamidi and Mohadeseh Taghinezhad. *Application of Superhypergraphs-Based Domination Number in Real World*. Infinite Study, 2023.
16. Berrocal Villegas Salomón Marcos, Montalvo Fritas Willner, Berrocal Villegas Carmen Rosa, Flores Fuentes Rivera María Yissel, Espejo Rivera Roberto, Laura Daysi Bautista Puma, and Dante Manuel Macazana Fernández. Using plithogenic n-superhypergraphs to assess the degree of relationship between information skills and digital competencies. *Neutrosophic Sets and Systems*, 84:513–524, 2025.
17. Shouxian Zhu. Neutrosophic n-superhypernetwork: A new approach for evaluating short video communication effectiveness in media convergence. *Neutrosophic Sets and Systems*, 85:1004–1017, 2025.
18. Yasar Nacaroglu, Nihat Akgunes, Sedat Pak, and I Naci Cangul. Some graph parameters of power set graphs. *Advances & Applications in Discrete Mathematics*, 26(2), 2021.
19. MA Shalu and S Devi Yamini. Counting maximal independent sets in power set graphs. *Indian Institute of Information Technology Design & Manufacturing (IIITD&M) Kancheepuram, India*, 2014.
20. Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.
21. Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.
22. Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.
23. Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.
24. Huda E Khali, Gonca D GÜNGÖR, and Muslim A Noah Zaina. Neutrosophic superhyper bi-topological spaces: Original notions and new insights. *Neutrosophic Sets and Systems*, 51(1):3, 2022.
25. Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.
26. Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.
27. Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.
28. Toufiq Parag and Ahmed Elgammal. Supervised hypergraph labeling. In *CVPR 2011*, pages 2289–2296. IEEE, 2011.
29. Maryam Mirzakhani and Jan Vondrák. Sperner's colorings, hypergraph labeling problems and fair division. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 873–886. SIAM, 2014.
30. Avah Banerjee, Maxwell Reeser, and Guoli Ding. Distributed matrix tiling using a hypergraph labeling formulation. In *Proceedings of the 23rd International Conference on Distributed Computing and Networking*, pages 62–71, 2022.
31. Zizhou ZHANG and Shaohua ZHANG. On the coprime labelings of hypergraph. *Wuhan University Journal of Natural Sciences*, 30(1):57–59, 2025.
32. Michał Tuczyński, Przemysław Wenus, and Krzysztof Wkesek. On cordial labeling of hypertrees. *Discrete Mathematics & Theoretical Computer Science*, 21(Graph Theory), 2019.
33. Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.
34. John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.
35. Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

 doi:10.20944/preprints202509.1593.v1

36. Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.
37. Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.
38. Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, (10):86–101, 2016.
39. Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.
40. Bui Cong Cuong and Vladik Kreinovich. Picture fuzzy sets-a new concept for computational intelligence problems. In *2013 third world congress on information and communication technologies (WICT 2013)*, pages 1–6. IEEE, 2013.
41. Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.
42. Young Bae Jun, Seok-Zun Song, and Seon Jeong Kim. Distances between hyper structures and length fuzzy ideals of bck/bci-algebras based on hyper structures. *Journal of Intelligent & Fuzzy Systems*, 35(2):2257–2268, 2018.
43. Yong Lin Liu, Hee Sik Kim, and J. Neggers. Hyperfuzzy subsets and subgroupoids. *J. Intell. Fuzzy Syst.*, 33:1553–1562, 2017.
44. Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.
45. Muhammad Azeem, Humera Rashid, Muhammad Kamran Jamil, Selma Gütmen, and Erfan Babaee Tirkolaee. Plithogenic fuzzy graph: A study of fundamental properties and potential applications. *Journal of Dynamics and Games*, pages 0–0, 2024.