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Article

A General Approach to Exact Polynomial Solutions: Demonstrations for Quintic and Sextic Equations and a Higher-Degree Conjecture

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Abstract

A novel decomposition of high-degree polynomials is presented in this paper by reformulating polynomial solving as a matrix decomposition problem, leveraging linear algebra to retrieve exact solutions whereas current methods rely on approximations. This approach is demonstrated on subfamilies of quintic and sextic polynomials, supported by numerical examples. The full matrix decompositions for each subfamily are provided in the appendix, from which the solutions naturally arise. This framework also suggests a conjecture for polynomials of degree nine and higher, thus outlining conditions for exact solvability and inviting further exploration.

Keywords: quintic equations; sextic equations; exact solutions; polynomial solving; linear algebra

MSC: 12D05; 12F10; 15A18; 11R32; 11C08

1. Introduction

Solving the polynomials through closed form specifically radicals is a problem in algebra with deep history [1]. Mathematicians even from the time of the Renaissance have been dealing with the challenge of finding closed form solutions to polynomials of increasing degree. Solving polynomials by radicals in the case of the quadratic, cubic, and quartic polynomials were discovered during the 16th century by mathematicians Scipione del Ferro, Niccolo Tartaglia, and Lodovico Ferrari [2]. But the general quintic doesn't yield to these techniques which led to the question of whether or not they are solvable by radicals in the first place or by extension, higher degree polynomials. The unsolvability of quintics was proved by Paolo Ruffini [3] partially at the start and Niels Abel completed the proof which concluded in the Abel-Ruffini theorem [4,5]. Evariste Galois was credited with providing clarity to their work that general polynomials of degree 5 and above cannot be solved with radicals in the 19th century. He demonstrated this with his own creation, Galois theory. Galois showed that the solvability of a polynomial in terms of its radicals is linked to the structure of its associated Galois group which is a group of symmetries over the roots of the polynomial in question [6]. If a polynomial can be solved with radicals, its Galois group must be a solvable group but with polynomials of degree 5 or greater, the Galois group is usually not solvable thus there is no solution by radicals [7,8]. This groundbreaking result established the unsolvability of the quintic polynomial with radicals thus creating a foundation for modern algebra.

Despite this result, particular cases of quintic and high degree polynomials are shown to have algebraic solutions although the general form does not. These cases go back to the fact that specific symmetries within the polynomial lead to solvable Galois groups. An example is that one can create quintics that have cyclic or abelian Galois groups. Klein's study on the Icosahedral group shows this link of symmetry and solvability where he found that restricting symmetry leads to solvable cases

[9] with future mathematicians building on top of it to demonstrate particular cases of high degree polynomials that are solvable [10].

In this paper, two major objectives are pursued. The first is to identify specific families for quintic and sextic polynomials that have algebraic solutions despite unsolvability for their general cases. Our approach leverages linear algebra methods to reduce the polynomial so that it is equivalent to solving a lower degree polynomial that has half of its degree. The second objective of this paper is to propose a conjecture regarding polynomials of degree 9 and higher where conditions are hypothesized that indicate solvability of these high degree polynomials. Our conjecture states that structured subfamilies of degree 9 and beyond may admit algebraic solutions under specific constraints. This proposition challenges the limitations set by Galois theory and provides a landscape for solvability of higher degrees with potential consequences for algebraic theory [11] and number theory [12]. To summarize, this work contributes to the ongoing exploration of polynomial solvability by:

1. Identifying specific families of solvable quintic and sextic polynomials showcasing where algebraic solutions exist despite limitations on the general form. After determining solvability conditions of each family, their matrix decompositions are retrieved and included in the appendix.

2. Formulating a conjecture for degree 9 and higher polynomials, which expand into boundaries of current solvability theories and provides a new road for research regarding algebraic solutions for polynomials.

3. Offering a framework to solve quintic and sextic polynomials which have rational roots with potential applications to higher degree polynomials backed by numerical examples. This highlights potential for implementing software based on the framework.

The rest of this paper is organized as follows: Section 2 provides related work on high degree polynomials such as quintics, sextics, and those with degrees higher degrees. Section 3 outlines the methodology used to identify algebraic solvable cases for both quintics and sextics as well as formalizing the general case. Sections 4 and 5 presents our proofs for specific cases, while Section 6 includes computational examples for completeness, and Section 7 introduces our conjecture for higher degrees. The paper concludes with a summary of the work with possible future research directions.

2. Related Work

2.1. Solvability and Galois Theory

In algebra, a polynomial is solvable by radicals if its roots can be expressed using a finite number of arithmetic operations such as additions, subtractions, multiplications, divisions and roots on the coefficients of the polynomial. Galois theory shows the required conditions for existence of algebraic solutions by analyzing symmetries of the roots of the polynomial. Galois produced the criterion of solvability by stating that a polynomial is solvable by radicals if and only if its associated Galois group is also a solvable group [7,8]. This links solvability to the group structure of permutations of the polynomial's roots: the Galois group. For quartic polynomials or lower, their Galois groups are always solvable which explains why they have solutions in terms of radicals. But Quintic polynomials or those of higher degree are usually non-solvable. Formally, symmetric group S_n for $n \geq 5$ is non-solvable and the general polynomial of degree n has S_n as its Galois group, thus it is unsolvable by radicals in the general case. This result is the Abel-Ruffini theorem which shows that there is no general formula in terms of radicals solving polynomials of degree 5 or higher [10,13]. While the general cases for polynomials of degree 5 and above lack algebraic solutions, certain particular cases have solutions. These cases of polynomial demonstrate special structural properties of their Galois group which arise from the fact that their symmetries reduce the complexity of their root arrangements. An example of this is a quintic polynomial whose Galois groups are abelian or cyclic which in consequence can be solved by radicals. An abelian group is one where all elements commute while a cyclic group is a group generated by a single element. Both are examples of solvable groups, making it possible for their associated polynomials to be solvable in terms of radicals [9].

An example of one of these solvable groups is the Icosahedral symmetry group, while usually non-solvable, specific configurations of the group lead to solvable cases for some quintics. Felix Klein explored this in his lectures of the Icosahedron [9], subsequent research has focused on finding explicit criteria for said specific cases. This can be by examining particular types of polynomials where the Galois group does not show the full complexity usually associated with its degree. This approach has led to numerous studies on specific classes of polynomials and their algebraic solutions [11,14].

2.2. Results for Degrees 5 and 6

For quintics, one line of research is finding structured forms of quintic polynomials which have solvable Galois groups. In these cases, the polynomial demonstrates special symmetry or factorization properties that reduced the associated Galois group to a cyclic or abelian form. Some forms commonly studied include cyclic quintic polynomials where the Galois group is cyclic of order 5, and abelian quintic polynomials which have a Galois group that is a direct product of cyclic groups. The order of a cyclic group is the number of elements in that group. Sextic polynomials present a similar complexity but allow for structured cases due to their composite nature. Take for example certain sextic polynomials which factor as a product of quadratics and cubics or other specific forms that lead to its associated Galois group being solvable. The analysis of sextics often involves techniques from Galois theory like descent, where one reduces the problem to simpler lower-degree polynomial factors. Studies have shown sextic polynomials with solvable Galois group can often be transformed through factorization or symmetrization [13]. In contrast to the theory, the scientific community is interested in retrieving the actual roots of the quintics and high degree polynomials albeit the theoretical limitations [15,16]. Nowadays, matrix methods are popular for calculating roots of arbitrary polynomials by approximation which is by retrieving the eigenvalues of the associated companion and generalized companion matrices. An example of this is the QR algorithm for retrieving eigenvalues and eigenvectors. Weierstrass's polynomial solving algorithm makes use of the QR algorithm along with some modifications, it is noted that the results are obtained by approximation. Further treatments of applying the QR algorithm in the context of polynomials or in general should be referenced in [17–22].

As solving quintic polynomials are a special case of solving polynomials of arbitrary degree, it can be seen that the standard approach for quintic polynomials is through solving with the same numerical methods discussed providing approximate solutions. A current well known algorithm for computing solutions to the quintic polynomials is that of King and Canfield [23] which is a modification Kiepert's algorithm [24]. Their approach involves Tschirnhaus transformations [25] which were first demonstrated by Kiepert to transform the quintic polynomial into the the Jacobi sextic form of $s^6 + \frac{10}{\Delta}s^3 - \frac{12g_2}{\Delta^2}s + \frac{5}{\Delta^2} = 0$ which are then solved using techniques from elliptic curves which involve a pair of infinite series. It is interesting to note that the Bring-Jerrard form $x^5 + ax + b$ is commonly used when approaching quintic polynomials as it can be shown that using the resultant, any quintic polynomial can be reduced to its Bring-Jerrard form [26–28]. For the case of the sextic polynomial, there are schemes to solve them approximately but they make use of advanced techniques such as complex geometry [29] and Kampé de Fériet functions defined by infinite series [30,31]. It can be seen that there is more to be desired with these methods as it would greatly benefit the mathematical community if there was a framework that both provides exact solutions and is very simple to implement.

2.3. Works on Degrees 7, 8, 9 and Higher

Similarly to quintics and sextics, there are indications that specific subfamilies of higher degree polynomials have algebraic solutions for specific cases. Recent work in algebraic theory suggests that increasing degrees of polynomials also lead to increasing opportunities to classify solvable polynomials based on structured Galois group with additional symmetry or decomposition properties [10,11]. There has been work on solving solvable septics through the use of decomposition methods into lower degree polynomials as in [32] or by relating the coefficients of the septic to its Galois group structure [33]. While for the case of octic polynomials, there is work presenting parameterizations with quintic and

sixth polynomials, thus making them easier to solve through employing six arbitrary parameters, this also implies that solvable cases can be found through explicit radicals [34]. Decomposition techniques similarly to the septic case are used to solve the octics [35], and interestingly, symmetry-breaking methods are explored in six-dimensional projective spaces, hinting at generalizations for even higher degrees [36]. This paper proposes a conjecture that polynomials of degree 9 and higher have specific cases which are solvable. This conjecture builds on patterns observed with lower degree polynomials and leverages the ideas developed in the paper which may provide a new frontier for solvability in higher dimensional representations.

3. Methodology

The central concept in this work involves utilizing the matrix representation of a polynomial to facilitate its decomposition into a pair of lower-degree polynomials, thereby simplifying its solution process. This decomposition requires that the polynomial's matrix representation have rank 2, a property which will be formalized later as a conjecture. To establish the framework, we first revisit the generalized theory from [37], as it provides the theoretical foundation upon which this paper builds. The discussion begins with the general case, followed by detailed analyses of the quintic and sextic cases, which lead into two supporting lemmas.

3.1. General Case

An n degree polynomial can be expressed in terms of a matrix product, a formulation that provides a powerful algebraic tool for analyzing its structure and properties. This representation generalizes techniques from the previous subsection and is expressed as follows:

$$P_n(x) = \sum_{i=0}^n b_n x^n = \begin{bmatrix} x^{\lfloor \frac{n}{2} \rfloor} \\ x^{\lfloor \frac{n}{2} \rfloor - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}^T \mathbf{A}_n \begin{bmatrix} x^{\lfloor \frac{n}{2} \rfloor} \\ x^{\lfloor \frac{n}{2} \rfloor - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}$$

Where \mathbf{A}_n is a symmetric matrix and the sum of anti-diagonals equals the coefficient of the polynomial $P_n(x)$, i.e., $b_{n+2-k} = \sum_{i+j=k} a_{i,j}$ for $k \in \{0, 1, \dots, n\}$. It is also noted the leading coefficient satisfies $b_n = 1$. We now we demonstrate the following Lemma,

Lemma 1. *The matrix \mathbf{A}_n is not unique; different ways of distributing the values of each coefficient on its corresponding anti-diagonal are possible.*

The zero polynomial can be expressed as a matrix equation, which ensures that the sum of elements along each anti-diagonal vanishes, maintaining the characteristic nullification of all its coefficients. The equation is as follows:

$$0 = \begin{bmatrix} x^{\lfloor \frac{n}{2} \rfloor} \\ x^{\lfloor \frac{n}{2} \rfloor - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}^T \mathbf{A}_n^0 \begin{bmatrix} x^{\lfloor \frac{n}{2} \rfloor} \\ x^{\lfloor \frac{n}{2} \rfloor - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}$$

Where the sum of anti-diagonals is null, i.e., $\forall k \in \{0, 1, \dots, n\} : \sum_{i+j=k} a_{i,j} = 0$. There are many ways to define explicitly the matrix \mathbf{A}_n^0 , one of them is as follows:

$$a_{i,j}^0 = \begin{cases} 0 & \text{if } i+j \leq 3 \text{ or } i+j \geq 2\lceil \frac{n}{2} \rceil + 1 \\ t_{i+j-3} & \text{if } i+j \text{ is even and } i \neq j \\ -2 \min(\lceil \frac{n}{2} \rceil - i, i-1) t_{i+j-3} & \text{if } i+j \text{ is even and } i = j \\ (-1)^{i+1} t_{i+j-3} & \text{if } i+j \text{ is odd and } \frac{i+j-1}{2} \text{ is even, } i \leq \lceil \frac{n}{2} \rceil + 1 \\ \left(-\frac{i+j-3}{2}\right)^{\delta(i-j+1)} t_{i+j-3} & \text{if } i+j \text{ is odd and } \frac{i+j-1}{2} \text{ are odd, } i \leq \lceil \frac{n}{2} \rceil + 1 \\ (-1)^{i+1} t_{i+j-3} & \text{if } i+j \text{ is odd and } \frac{2\lceil \frac{n}{2} \rceil - i - j - 1}{2} \text{ is even, } i > \lceil \frac{n}{2} \rceil + 1 \\ \left(-\frac{i+j-3}{2}\right)^{\delta(i-j+1)} t_{i+j-3} & \text{if } i+j \text{ and } \frac{2\lceil \frac{n}{2} \rceil - i - j - 1}{2} \text{ are odd, } i > \lceil \frac{n}{2} \rceil + 1 \end{cases}, \forall i > j: a_{i,j}^0 = a_{j,i}^0$$

Thus it is observed the symmetric matrix \mathbf{A}_n^0 has $2\lceil \frac{n}{2} \rceil - 3$ parameters $t_1, t_2, \dots, t_{2\lceil \frac{n}{2} \rceil - 3}$. The number of parameters can be increased or decreased, so this decomposition is not unique. Below are two examples of the zero matrix \mathbf{A}_5^0 ,

Example 1.

$$\mathbf{A}_5^0 = \begin{bmatrix} 0 & 0 & t_1 & t_2 & t_2 \\ 0 & -2t_1 & -t_2 & -t_3 & t_4 \\ t_1 & -t_2 & -4t_3 & -t_4 & t_5 \\ t_2 & t_3 & -t_4 & -2t_5 & 0 \\ t_3 & t_4 & t_5 & 0 & 0 \end{bmatrix}$$

Example 2.

$$\mathbf{A}_5^0 = \begin{bmatrix} 0 & 0 & t_1 & t_2 & t_3 \\ 0 & -2t_1 & -t_2 & -t_3 & t_4 \\ t_1 & -t_2 & 0 & -t_4 & t_5 \\ t_2 & -t_3 & -t_4 & -2t_5 & 0 \\ t_3 & t_4 & t_5 & 0 & 0 \end{bmatrix}$$

One can verify these examples by checking if the sum in each anti-diagonal goes to zero. Therefore, each polynomial $P_n(x)$ of degree n can be written as a matrix product:

$$P_n(x) = \sum_{i=0}^n b_n x^n = \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} \\ x^{\lceil \frac{n}{2} \rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}^T (\mathbf{A}_n + \mathbf{A}_n^0) \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} \\ x^{\lceil \frac{n}{2} \rceil - 1} \\ \vdots \\ x \\ 1 \end{bmatrix}$$

Where the notation can be simplified even further,

$$\mathbf{X} = \begin{bmatrix} x^{\lceil \frac{n}{2} \rceil} & x^{\lceil \frac{n}{2} \rceil - 1} & \dots & x & 1 \end{bmatrix}^T$$

Which results in,

$$P_n(x) = \mathbf{X}^T (\mathbf{A}_n + \mathbf{A}_n^0) \mathbf{X}$$

Forcing the rank of the matrix $\mathbf{A}_n + \mathbf{A}_n^0$ to be equal to two: for polynomials of degree 3 and 4, this can always be forced. However, for higher dimensions, we will encounter a hard system of nonlinear equations that need to be solved. Extra assumptions about the coefficients of the polynomial are needed to solve the system and, as a consequence, the rank of the matrix will be equal to two. If the rank of $\mathbf{A}_n + \mathbf{A}_n^0$ is two, at most two non-zero eigenvalues can be found in the orthogonal diagonalization of $\mathbf{A}_n + \mathbf{A}_n^0 = \mathbf{VDV}^T$,

$$P_n(x) = \mathbf{X}^T \mathbf{VDV}^T \mathbf{X}$$

We note $\mathbf{Y} = \mathbf{V}^T \mathbf{X}$, so:

$$P_n(x) = \mathbf{Y}^T \mathbf{D} \mathbf{Y} = \mathbf{Y}^T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{Y}$$

Resulting in,

$$P_n(x) = \lambda_1 \left(\mathbf{V}_{1,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{1,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{1, \lceil \frac{n}{2} \rceil + 1} \right)^2 + \lambda_2 \left(\mathbf{V}_{2,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{2,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{2, \lceil \frac{n}{2} \rceil + 1} \right)^2$$

Where λ_1 and λ_2 are the eigenvalues of $\mathbf{A}_n + \mathbf{A}_n^0$, and $(\mathbf{V}_{1,1}, \mathbf{V}_{1,2}, \dots, \mathbf{V}_{1, \lceil \frac{n}{2} \rceil + 1})$ and $(\mathbf{V}_{2,1}, \mathbf{V}_{2,2}, \dots, \mathbf{V}_{2, \lceil \frac{n}{2} \rceil + 1})$ are the corresponding orthonormal eigenvectors. Thus the solution of $P_n(x) = 0$ can be found easily by solving polynomials of degrees $\lceil \frac{n}{2} \rceil$ as follows:

$$\begin{aligned} & \lambda_1 \left(\mathbf{V}_{1,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{1,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{1, \lceil \frac{n}{2} \rceil + 1} \right)^2 \\ & = -\lambda_2 \left(\mathbf{V}_{2,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{2,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{2, \lceil \frac{n}{2} \rceil + 1} \right)^2 \end{aligned}$$

Therefore, the preceding analysis leads us to the following key relation,

$$\begin{aligned} & \mathbf{V}_{1,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{1,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{1, \lceil \frac{n}{2} \rceil + 1} \\ & = \pm \sqrt{\frac{-\lambda_2}{\lambda_1}} \left(\mathbf{V}_{2,1} x^{\lceil \frac{n}{2} \rceil} + \mathbf{V}_{2,2} x^{\lceil \frac{n}{2} \rceil - 1} + \cdots + \mathbf{V}_{2, \lceil \frac{n}{2} \rceil + 1} \right) \end{aligned}$$

This equation succinctly captures the relationship between the two orthonormal eigenvectors, \mathbf{V}_1 and \mathbf{V}_2 , corresponding to the eigenvalues λ_1 and λ_2 of the matrix $\mathbf{A}_n + \mathbf{A}_n^0$. The presence of the square root term $\sqrt{\frac{-\lambda_2}{\lambda_1}}$ signifies the relative scaling between these two eigenvectors, which provides a critical step in the decomposition of the original polynomial into simpler, solvable components.

3.2. Quintic and Sextic Case

A general monic quintic polynomial is of the form,

$$P_5(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e$$

Where a, b, c, d, e are the coefficients that define the polynomial. This form highlights the primary structure of the quintic polynomial, with the leading coefficient normalized to 1 for simplicity. Proceeding with the analysis, the polynomial can be represented as follows where $\mathbf{X} = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix}^T$,

$$P_5(x) = \begin{bmatrix} x^3 & x^2 & x & 1 \end{bmatrix} \mathbf{A} \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \mathbf{X}^T \mathbf{A} \mathbf{X}$$

Where \mathbf{A} is of the form,

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & a & \frac{b}{2} & 0 \\ 0 & \frac{b}{2} & c & \frac{d}{2} \\ 0 & 0 & \frac{d}{2} & e \end{bmatrix}$$

It can be seen that evaluating the matrix product shows that it is equivalent to the polynomial as formulated in the previous subsection. Since the matrix \mathbf{A} is real and symmetric, it indeed can be orthogonally diagonalized,

$$\mathbf{A} = \mathbf{V}\mathbf{D}\mathbf{V}^T$$

The matrix \mathbf{V} consists of the orthonormal eigenvectors of \mathbf{A} as its columns, e.g. $\mathbf{V} = [\mathbf{V}_1 \ \mathbf{V}_2 \ \mathbf{V}_3 \ \mathbf{V}_4]$ and \mathbf{D} is a diagonal matrix where the diagonal entries are the eigenvalues λ_i of \mathbf{A} . Now use the following substitution,

$$\mathbf{V}^T \mathbf{X} = [\mathbf{V}_1 \ \mathbf{V}_2 \ \mathbf{V}_3 \ \mathbf{V}_4]^T \begin{bmatrix} x^3 \\ x^2 \\ x \\ 1 \end{bmatrix} = \mathbf{Y} = \begin{bmatrix} y_3 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix}$$

With y_i as polynomials of degree 3 in terms of x , resulting in the form,

$$P_5(x) = \mathbf{Y}^T \mathbf{D} \mathbf{Y} = \begin{bmatrix} y_3 & y_2 & y_1 & y_0 \end{bmatrix} \begin{bmatrix} \lambda_4 & 0 & 0 & 0 \\ 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_1 \end{bmatrix} \begin{bmatrix} y_3 \\ y_2 \\ y_1 \\ y_0 \end{bmatrix} = \sum_{i=1}^4 \lambda_i y_i^2 = 0$$

Consider when the matrix \mathbf{A} is of rank 2, this means we have only two nonzero eigenvalues, which significantly simplifies the polynomial representation and solution process. A rank-2 matrix implies that the polynomial $P_n(x)$ can be reduced to a system involving just two independent components, corresponding to the orthonormal eigenvectors associated with these nonzero eigenvalues,

$$\sum_{i=1}^2 \lambda_i y_i^2 = 0 \Rightarrow \lambda_3 y_3^2 = \lambda_4 y_4^2 \Rightarrow |y_3| = |y_4| \sqrt{\frac{-\lambda_4}{\lambda_3}}$$

Which implies that the system can be solved as both y_1 and y_2 are at most degree 3 polynomials. Expanding the above leads to:

$$\begin{aligned} \lambda_3 (y_3)^2 + \lambda_4 (y_4)^2 &= 0 \\ \Rightarrow \lambda_3 (\mathbf{V}_3 \mathbf{X})^2 + \lambda_4 (\mathbf{V}_4 \mathbf{X})^2 &= 0 \end{aligned}$$

Upon simplification,

$$\begin{aligned} \mathbf{V}_3 \mathbf{X} &= \pm \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_4 \mathbf{X} \\ \Rightarrow \mathbf{V}_{31} x^3 + \mathbf{V}_{32} x^2 + \mathbf{V}_{33} x + \mathbf{V}_{34} &= \pm \sqrt{\frac{-\lambda_4}{\lambda_3}} (\mathbf{V}_{41} x^3 + \mathbf{V}_{42} x^2 + \mathbf{V}_{43} x + \mathbf{V}_{44}) \end{aligned}$$

Thus one ends up with the following closed form solutions where solving the quartics provide at most 6 solutions for the quintic polynomial in this example where one of the solutions are erroneous if so. This means solving Equation (1) provides 3 solutions and Equation (2) provides the other 3.

$$(\mathbf{V}_{31} + \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{41}) x^3 + (\mathbf{V}_{32} + \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{42}) x^2 + (\mathbf{V}_{33} + \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{43}) x + (\mathbf{V}_{34} + \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{44}) = 0 \quad (1)$$

$$(\mathbf{V}_{31} - \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{41}) x^3 + (\mathbf{V}_{32} - \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{42}) x^2 + (\mathbf{V}_{33} - \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{43}) x + (\mathbf{V}_{34} - \sqrt{\frac{-\lambda_4}{\lambda_3}} \mathbf{V}_{44}) = 0 \quad (2)$$

Solving the two cases in the previous work provided the full solution set, but it will be seen that there is no need for this when solving quintics. The matrix \mathbf{A} can be reduced to rank 2, by introducing the construction which does not change the quintic equation:

$$P_5(x) = \mathbf{A}_{s,t,r} = \begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & a - 2s & \frac{b}{2} - t & -r \\ s & \frac{b}{2} - t & c + 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

Evaluating the matrix product for $P_5(x)$ shows that it is equivalent even if we introduce the construction for $\mathbf{A}_{s,t,r}$. Thus the goal is to find values of r , s , and t that force the matrix to be rank 2 then use this new modified matrix to solve the polynomial. A similar matrix can be used for the sextic defined by the monic sextic polynomial of the form:

$$P_6(x) = x^6 + fx^5 + ax^4 + bx^3 + cx^2 + dx + e$$

This is true since they use the same steps with an adjustment to the matrix \mathbf{A} . Specifically, the methodology used for the quintic case can be seamlessly extended to handle the sextic polynomial by incorporating one additional degree term. This extension follows naturally from the symmetry and structure of the matrix formulation. As in the quintic case, the goal is to decompose the polynomial into simpler components by representing it in matrix form, but here the matrix must account for the higher degree:

$$P_6(x) = \mathbf{A}_{s,t,r} = \begin{bmatrix} 1 & \frac{f}{2} & s & t \\ \frac{f}{2} & a - 2s & \frac{b}{2} - t & -r \\ s & \frac{b}{2} - t & c + 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

This matrix encapsulates the coefficients of the sextic polynomial in a manner similar to the quintic case, but with slight modifications to reflect the additional terms. Now that we have built up the necessary theory, we now state the following lemmas and prove them in the following section.

4. Main Results

For both lemmas it is noted for notation that with an example subcondition such as $b = 2b'$ for a quintic polynomial say, then b' is considered the coefficient of x^3 , e.g. $p(x) = fx^5 + ax^4 + b'x^3 + cx^2 + dx + e$.

Lemma 2. Let $p(x) = fx^5 + ax^4 + bx^3 + cx^2 + dx + e$ be a quintic equation with real coefficients. If the coefficients f, a, b, c, d, e satisfy one of the following conditions, then the roots of the quintic equation can be expressed in radicals:

1. Condition One: $f = 1, a = b = 0$, and $c^2d + e^2 = 0$
2. Condition Two: $f = 1, a = b = c = 0$, and $d^5 = e^4$
3. Condition Three: $f = 1, a = b = c = 0$, and $e^4 + 6^4d^5 = 0$
4. Condition Four: $f = 1, a = 0$ and $c^2d + e^2 - bec = 0$
5. Condition Five: $f = 1, a = 0, e = c = -2\sqrt{\frac{-b}{4}}$, and $d + 1 = b$
6. Condition Six: $f = 1, a = 0, c = e + be^{\frac{1}{3}} + e^{\frac{2}{3}}$, and $d = -e^{\frac{2}{3}} - be^{\frac{1}{3}} - 2e$
7. Condition Seven: $f = 2, a' = 2a, b' = 2b, d' = 2d, c - 2(\sqrt{-b})^3 + 2(\frac{d}{\sqrt{-b}}) = 0$, and $e + 2(\sqrt{-b})^5 + 2d'\sqrt{-b} = 0$
8. Condition Eight: $f = 2, a' = 2a, b' = 2b, d' = 2d$, and $c + 2(\frac{e}{2d})^3 + 2b(\frac{e}{2d}) + 2(\frac{-e}{\sqrt{d}}) - 2(\frac{-e}{\sqrt{d}})(\sqrt{d}) = 0$

Lemma 3. Let $p(x) = gx^6 + fx^5 + ax^4 + bx^3 + cx^2 + dx + e$ be a sextic equation with real coefficients. If the coefficients g, f, a, b, c, d, e satisfy one of the following conditions, then the roots of the sextic equation can be expressed in radicals:

1. Condition One: $g = 1, f = a = b = c = 0$, and $d = -2\left(\frac{-e}{2}\right)^{\frac{1}{3}}\sqrt{e}$
2. Condition Two: $g = 1, f = a = b = c = 0$, and $e^5 2^8 = 5^5 d^6$
3. Condition Three: $g = 1, f = 0, a \neq 0, b' = 2b, d' = 2d, b = \sqrt{e}$ and $d^2 = ce$

4.1. Proof of Lemma 2, Condition One

A matrix of the form below is considered

$$\mathbf{A}_{t,r} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & t \\ \frac{1}{2} & 0 & -t & -r \\ 0 & -t & c + 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

Applying gaussian elimination,

$$\begin{bmatrix} \frac{1}{2} & 0 & -t & -r \\ 0 & \frac{1}{2} & 0 & t \\ 0 & 0 & c + 2r & \frac{d + 4t^2}{2} \\ 0 & 0 & \frac{d + 4t^2}{2} & 4rt + e \end{bmatrix}$$

To zero out the bottom two rows, it is required to solve the following system of equations,

$$c + 2r = 0 \quad (3)$$

$$d + 4t^2 = 0 \quad (4)$$

$$4rt + e = 0 \quad (5)$$

Solving Equation (3) and plugging it to Equation (5),

$$r = -\frac{c}{2}, \quad t = \frac{e}{2c} \quad (6)$$

Plugging these results into Equation (4):

$$\begin{aligned} d + 4\left(\frac{e}{2c}\right)^2 &= 0 \\ \Rightarrow d + \frac{e^2}{c^2} &= 0 \end{aligned} \quad (7)$$

Multiplying both sides by c ,

$$\Rightarrow c^2 d + e^2 = 0$$

Thus the polynomial in this example is solvable given that the above constraint of its coefficients is satisfied. As in the previous example, the nonzero eigenvalues are considered which are Equation (A3) and Equation (A4) with the normalized versions of the corresponding eigenvectors in Equation (A7) and Equation (A9). Now these values can be plugged into Equation (1) and Equation (2) where the only difference are the formulas defining the eigenvalues and eigenvectors. \square

4.2. Proof of Lemma 2, Condition Two

A general matrix of the form below is used,

$$\mathbf{A}_{s,t,r} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & \frac{b}{2} - t & -r \\ -s & \frac{b}{2} - t & c + 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

An arbitrary assumption $r = 0$ is made.

$$\mathbf{A}_{s,t} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & \frac{b}{2} - t & 0 \\ -s & \frac{b}{2} - t & c & \frac{d}{2} \\ t & 0 & \frac{d}{2} & e \end{bmatrix}$$

Upon applying gaussian elimination, the following matrix is the result,

$$\begin{bmatrix} \frac{1}{2} & 2s & \frac{b-2t}{2} & 0 \\ 0 & \frac{1}{2} & -s & t \\ 0 & 0 & c + 8s^3 + 2bs - 4st & \frac{d + 4t^2 - 2bt - 16s^2t}{2} \\ 0 & 0 & \frac{d + 4t^2 - 2bt - 16s^2t}{2} & 8st^2 + e \end{bmatrix}$$

Indicating that the following system has to be solved in order to force Rank 2.

$$c + 8s^3 + 2bs - 4st = 0$$

$$d + 4t^2 - 2bt - 16s^2t = 0$$

$$8st^2 + e = 0$$

This system can be simplified further as it is recalled that the quintic considered in this subsection has coefficients satisfying $b = 0, c = 0$, and $d \neq 0$.

$$8s^3 - 4st = 0 \tag{8}$$

$$d + 4t^2 - 16s^2t = 0 \tag{9}$$

$$8st^2 + e = 0 \tag{10}$$

Solving Equation (8) one gets the following,

$$t = 2s^2 \tag{11}$$

And applying Equation (11) to Equation (9), this results in

$$4t^2 - 8t(t) + d = 0 \Rightarrow d = 4t^2 \tag{12}$$

Substituting into Equation (10),

$$2ds + e = 0 \Rightarrow s = \frac{-e}{2d} \tag{13}$$

Plugging in the values of s and t from Equation (13) and Equation (12) into Equation (11),

$$\pm \sqrt{\frac{d}{4}} = \frac{e^2}{2d^2} \Rightarrow \frac{d}{4} = \frac{e^4}{4d^4} \Rightarrow d^5 = e^4 \quad (14)$$

Thus, the quintic is only solvable when the coefficients satisfy Equation (14). Plugging in Equation (12) and Equation (13) into $\mathbf{A}_{s,t}$, one retrieves nonzero eigenvalues defined in Equation (A13) and Equation (A14) along with the normalized versions of their corresponding eigenvectors in Equation (A17) and Equation (A19). These equations can be plugged into Equation (1) and Equation (2) as before, deriving a closed form solution. \square

4.3. Proof of Lemma 2, Condition Three

To solve the quintic, a matrix of the form below is considered,

$$\mathbf{A}_{s,t,r} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & -t & -r \\ -s & -t & 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

Applying gaussian elimination,

$$\left[\begin{array}{ccc|c} \frac{1}{4} & 0 & \frac{4s^2 - t}{2} & \frac{-r - 4st}{2} \\ 0 & \frac{1}{4} & -\frac{s}{2} & \frac{t}{2} \\ 0 & 0 & \frac{r + 4s^3 - 2st}{2} & \frac{d - 4rs + 4t^2 - 16s^2t}{8} \\ 0 & 0 & \frac{d - 4rs + 4t^2 - 16s^2t}{8} & \frac{8st^2 + 4rt + e}{4} \end{array} \right]$$

Thus the following system has to be solved,

$$r + 4s^3 - 2st = 0 \quad (15)$$

$$d - 4rs + 4t^2 - 16s^2t = 0 \quad (16)$$

$$e + 8st^2 + 4rt = 0 \quad (17)$$

A condition is assumed to simplify solving of the above system,

$$d = 4rs \quad (18)$$

Plugging this into Equation (16), simplifies to

$$4t^2 - 16s^2t = 0 \Rightarrow t = 4s^2 \quad (19)$$

Inserting Equation (19) into Equation (15),

$$4s^3 + r - 2s(4s^2) = 0 \Rightarrow r = 4s^3 \quad (20)$$

Using both Equation (19) and Equation (20) into Equation (17),

$$e + 8s(4s^2)^2 + 4(4s^3)(4s^2) = 0 \Rightarrow e + 192s^5 = 0 \quad (21)$$

Lastly, Equation (20) has to be inserted into Equation (18) to represent it completely in terms of s

$$d = 4s(4s^3) = 16s^4 \quad (22)$$

These set of equations imply that plugging Equation (22) into Equation (21) leads to the condition,

$$e^4 + 6^4 d^5 = 0 \quad (23)$$

Assuming the coefficients of the quintic satisfy Equation (23), plugging in Equations (19)-(22) into the matrix $\mathbf{A}_{s,t,r}$ results in the nonzero Eigenvalues defined in Equation (A23) and Equation (A24) with the normalized versions of the Eigenvectors defined in Equations (A27)-(A30). Plugging them into Equation (1) and Equation (1) yields the roots of the quintic. \square

4.4. Proof of Lemma 2, Condition Four

The matrix to apply gaussian elimination on is of the form,

$$\mathbf{A}_{t,r} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & t \\ \frac{1}{2} & 0 & \frac{b}{2} - t & -r \\ 0 & \frac{b}{2} - t & c + 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

Using gaussian elimination,

$$\begin{bmatrix} \frac{1}{2} & 0 & \frac{b-2t}{2} & -r \\ 0 & \frac{1}{2} & 0 & t \\ 0 & 0 & c+2r & \frac{d+4t^2-2bt}{2} \\ 0 & 0 & \frac{d+4t^2-2bt}{2} & 4rt+e \end{bmatrix}$$

As in the previous section, to zero out the bottom two rows to achieve rank 2, the following system of equations must be solved.

$$c + 2r = 0 \quad (24)$$

$$d + 4t^2 - 2bt = 0 \quad (25)$$

$$4rt + e = 0 \quad (26)$$

It can be seen that by solving Equation (24) and plugging it into Equation (26), one gets the following solutions for r and t .

$$r = \frac{-c}{2}, \quad t = \frac{e}{2c} \quad (27)$$

Plugging this into Equation (25),

$$\begin{aligned} d + 4\left(\frac{e}{2c}\right)^2 - 2b\frac{e}{2c} &= 0 \\ \Rightarrow d + \frac{e^2}{c^2} - \frac{be}{c} &= 0 \end{aligned} \quad (28)$$

Multiplying both sides by c^2 ,

$$\Rightarrow c^2 d + e^2 - bec = 0 \quad (29)$$

Thus, if the coefficients of the polynomial $x^5 + bx^3 + cx^2 + dx + e = 0$ satisfy Equation (29), one can solve the quintic. In the appendix, the corresponding eigenvalues and eigenvectors are attained. Thus the solutions to the quintic can be extracted using the nonzero eigenvalues in Equation (A33) and Equation (A34) with their corresponding normalized versions of the eigenvectors defined in Equation (A37) and Equation (A39). This is done by plugging them into Equation (1) and Equation (2) then solving them. \square

4.5. Proof of Lemma 2, Condition Five

A matrix of the form below is considered,

$$\mathbf{A}_{s,t} = \begin{bmatrix} 0 & \frac{1}{2} & -s & t \\ \frac{1}{2} & 2s & \frac{b}{2} - t & 0 \\ -s & \frac{b}{2} - t & c & \frac{d}{2} \\ t & 0 & \frac{d}{2} & e \end{bmatrix}$$

With gaussian elimination resulting in the matrix,

$$\begin{bmatrix} \frac{1}{4} & 0 & \frac{b+8s^2-2t}{4} & -2st \\ 0 & \frac{1}{4} & \frac{-s}{2} & \frac{t}{2} \\ 0 & 0 & \frac{c+8s^3+2bs-4st}{4} & \frac{d+4t^2-2bt-16s^2t}{8} \\ 0 & 0 & \frac{d+4t^2-2bt-16s^2t}{8} & \frac{8st^2+e}{4} \end{bmatrix}$$

The system to solve is thus,

$$c + 8s^3 + 2bs - 4st = 0 \quad (30)$$

$$d + 4t^2 - 2bt - 16s^2t = 0 \quad (31)$$

$$8st^2 + e = 0 \quad (32)$$

Approaching Equation (32),

$$s = \frac{-e}{8t^2} \quad (33)$$

Plugging this result into Equation (30),

$$c - \frac{e^3}{64t^6} - \frac{be}{4t^2} + \frac{e}{2t} = 0 \Rightarrow 64t^6c - e^3 - 16bet^4 + 32et^5 = 0 \quad (34)$$

Equation (34) shows that one has to solve a sextic in order to find the required conditions to determine the solvability and solutions of the polynomial in this subsection. But what if Equation (33) is plugged into Equation (31)?

$$d + 4t^2 - 2bt - 16\left(\frac{-e}{8t^2}\right)^2t = 0 \Rightarrow d + 4t^2 - 2bt - \frac{e^2}{4t^3} = 0 \quad (35)$$

Multiplying both sides of Equation (35) by t^3 ,

$$t^3d + 4t^5 - 2bt^4 - \frac{e^2}{4} = 0 \quad (36)$$

Thus a quintic has to be evaluated to proceed further. Proceeding further for either case when encountering a quintic or sextic requires that they are solvable in the first place which is another issue altogether. Thus it is best to make some assumptions for s and t to proceed further without complication. Before proceeding with any steps, it is observed that the substitution $b = -4s^2$ simplifies Equation (30) and Equation (31). Applying this substitution to Equation (31),

$$d + 4t^2 - 2bt + 4(-4s^2)t = 0 \Rightarrow d + 4t^2 + 2bt = 0 \quad (37)$$

Now using this substitution on Equation (30),

$$c - 2(-4s^2) + 2bs - 4st = 0 \Rightarrow c - 4st = 0 \quad (38)$$

Another substitution is now made $t = \frac{-1}{2}$ to simplify Equation (38)

$$c - 4s\left(\frac{-1}{2}\right) = 0 \Rightarrow c + 2s = 0 \quad (39)$$

Also simplifying Equation (32)

$$8s\left(\frac{-1}{2}\right)^2 + e = 0 \Rightarrow e + 2s = 0 \quad (40)$$

Equation (39) and Equation (40) imply the requirement that $e = c = -2s$. Plugging the substitution for t into Equation (41),

$$d + 4\left(\frac{-1}{2}\right)^2 + 2b\frac{-1}{2} = 0 \Rightarrow d + 1 = b \quad (41)$$

Thus when the polynomial's coefficients satisfy $e = c = -2\sqrt{\frac{-b}{4}}$ and $d + 1 = b$, one can substitute $s = \sqrt{\frac{-b}{4}}$ and $t = \frac{-1}{2}$ into the matrix to retrieve the eigenvalues defined in Equation (A43) and Equation (A44) along with their corresponding normalized versions of the eigenvectors in Equation (A48) and Equation (A50). Substituting the eigenvalues and eigenvectors into Equation (1) and Equation (2). The implication of making use of assumptions through substitutions shows that one can have different configurations for s and t along with their corresponding conditions. Meaning that it is possible to solve the polynomial with different solutions for s and t given that the polynomial satisfies the respective conditions. \square

4.6. Proof of Lemma 2, Condition Six

Another assumption, $s = t$ is considered to solve Equations (30), (31), and (32). This results in,

$$c + 8t^3 + 2bt - 4t^2 = 0 \quad (42)$$

$$d + 4t^2 - 2bt - 16t^3 = 0 \quad (43)$$

$$8t^3 + e = 0 \quad (44)$$

Solving Equation (44),

$$t = \left(\frac{-e}{8}\right)^{\frac{1}{3}} \quad (45)$$

Plugging this into Equation (42) and Equation (43),

$$c = -8\left(\left(\frac{-e}{8}\right)^{\frac{1}{3}}\right)^3 - 2b\left(\frac{-e}{8}\right)^{\frac{1}{3}} + 4\left(\left(\frac{-e}{8}\right)^{\frac{1}{3}}\right)^2 \Rightarrow c = e + be^{\frac{1}{3}} + e^{\frac{2}{3}} \quad (46)$$

$$d = -4\left(\left(\frac{-e}{8}\right)^{\frac{1}{3}}\right)^2 + 2b\left(\frac{-e}{8}\right)^{\frac{1}{3}} + 16\left(\left(\frac{-e}{8}\right)^{\frac{1}{3}}\right)^3 \Rightarrow d = -e^{\frac{2}{3}} - be^{\frac{1}{3}} - 2e \quad (47)$$

Thus when the quintic's coefficients satisfies Equation (46) and Equation (47), setting $s = t = \left(\frac{-e}{8}\right)^{\frac{1}{3}}$ reduces the matrix $\mathbf{A}_{s,t}$ to rank 2 and thus solvable. Thus the eigenvalues defined by Equation (A53) and Equation (A54) with their corresponding normalized versions of the eigenvectors defined by Equations (A57)-(A60) provide the roots of the quintic when plugged into Equation (1) and Equation (2). \square

4.7. Proof of Lemma 2, Condition Seven

The quintic is presented using the following matrix,

$$A_{s,t,r} = \begin{bmatrix} 0 & 1 & -s & t \\ 1 & 2s & b-t & -r \\ -s & b-t & c+2r & d \\ t & -r & d & e \end{bmatrix}$$

Applying gaussian elimination,

$$\begin{bmatrix} 1 & 0 & b + 2s^2 - t & -r - 2st \\ 0 & 1 & -s & t \\ 0 & 0 & c + 2s^3 + 2bs + 2r - 2st & d - rs + t^2 - bt - 2s^2t \\ 0 & 0 & d - rs + t^2 - bt - 2s^2t & e + 2st^2 + 2rt \end{bmatrix}$$

Thus have to solve the following system to achieve rank 2,

$$c + 2s^3 + 2bs + 2r - 2st = 0 \quad (48)$$

$$d - rs + t^2 - bt - 2s^2t = 0 \quad (49)$$

$$e + 2st^2 + 2rt = 0 \quad (50)$$

Assume the following conditions,

$$d = rs \quad (51)$$

$$b = -s^2 \quad (52)$$

Using the above conditions, Equation (49) evaluates to the following,

$$t^2 - s^2t = 0 \Rightarrow t = s^2 \quad (53)$$

Plugging Equation (53) and Equation (52) into Equation (48) and Equation (50) as needed,

$$c + 2s^3 + 2bs + 2r - 2s^3 = 0 \Rightarrow c - 2s^3 + 2r = 0 \quad (54)$$

$$e + 2s^5 + 2rs^2 = 0 \Rightarrow e + 2s^5 - 2br = 0 \quad (55)$$

To simplify further, Equation (51) and Equation (52) are equivalent to the below respectively,

$$r = \frac{d}{\sqrt{-b}} \quad (56)$$

$$s = \sqrt{-b} \quad (57)$$

Applying this back to Equation (54) and Equation (55).

$$c - 2(\sqrt{-b})^3 + 2\left(\frac{d}{\sqrt{-b}}\right) = 0 \quad (58)$$

$$e + 2(\sqrt{-b})^5 + 2\left(\frac{d}{\sqrt{-b}}\right)(\sqrt{-b})^2 = 0 \Rightarrow e + 2(\sqrt{-b})^5 + 2d\sqrt{-b} = 0 \quad (59)$$

Given that the coefficients of this polynomial satisfy Equation (58) and Equation (59), then substituting Equations (51) - (57) into the matrix $\mathbf{A}_{s,t,r}$, the quintic can be solved by plugging in the computed nonzero eigenvalues in Equation (A63) and Equation (A64) with the normalized versions of the corresponding eigenvectors defined by Equations (A67)-(A70) into Equation (1) and Equation (2). \square

4.8. Proof of Lemma 2, Condition Eight

Now consider the below alternate conditions,

$$e = -rt \quad (60)$$

$$b = 2t \quad (61)$$

Plugging the first condition into Equation (50),

$$2st^2 + rt = 0 \Rightarrow r = -2st \quad (62)$$

Substituting the above into Equation (49) along with Equation (61),

$$d - t^2 = 0 \Rightarrow d = t^2 \quad (63)$$

Putting the above into Equation (60),

$$r = \frac{-e}{\sqrt{d}} \quad (64)$$

Making use of both Equation (63) and Equation (64) into Equation (62),

$$s = \frac{r}{-2t} = \frac{\frac{-e}{\sqrt{d}}}{-2(\sqrt{d})} \Rightarrow s = \frac{e}{2d} \quad (65)$$

Plugging in Equations (63)-(65) into Equation (48),

$$c + 2\left(\frac{e}{2d}\right)^3 + 2b\left(\frac{e}{2d}\right) + 2\left(\frac{-e}{\sqrt{d}}\right) - 2\left(\frac{-e}{\sqrt{d}}\right)(\sqrt{d}) = 0 \quad (66)$$

Given that the coefficients of the polynomial for the complex condition in Equation (66), the solution is defined by the Eigenvalues in Equation (A73) and Equation (A74) with their corresponding normalized versions of the Eigenvectors defined in Equations (A77)-(A80) being plugged into Equation (1) and Equation (2). \square

4.9. Proof of Lemma 3, Condition One

A matrix of the form below is used where $r = 0$,

$$\mathbf{A}_{s,t} = \begin{bmatrix} 1 & 0 & -s & t \\ 0 & 2s & -t & 0 \\ -s & -t & 0 & \frac{d}{2} \\ t & 0 & \frac{d}{2} & e \end{bmatrix}$$

Applying gaussian elimination and assuming $s \neq 0$,

$$\begin{bmatrix} 1 & 0 & -s & t \\ 0 & 2s & -t & 0 \\ 0 & 0 & -2s^3 - t^2 & ds + 2s^2t \\ 0 & 0 & ds + 2s^2t & -2st^2 + 2se \end{bmatrix}$$

This leads to the system to be solved

$$-2s^3 - t^2 = 0 \quad (67)$$

$$ds + 2s^2t = 0 \quad (68)$$

$$-2st^2 + 2se = 0 \quad (69)$$

From Equation (69), it can be seen that,

$$t^2 = e \Rightarrow t = \pm\sqrt{e} \quad (70)$$

Plugging Equation (70) into Equation (67),

$$-2s^3 - e = 0 \Rightarrow s = \left(\frac{-e}{2}\right)^{\frac{1}{3}} \quad (71)$$

Plugging Equation (70) and Equation (71) into Equation (68),

$$d + 2\left(\frac{-e}{2}\right)^{\frac{1}{3}}\sqrt{e} = 0 \Rightarrow d = -2\left(\frac{-e}{2}\right)^{\frac{1}{3}}\sqrt{e} \quad (72)$$

Therefore when the sextic of form $x^6 + dx + e = 0$ satisfies the above Equation (72), it is solvable by plugging in the nonzero eigenvalues defined in Equation (A83) and Equation (A84) with the normalized versions of the corresponding eigenvectors in Equation (A88) and Equation (A90). Plugging them into Equation (1) and Equation (2) provides the closed form solution. \square

4.10. Proof of Lemma 3, Condition Two

A similar matrix is used as in the previous subsection but $r \neq 0$.

$$\mathbf{A}_{s,t,r} = \begin{bmatrix} 1 & 0 & -s & t \\ 0 & 2s & -t & -r \\ -s & -t & 2r & \frac{d}{2} \\ t & -r & \frac{d}{2} & e \end{bmatrix}$$

By gaussian elimination with $s \neq 0$,

$$\left[\begin{array}{cccc} 2s & 0 & -2s^2 & 2st \\ 0 & 2s & -t & -r \\ 0 & 0 & -2s^3 + 4rs - t^2 & ds - rt + 2s^2t \\ 0 & 0 & ds - rt + 2s^2t & -r^2 - 2st^2 + 2se \end{array} \right]$$

Thus, the below system is considered,

$$-2s^3 + 4rs - t^2 = 0 \quad (73)$$

$$ds - rt + 2s^2t = 0 \quad (74)$$

$$-r^2 + 2es - 2st^2 = 0 \quad (75)$$

The difference compared to the previous subsection's general case is assuming a different condition to simplify solving of system,

$$d = -st \quad (76)$$

Substituting into Equation (74),

$$(-st)s - rt + 2s^2t = 0 \Rightarrow r = s^2 \quad (77)$$

Applying Equation (77) into Equation (73),

$$-2s^3 + 4s(s^2) - t^2 = 0 \Rightarrow \sqrt{2s^3} = t \quad (78)$$

Using both Equation (77) and Equation (78) into Equation (75),

$$-(s^2)^2 + 2es - 2s(2s^3) = 0 \Rightarrow e = \frac{5s^3}{2} \quad (79)$$

Finally, to represent Equation (76) in terms of s ,

$$d = -s(\sqrt{2s^3}) \quad (80)$$

Making use of both Equation (79) and Equation (80) imply the coefficients of the sextic have to satisfy the condition,

$$e^5 = \left(\frac{5}{2}\right)^5 \frac{d^6}{2^3} \Rightarrow e^5 = \frac{5^5 d^6}{2^8} \quad (81)$$

Thus given that the above condition defined by Equation (81) is satisfied, plugging Equations (77)-(81) into $\mathbf{A}_{s,t,r}$ yields the eigenvalues in Equation (A93) and Equation (A94) with their respective normalized versions of the eigenvectors defined by Equations (A97)-(A100). And plugging these into Equation (1) and Equation (2) yields the roots to the sextic. \square

4.11. Proof of Lemma 3, Condition Three

The sextic can be represented by the matrix,

$$\mathbf{A}_{s,t,r} = \begin{bmatrix} 1 & 0 & -s & t \\ 0 & a+2s & b-t & -r \\ -s & b-t & c+2r & d \\ t & -r & d & e \end{bmatrix}$$

Where by gaussian elimination with the added requirement $a \neq -2s$,

$$\left[\begin{array}{cccc} a+2s & 0 & -2s^2 - as & at + 2st \\ 0 & a+2s & b-t & -r \\ 0 & 0 & -b^2 + ac - 2s^3 - as^2 + 2cs + 2ar + 4rs - t^2 + 2bt & ad + 2ds + br + 2s^2t + ast - rt \\ 0 & 0 & ad + 2ds + br + 2s^2t + ast - rt & ae + 2fs - r^2 - at^2 - 2st^2 \end{array} \right]$$

Thus achieving rank 2 requires solving the following system,

$$-b^2 + ac - 2s^3 - as^2 + 2cs + 2ar + 4rs - t^2 + 2bt = 0$$

$$ad + 2ds + br + 2s^2t + ast - rt = 0$$

$$ae + 2es - r^2 - at^2 - 2st^2 = 0$$

It is noted that this system can be expressed in terms of factors,

$$(a+2s)(e-t^2) - r^2 = 0 \quad (82)$$

$$(a+2s)(d+st) + r(b-t) = 0 \quad (83)$$

$$(a+2s)(c-s^2+2r) - (b-t)^2 = 0 \quad (84)$$

Assume the conditions,

$$b = t \quad (85)$$

$$r = 0 \quad (86)$$

Resulting in the system becoming,

$$(a+2s)(e-t^2) = 0 \quad (87)$$

$$(a+2s)(d+st) = 0 \quad (88)$$

$$(a+2s)(c-s^2) = 0 \quad (89)$$

Since $a \neq -2s$, Equations (87)-(89) become,

$$e = t^2 \Rightarrow t = \sqrt{e} \Rightarrow b = \sqrt{e} \quad (90)$$

$$d = -st \quad (91)$$

$$c = s^2 \Rightarrow s = \sqrt{c} \quad (92)$$

This also implies $a \neq -2\sqrt{c}$. Plugging Equation (90) and Equation (92) into Equation (91),

$$d = -\sqrt{ce} \Rightarrow d^2 = ce \quad (93)$$

Thus when the sextic's coefficients satisfy Equation (93) and Equation (90) along with $a \neq -2\sqrt{c}$, the sextic can be solved using Equation (1) and Equation (2) by plugging in the eigenvalues in Equations (A103) and (A104) along with their normalized versions of the eigenvectors in Equations (A107) and (A108) which are found by plugging Equations (90)-(92) into $\mathbf{A}_{s,t,r}$. \square

We now demonstrate using the techniques in the previous to solve polynomials with integer coefficients.

5. Computational Examples

Example 3. Find the roots of the following quintic polynomial, $x^5 - 3x^4 - 23x^3 + 51x^2 + 94x - 120$

Plugging in values for the coefficients, the following matrix is formed where values of s, t, r have to be found to force rank 2,

$$\mathbf{A}_{s,t,r} = \begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -3 - 2 \cdot s & \frac{-23}{2} - t & -r \\ s & \frac{-23}{2} - t & 51 + 2r & \frac{94}{2} \\ t & -r & \frac{94}{2} & -120 \end{bmatrix}$$

To figure out how to set $r, s,$ and t so that the rank of the matrix 2, gaussian elimination will be performed on the symbolic matrix. Upon applying gaussian elimination, one gets the row reduced echelon form,

$$\begin{bmatrix} \frac{1}{2} & -2s - 3 & \frac{-2t - 23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 2r - 8s^3 - 12s^2 + 46s + 4st + 51 & 2rs + 2t^2 - 8s^2t - 12st + 23t + 47 \\ 0 & 0 & 2rs + 2t^2 - 8s^2t - 12st + 23t + 47 & -8st^2 - 12t^2 + 4rt - 120 \end{bmatrix}$$

Assuming the following equation is true

$$2r - 8s^3 - 12s^2 + 46s + 4st + 51 = 0 \quad (94)$$

$$\begin{bmatrix} \frac{1}{2} & -2s - 3 & \frac{-2t - 23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 2rs + 2t^2 - 8s^2t - 12st + 23t + 47 \\ 0 & 0 & 2rs + 2t^2 - 8s^2t - 12st + 23t + 47 & -8st^2 - 12t^2 + 4rt - 120 \end{bmatrix}$$

Now add the following equation as an extra constraint

$$2rs + 2t^2 - 8s^2t - 12st + 23t + 47 = 0 \quad (95)$$

$$\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8st^2-12t^2+4rt-120 \end{bmatrix}$$

Adding the final constraint one achieves,

$$-8st^2 - 12t^2 + 4rt - 120 = 0 \quad (96)$$

$$\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank} \left(\begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ s & \frac{-2t-23}{2} & 2r+51 & 47 \\ t & -r & 47 & -120 \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} \frac{1}{2} & -2s-3 & \frac{-2t-23}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = 2$$

The system of three equations being satisfied (Equations (1)-(3)) are noted as they will be the key conditions for r, s, t to set the matrix to rank 2. This system has 3 equations and 3 variables thus is solvable.

$$\begin{aligned} 2r - 8s^3 - 12s^2 + 46s + 4st + 51 &= 0, \\ 2rs + 2t^2 - 8s^2t - 12st + 23t + 47 &= 0, \\ -8st^2 - 12t^2 + 4rt - 120 &= 0 \end{aligned}$$

Inputting the above system to an algebraic solver provides 10 sets of possible solutions for r, s , and t . Plugging these values into $\mathbf{A}_{s,t,r}$, the appropriate eigenvalues and eigenvectors of the nonzero eigenvalues are retrieved and plugged into the equation $y_1 = y_2 \sqrt{\frac{-\lambda_2}{\lambda_1}}$. Where it is recalled that y_i is the product of the matrix consisting of the eigenvectors as its columns and \mathbf{X} the vector containing powers of x . Thus for each r, s, t , a solution is derived for the quintic meaning that there are 10 sets of solutions. Recall that it is also required to solve the case when signs are different as shown in $y_1 = -y_2 \sqrt{\frac{-\lambda_2}{\lambda_1}}$. But this is not required due to having multiple solutions with implies that solution values that appear more than once belong to the proper solution set which is the roots of the polynomial. Thus the roots are $[-4, -2, 1, 3, 5]$, the polynomial formed from these roots is equal to the polynomial that had to be solved in the example thus confirming that the roots are correct.

Example 4. Find the roots of the following quintic polynomial, $x^5 - 57x^4 + 1227x^3 - 12547x^2 + 61236x - 114660$

Proceeding as before,

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -57-2 \cdot s & \frac{1227}{2} - t & -r \\ s & \frac{1227}{2} - t & -12547 + 2r & \frac{61236}{2} \\ t & -r & \frac{61236}{2} & -114660 \end{bmatrix}$$

With gaussian elimination resulting in,

$$\begin{bmatrix} \frac{1}{2} & -2s - 57 & \frac{-2t + 1227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 2r - 8s^3 - 228s^2 - 2454s + 4st - 12547 & 2rs + 2t^2 - 8s^2t - 228st - 1227t + 30618 \\ 0 & 0 & 2rs + 2t^2 - 8s^2t - 228st - 1227t + 30618 & -8st^2 - 228t^2 + 4rt - 114660 \end{bmatrix}$$

Assuming the following equation is true

$$2r - 8s^3 - 228s^2 - 2454s + 4st - 12547 = 0 \quad (97)$$

$$\begin{bmatrix} \frac{1}{2} & -2s - 57 & \frac{-2t + 1227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 2rs + 2t^2 - 8s^2t - 228st - 1227t + 30618 \\ 0 & 0 & 2rs + 2t^2 - 8s^2t - 228st - 1227t + 30618 & -8st^2 - 228t^2 + 4rt - 114660 \end{bmatrix}$$

Now adding the below equation as an extra constraint

$$2rs + 2t^2 - 8s^2t - 228st - 1227t + 30618 = 0 \quad (98)$$

$$\begin{bmatrix} \frac{1}{2} & -2s - 57 & \frac{-2t + 1227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8st^2 - 228t^2 + 4rt - 114660 \end{bmatrix}$$

Adding the final constraint one achieves,

$$-8st^2 - 228t^2 + 4rt - 114660 = 0 \quad (99)$$

$$\begin{bmatrix} \frac{1}{2} & -2s - 57 & \frac{-2t + 1227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{rank} \left(\begin{bmatrix} 0 & \frac{1}{2} & s & t \\ \frac{1}{2} & -2s - 57 & \frac{-2t + 1227}{2} & -r \\ s & \frac{-2t + 1227}{2} & 2r - 12547 & 30618 \\ t & -r & 30618 & -114660 \end{bmatrix} \right) = \text{rank} \left(\begin{bmatrix} \frac{1}{2} & -2s - 57 & \frac{-2t + 1227}{2} & -r \\ 0 & \frac{1}{2} & s & t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right) = 2$$

Thus the system to solve from Equations (4)-(6) is

$$\begin{aligned} 2r - 8s^3 - 228s^2 - 2454s + 4st - 12547 &= 0 \\ 2rs + 2t^2 - 8s^2t - 228st - 1227t + 30618 &= 0 \\ -8st^2 - 228t^2 + 4rt - 114660 &= 0 \end{aligned}$$

The solutions for r, s, t are again multiple sets and this leads to multiple solutions as in the previous example. It is noted that some solutions don't correctly lead to rank 2 hence there may be less possible solutions to the polynomial compared to the previous case. Thus it can be calculated that the roots to

the quintic in this example is [6, 7, 10, 13, 21]. Again, the polynomial formed from these roots is equal to the polynomial that had to be solved in the example thus confirming that the roots are correct. For completeness, the solutions for when $y_1 = -y_2 \sqrt{\frac{-\lambda_2}{\lambda_1}}$ can be used for each corresponding solution set leading to calculating a single set of roots in one iteration.

It can be seen from the previous examples that there is a relationship between the system of equations to solve and the corresponding polynomial. For a quintic polynomial $P_5(x)$, the following matrix can be forced to rank 2:

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{2} & r & s \\ \frac{1}{2} & a - 2r & \frac{b}{2} - s & -t \\ r & \frac{b}{2} - s & c + 2t & \frac{d}{2} \\ s & -t & \frac{d}{2} & e \end{bmatrix}$$

Given the system of equations below are satisfied which are expressed in terms of r, s, t where all the coefficients of the quintic are nonzero.

$$\begin{aligned} 2r - 8s^3 + 4as^2 - 2bs + 4st + c &= 0 \\ 2rs + 2t^2 - 8s^2t + 4ast - bt + \frac{d}{2} &= 0 \\ -8st^2 + 4at^2 + 4rt + e &= 0 \end{aligned}$$

Example 5. Decompose the following sextic polynomial so that it's easier to solve, $x^6 - 11x^4 + 2x^3 + x^2 - 2x + 1$

It is observed that the sextic polynomial satisfies Lemma 3 Condition Three, which satisfies the conditions:

$$\begin{aligned} d^2 &= ce \\ b &= \sqrt{e} \\ a &\neq -2\sqrt{c} \end{aligned}$$

Since plugging in the coefficients of the sextic,

$$\begin{aligned} d^2 = ce &\Rightarrow (1)^2 = 1(1) \\ b = \sqrt{e} &\Rightarrow 1 = \sqrt{1} \\ a \neq -2\sqrt{c} &\Rightarrow -11 \neq -2\sqrt{1} \end{aligned}$$

Thus the eigenvalues in Equations (A103) and (A104) along with their normalized versions of the eigenvectors in Equations (A107) and (A108) can be used. This results in the eigenvalues,

$$\begin{aligned} \lambda_3 &= a + 2\sqrt{c} = -11 + 2 = -9 \\ \lambda_4 &= 1 + c + b^2 = 1 + 1 + 1 = 3 \end{aligned}$$

With the respective eigenvectors,

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{b} \\ 0 \\ -\frac{\sqrt{c}}{b} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

Where normalized,

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Then Equation (1) and Equation (2) are used to decompose the sextic as follows:

$$0x^3 + (1)x^2 + 0x + 0 = \pm \sqrt{\frac{1}{3}} \left(\frac{1}{\sqrt{3}}x^3 + 0x^2 - \frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}} \right)$$

Thus a pair of cubics are acquired,

$$3x^2 = (x^3 - x + 1) \Rightarrow x^3 - 3x^2 - x + 1 = 0$$

$$3x^2 = -(x^3 - x + 1) \Rightarrow x^3 + 3x^2 - x + 1 = 0$$

Where solving each cubic gives gives the roots of the sextic, to confirm equivalence, since they have the same roots as the sextic then their product should be equivalent to the original sextic,

$$(x^3 - 3x^2 - x + 1)(x^3 + 3x^2 - x + 1) = x^6 - 11x^4 + 2x^3 + x^2 - 2x + 1$$

6. Conjecture on Higher Degrees

From the discussions in the previous sections, the following can be conjectured.

Conjecture: For any polynomial $P_n(x)$ of degree $n \geq 9$ where $\lceil \frac{n}{2} \rceil$ is even, by choosing the right two matrices \mathbf{A}_n and \mathbf{A}_n^0 , it can be expressed as a linear combination of the squares of two polynomials of degree $\lceil \frac{n}{2} \rceil$, such that

$$P_n(x) = \lambda_1 \left(\sum_{i=0}^{\lceil \frac{n}{2} \rceil} a_i x^i \right)^2 + \lambda_2 \left(\sum_{i=0}^{\lceil \frac{n}{2} \rceil} b_i x^i \right)^2$$

Where λ_1 and λ_2 are constants, and a_i and b_i are coefficients for the polynomials of degree $\lceil \frac{n}{2} \rceil$. Additionally, if $P_n(x)$ is monic (i.e., the leading coefficient of $P_n(x)$ is 1) and n is odd, then it must satisfy the condition

$$\lambda_1 a_{\frac{n+1}{2}}^2 + \lambda_2 b_{\frac{n+1}{2}}^2 = 0$$

This conjecture draws inspiration from observing the solvability of lower degree polynomials and naturally extends the framework to higher degrees. By carefully analyzing the rank of the matrix representations \mathbf{A}_n and \mathbf{A}_n^0 , solvable cases can be identified within the generally unsolvable domain of polynomials of degree $n \geq 9$. It is noted that the additional condition for the odd case as mentioned in the conjecture arises from the fact that the odd degree polynomial without setting the extra condition would have an extra degree term. For example, considering a quintic, without satisfying the condition, a term with factor x^6 would be included which is not desired, hence the importance of the odd condition. The implication of this conjecture is that there are possible applications to symbolic computation, algebraic geometry, and developing efficient algorithms to solve high degree polynomials.

7. Discussion

The lemmas presented in this paper advance significantly the understanding of algebraic solutions of higher degree polynomials. By showing specific cases of quintic and sextic polynomials with

solvable structures, it is demonstrated that even within the generally unsolvable realm of higher-degree polynomials, there exist solvable subfamilies. These cases offer concrete examples of how certain structural properties influence solvability. Furthermore, the derived conjecture for polynomials of degree 9 and higher extends these insights by suggesting a broader framework for solvability based on matrix decompositions. This framework not only is a bridge between theoretical algebra and computational approaches but also introduces a new definition of solvability within polynomial theory.

This work builds on foundational concepts in algebra, by exploring new methods for identifying solvable cases of quintic and sextic polynomials. While Galois theory shows that general polynomials of degree 5 or higher lack algebraic solutions due to their non-solvable Galois groups, a novel framework is introduced based on matrix representations and rank constraints to systematically identify solvable subfamilies. Unlike traditional approaches such as Bring-Jerrard forms or Klein's work on the Icosahedral group, the framework demonstrated in the paper leverages the eigenstructure of symmetric matrices associated with the polynomial. By reducing the matrix rank and expressing polynomials as a sum of squares, a direct solution is provided for specific cases. This approach not only clarifies the structural properties that enable solvability but also extends these principles to higher-degree polynomials through the conjecture, offering a new avenue for research in algebraic solvability and symbolic computation.

Despite its contributions, this study has certain limitations. First, the identified solvable cases rely on specific assumptions about the coefficients of the polynomials, which may limit their applicability to broader classes of polynomials. Additionally, the approach requires the polynomial's matrix representation to achieve a rank-2 form, which can lead to a complex system of nonlinear equations that may not be easily solvable. There is also the computational limitation that computing eigenstructures for higher degree polynomials would be demanding on resources. The proposed conjecture, while promising, remains unproven in its entirety and is based on observed patterns rather than a formal, generalized proof. Finally, what has been explored in this paper has been constrained to degrees 5, 6, and 9 or higher, leaving an opportunity for further investigation into degrees 7 and 8.

Future research could focus on several promising directions. One possibility is the rigorous proof of the conjecture for polynomials of degree 9 and higher. This could involve exploring additional matrix decomposition techniques or developing new methods for identifying solvable subfamilies. Another natural direction is to make use of the framework to analyze solvability for degrees 7 and 8, which may reveal additional insights into the boundaries of algebraic solvability. Finally, exploring the interplay between matrix-based approaches and advanced computational algebra systems could lead to more efficient algorithms for identifying and solving higher-degree polynomials.

8. Conclusion

In this study, we have explored conditions under which specific classes of quintic and sextic polynomials have algebraic solutions, despite the general unsolvability as shown by Galois theory. Using matrix representations and polynomial solving, we identified particular forms of quintics and sextics with solvable structures and provided explicit solutions for these cases. This analysis highlights the unique structural properties within these polynomial families that enable algebraic solvability.

Furthermore, we proposed a conjecture for polynomials of degree 9 and higher, suggesting that carefully structured subfamilies may also have algebraic solutions. This conjecture, grounded in observed patterns of symmetry and decomposability, extends current solvability theories and opens pathways for further investigation into polynomial structures at higher degrees.

Our findings contribute to the broader understanding of polynomial theory by challenging the boundaries of algebraic solvability and offering a systematic approach to identify solvable forms as well as provide their solutions. This work not only enhances theoretical insights into polynomial solvability but also provides an essential framework for future research into high degree polynomial solutions, with implications for algebraic theory. Future research could aim to test and refine our

conjecture, exploring applications in higher-dimensional representations and alternative polynomial forms. Due to the generality of this framework, future research would also involve extracting more particular cases as the paper only has gone through a portion of them due to assumptions made when solving the polynomial system.

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

Appendix A Closed Form Equations for Lemma 2 Condition One

The Eigenvalues:

$$\lambda_1 = 0 \quad (\text{A1})$$

$$\lambda_2 = 0 \quad (\text{A2})$$

$$\lambda_3 = \frac{c^2e - \sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4}}{2c^2} \quad (\text{A3})$$

$$\lambda_4 = \frac{c^2e + \sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4}}{2c^2} \quad (\text{A4})$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -c \\ -\frac{e}{c} \\ 0 \\ 1 \end{bmatrix} \quad (\text{A5})$$

$$\mathbf{V}_2 = \begin{bmatrix} \frac{e}{c} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{A6})$$

$$\mathbf{V}_{31_{\text{numerator}}} = c \left(c^4(-e) - c^2e^3 + c^2\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} + e^2\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} \right)$$

$$\mathbf{V}_{31_{\text{denominator}}} = \frac{-2c^6e - 2c^4e^3 - c^4e - 2c^2e^3 + c^4\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} + 2c^2e^2\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} - e^5}{2c^2e^2\sqrt{c^6 + c^4e^2 + c^4 + 2c^2e^2 + e^4} - e^5}$$

$$\begin{aligned}
V_{32_{\text{numerator}}} &= c \left(c^6 + c^4 e^2 + c^4 + 2c^2 e^2 - c^2 e \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^4 \right) \\
V_{32_{\text{denominator}}} &= \frac{2c^6 e + 2c^4 e^3 + c^4 e + 2c^2 e^3 - c^4 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} - 2c^2 e^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^5}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}} \\
V_{33_{\text{numerator}}} &= \frac{c^4 (-e^2) - c^2 e^4 + c^2 e \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}} \\
V_{33_{\text{denominator}}} &= \frac{-2c^6 e - 2c^4 e^3 - c^4 e - 2c^2 e^3 + c^4 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + 2c^2 e^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} - e^5}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}}
\end{aligned}$$

$$V_{34} = 1 \tag{A7}$$

Where

$$V_3 = \begin{bmatrix} \frac{V_{31_{\text{numerator}}}}{V_{31_{\text{denominator}}}} \\ \frac{V_{32_{\text{numerator}}}}{V_{32_{\text{denominator}}}} \\ \frac{V_{33_{\text{numerator}}}}{V_{33_{\text{denominator}}}} \\ V_{34} \end{bmatrix} \tag{A8}$$

$$\begin{aligned}
V_{41_{\text{numerator}}} &= c \left(c^4 e + c^2 e^3 + c^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} \right) \\
V_{41_{\text{denominator}}} &= \frac{2c^6 e + 2c^4 e^3 + c^4 e + 2c^2 e^3 + c^4 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + 2c^2 e^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^5}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}} \\
V_{42_{\text{numerator}}} &= c \left(c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + c^2 e \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^4 \right) \\
V_{42_{\text{denominator}}} &= \frac{2c^6 e + 2c^4 e^3 + c^4 e + 2c^2 e^3 + c^4 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + 2c^2 e^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^5}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}} \\
V_{43_{\text{numerator}}} &= \frac{c^4 e^2 + c^2 e^4 + c^2 e \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}} \\
V_{43_{\text{denominator}}} &= \frac{2c^6 e + 2c^4 e^3 + c^4 e + 2c^2 e^3 + c^4 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + 2c^2 e^2 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4} + e^5}{e^3 \sqrt{c^6 + c^4 e^2 + c^4 + 2c^2 e^2 + e^4}}
\end{aligned}$$

$$V_{44} = 1 \tag{A9}$$

Where

$$V_4 = \begin{bmatrix} \frac{V_{41_{\text{numerator}}}}{V_{41_{\text{denominator}}}} \\ \frac{V_{42_{\text{numerator}}}}{V_{42_{\text{denominator}}}} \\ \frac{V_{43_{\text{numerator}}}}{V_{43_{\text{denominator}}}} \\ V_{44} \end{bmatrix} \tag{A10}$$

Appendix B Closed Form Equations for Lemma 2 Condition Two

The Eigenvalues:

$$\lambda_1 = 0 \quad (\text{A11})$$

$$\lambda_2 = 0 \quad (\text{A12})$$

$$\lambda_3 = \frac{1}{2} \left(-\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e} \right) \quad (\text{A13})$$

$$\lambda_4 = \frac{1}{2} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e} \right) \quad (\text{A14})$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -2e^{3/5} \\ -e^{2/5} \\ 0 \\ 1 \end{bmatrix} \quad (\text{A15})$$

$$\mathbf{V}_2 = \begin{bmatrix} -e^{2/5} \\ -\sqrt[5]{e} \\ 1 \\ 0 \end{bmatrix} \quad (\text{A16})$$

$$\mathbf{V}_{31_{\text{numerator}}} = 2e^{6/5} + 3e^{4/5} + 3e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{3/5} + 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\sqrt[5]{e} + 1$$

$$\mathbf{V}_{31_{\text{denominator}}} = (e^{4/5} - 1)e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} - e + \sqrt[5]{e} \right)$$

$$\mathbf{V}_{32_{\text{numerator}}} = e^{9/5} + 4e^{7/5} + 6e^{3/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{4/5} + 4\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + 4e + 3\sqrt[5]{e}$$

$$\mathbf{V}_{32_{\text{denominator}}} = (e^{4/5} - 1)e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} - e + \sqrt[5]{e} \right)$$

$$\mathbf{V}_{33_{\text{numerator}}} = e^{9/5} + 2e^{7/5} + 3e^{3/5} + 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e + 2\sqrt[5]{e}$$

$$\mathbf{V}_{33_{\text{denominator}}} = (e^{4/5} - 1)\sqrt[5]{e} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} - e + \sqrt[5]{e} \right)$$

$$\mathbf{V}_{34} = 1 \quad (\text{A17})$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{\text{numerator}}}}{\mathbf{V}_{31_{\text{denominator}}}} \\ \frac{\mathbf{V}_{32_{\text{numerator}}}}{\mathbf{V}_{32_{\text{denominator}}}} \\ \frac{\mathbf{V}_{33_{\text{numerator}}}}{\mathbf{V}_{33_{\text{denominator}}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A18})$$

$$\mathbf{V}_{41_{\text{numerator}}} = -2e^{6/5} - 3e^{4/5} - 3e^{2/5} + \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{3/5} + 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}\sqrt[5]{e} - 1$$

$$\mathbf{V}_{41_{\text{denominator}}} = (e^{4/5} - 1)e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e} \right)$$

$$\begin{aligned}
V_{42_{\text{numerator}}} &= e^{9/5} + 4e^{7/5} + 6e^{3/5} - \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{4/5} - \\
&\quad 4\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{2/5} - \\
&\quad \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + 4e + 3\sqrt[5]{e} \\
V_{42_{\text{denominator}}} &= \left(e^{4/5} - 1\right)e^{2/5} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right) \\
V_{43_{\text{numerator}}} &= e^{9/5} + 2e^{7/5} + 3e^{3/5} - 2\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1}e^{2/5} - \\
&\quad \sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e + 2\sqrt[5]{e} \\
V_{43_{\text{denominator}}} &= \left(e^{4/5} - 1\right)\sqrt[5]{e} \left(\sqrt{e^{8/5} + 2e^{6/5} + 2e^{4/5} + 2e^{2/5} + e^2 + 1} + e - \sqrt[5]{e}\right) \\
V_{44} &= 1
\end{aligned} \tag{A19}$$

Where

$$V_4 = \begin{bmatrix} \frac{V_{41_{\text{numerator}}}}{V_{41_{\text{denominator}}}} \\ \frac{V_{42_{\text{numerator}}}}{V_{42_{\text{denominator}}}} \\ \frac{V_{43_{\text{numerator}}}}{V_{43_{\text{denominator}}}} \\ V_{44} \end{bmatrix} \tag{A20}$$

Appendix C Closed Form Equations for Lemma 2 Condition Three

The Eigenvalues:

$$\lambda_1 = 0 \tag{A21}$$

$$\lambda_2 = 0 \tag{A22}$$

$$\lambda_3 = \frac{1}{2} \left(-\sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e} e^3 + 416 \sqrt[3]{2} \sqrt[3]{e} e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e} e + 4 \sqrt[3]{2} \sqrt[3]{e} e^2 + 1} + 48 \sqrt[3]{2} e \sqrt[3]{e} e^2 - 2^{2/3} \sqrt[3]{e} - 4e \right) \tag{A23}$$

$$\lambda_4 = \frac{1}{2} \left(\sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e} e^3 + 416 \sqrt[3]{2} \sqrt[3]{e} e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e} e + 4 \sqrt[3]{2} \sqrt[3]{e} e^2 + 1} + 48 \sqrt[3]{2} e \sqrt[3]{e} e^2 - 2^{2/3} \sqrt[3]{e} - 4e \right) \tag{A24}$$

Corresponding Eigenvectors:

$$V_1 = \begin{bmatrix} -20e \\ -4 \sqrt[3]{2} \sqrt[3]{e} e^2 \\ 0 \\ 1 \end{bmatrix} \tag{A25}$$

$$V_2 = \begin{bmatrix} 0 \\ -2^{2/3} \sqrt[3]{e} \\ 1 \\ 0 \end{bmatrix} \tag{A26}$$

$$\begin{aligned}
V_{31_{\text{numerator}}} &= -7680 \sqrt[3]{e} e^2 e^3 - 160 \cdot 2^{2/3} e^3 - 40 \sqrt[3]{2} \sqrt[3]{e} e^2 + \\
&\quad 80 \cdot 2^{2/3} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e} e^3 + 416 \sqrt[3]{2} \sqrt[3]{e} e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e} e + 4 \sqrt[3]{2} \sqrt[3]{e} e^2 + 1} e^2 + \\
&\quad 10 \sqrt[3]{2} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e} e^3 + 416 \sqrt[3]{2} \sqrt[3]{e} e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e} e + 4 \sqrt[3]{2} \sqrt[3]{e} e^2 + 1} e + \\
&\quad 5 \sqrt[3]{e} e^2 \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e} e^3 + 416 \sqrt[3]{2} \sqrt[3]{e} e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e} e + 4 \sqrt[3]{2} \sqrt[3]{e} e^2 + 1} + 40 \sqrt[3]{e} e^2 e + 5 \cdot 2^{2/3} e
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{31_{denominator}} &= 20\sqrt[3]{e^2}(-9216e^4 - 512 \cdot 2^{2/3} \sqrt[3]{e}e^3 - 248\sqrt[3]{2}\sqrt[3]{e^2}e^2 - 24e^2 + \\
&\quad 96 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 2\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + \\
&\quad \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e - 2^{2/3} \sqrt[3]{e}e - \sqrt[3]{2}\sqrt[3]{e^2}}) \\
\mathbf{V}_{32_{numerator}} &= -3840 \cdot 2^{2/3} \sqrt[3]{e^2}e^4 - 1760\sqrt[3]{2}e^4 - 1040\sqrt[3]{e}e^3 - 160 \cdot 2^{2/3} \sqrt[3]{e^2}e^2 - 30\sqrt[3]{2}e^2 + \\
&\quad 80\sqrt[3]{2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^3} - \\
&\quad 40\sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} - \\
&\quad 10 \cdot 2^{2/3} \sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} - 5\sqrt[3]{e}e \\
\mathbf{V}_{32_{denominator}} &= 20e\sqrt[3]{e}(9216e^4 + 512 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 24e^2 - \\
&\quad 96 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} - \\
&\quad 2\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} - \\
&\quad \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e + 2^{2/3} \sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e^2}}) \\
\mathbf{V}_{33_{numerator}} &= -1536\sqrt[3]{2}\sqrt[3]{e^2}e^3 - 704e^3 - 208 \cdot 2^{2/3} \sqrt[3]{e}e^2 + \\
&\quad 32 \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} - \\
&\quad 8 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} - \\
&\quad 4\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} - \\
&\quad 64\sqrt[3]{2}\sqrt[3]{e^2}e - 12e - 2^{2/3} \sqrt[3]{e}e \\
\mathbf{V}_{33_{denominator}} &= 4(-9216e^4 - 512 \cdot 2^{2/3} \sqrt[3]{e}e^3 - 248\sqrt[3]{2}\sqrt[3]{e^2}e^2 - 24e^2 + \\
&\quad 96 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 2\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + \\
&\quad \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e - 2^{2/3} \sqrt[3]{e}e - \sqrt[3]{2}\sqrt[3]{e^2}})
\end{aligned}$$

$$\mathbf{V}_{34} = 1 \quad (\text{A27})$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ -\frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A28})$$

$$\begin{aligned}
\mathbf{V}_{41_{numerator}} &= -7680\sqrt[3]{e^2}e^3 - 160 \cdot 2^{2/3}e^3 - 40\sqrt[3]{2}\sqrt[3]{e}e^2 - \\
&\quad 80 \cdot 2^{2/3} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} - \\
&\quad 10\sqrt[3]{2}\sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} - \\
&\quad 5\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + 40\sqrt[3]{e^2}e + 5 \cdot 2^{2/3}e
\end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{41_{denominator}} &= 20\sqrt[3]{e^2}(9216e^4 + 512 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 24e^2 + \\
&\quad 96 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 2\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + \\
&\quad \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e + 2^{2/3} \sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e^2}}) \\
\mathbf{V}_{42_{numerator}} &= -3840 \cdot 2^{2/3} \sqrt[3]{e^2}e^4 - 1760\sqrt[3]{2}e^4 - 1040\sqrt[3]{e}e^3 - 160 \cdot 2^{2/3} \sqrt[3]{e^2}e^2 - 30\sqrt[3]{2}e^2 - \\
&\quad 80\sqrt[3]{2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^3} + \\
&\quad 40\sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 10 \cdot 2^{2/3} \sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} - 5\sqrt[3]{e}e \\
\mathbf{V}_{42_{denominator}} &= 20e\sqrt[3]{e}(9216e^4 + 512 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 24e^2 + \\
&\quad 96 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 2\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + \\
&\quad \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e + 2^{2/3} \sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e^2}}) \\
\mathbf{V}_{43_{numerator}} &= -1536\sqrt[3]{2}\sqrt[3]{e^2}e^3 - 704e^3 - 208 \cdot 2^{2/3} \sqrt[3]{e}e^2 - \\
&\quad 32 \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 8 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + \\
&\quad 4\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1} - \\
&\quad 64\sqrt[3]{2}\sqrt[3]{e^2}e - 12e - 2^{2/3} \sqrt[3]{e}e \\
\mathbf{V}_{43_{denominator}} &= 4(9216e^4 + 512 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 248\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 24e^2 + \\
&\quad 96 \cdot 2^{2/3} \sqrt[3]{e} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e^2} + \\
&\quad 2\sqrt[3]{2}\sqrt[3]{e^2} \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e} + \\
&\quad \sqrt{2304 \cdot 2^{2/3} \sqrt[3]{e}e^3 + 416\sqrt[3]{2}\sqrt[3]{e^2}e^2 + 224e^2 + 24 \cdot 2^{2/3} \sqrt[3]{e}e + 4\sqrt[3]{2}\sqrt[3]{e^2} + 1e + 2^{2/3} \sqrt[3]{e}e + \sqrt[3]{2}\sqrt[3]{e^2}})
\end{aligned}$$

$$\mathbf{V}_{44} = 1 \quad (\text{A29})$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} -\frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ -\frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ -\frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix} \quad (\text{A30})$$

Appendix D Closed Form Equations for Lemma 2 Condition Four

The Eigenvalues:

$$\lambda_1 = 0 \quad (\text{A31})$$

$$\lambda_2 = 0 \quad (\text{A32})$$

$$\lambda_3 = \frac{ce^{3/2} - \sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)}}{2c\sqrt[3]{e}} \quad (\text{A33})$$

$$\lambda_4 = \frac{ce^{3/2} + \sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)}}{2c\sqrt{e}} \quad (\text{A34})$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -c \\ -\frac{e}{c} \\ 0 \\ 1 \end{bmatrix} \quad (\text{A35})$$

$$\mathbf{V}_2 = \begin{bmatrix} -\frac{2^{2/3}c\sqrt[3]{e}}{\sqrt{e}} \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (\text{A36})$$

$$\mathbf{V}_{31_{\text{numerator}}} = c^3e^{3/2} - e^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} - c^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + ce^{7/2}$$

$$\mathbf{V}_{31_{\text{denominator}}} = -2c^4e^{3/2} - 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e^2} - 2c^2e^{7/2} - 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e^2} - c^2e^{3/2} + 2ce^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + c^3\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} - e^{7/2}$$

$$\mathbf{V}_{32_{\text{numerator}}} = c\left(2\sqrt[3]{2}c^4\sqrt[3]{e^2} + c^4e + c^2e^3 + 2\sqrt[3]{2}c^2e^2\sqrt[3]{e^2} - ce^{3/2}\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + c^2e + e^3\right)$$

$$\mathbf{V}_{32_{\text{denominator}}} = \sqrt{e}(2c^4e^{3/2} + 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e^2} + 2c^2e^{7/2} + 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e^2} + c^2e^{3/2} - 2ce^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} - c^3\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + e^{7/2})$$

$$\mathbf{V}_{33_{\text{numerator}}} = -2^{2/3}c^4e^2\sqrt[3]{e} + 2^{2/3}c^2e^4\sqrt[3]{e} - 2^{2/3}ce^{5/2}\sqrt[3]{e}\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} - 2^{2/3}c^3\sqrt{e}\sqrt[3]{e}\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)}$$

$$\mathbf{V}_{33_{\text{denominator}}} = -c^4e^{5/2} - c^2e^{9/2} - e^{3/2}(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) + 2ce^3\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + c^3e\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)}$$

$$\mathbf{V}_{34} = 1 \quad (\text{A37})$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} -\frac{\mathbf{V}_{31_{\text{numerator}}}}{\mathbf{V}_{31_{\text{denominator}}}} \\ \frac{\mathbf{V}_{32_{\text{numerator}}}}{\mathbf{V}_{32_{\text{denominator}}}} \\ -\frac{\mathbf{V}_{33_{\text{numerator}}}}{\mathbf{V}_{33_{\text{denominator}}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A38})$$

$$\mathbf{V}_{41_{\text{numerator}}} = -c^3e^{3/2} - e^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} - c^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} - ce^{7/2}$$

$$\mathbf{V}_{41_{\text{denominator}}} = 2c^4e^{3/2} + 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e^2} + 2c^2e^{7/2} + 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e^2} + c^2e^{3/2} + 2ce^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + c^3\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} + e^{7/2}$$

$$\begin{aligned}
\mathbf{V}_{42_{\text{numerator}}} &= c(2\sqrt[3]{2}c^4\sqrt[3]{e^2} + c^4e + c^2e^3 + 2\sqrt[3]{2}c^2e^2\sqrt[3]{e^2}) \\
&\quad ce^{3/2}\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) + c^2e + e^3} \\
\mathbf{V}_{42_{\text{denominator}}} &= \sqrt{e}(2c^4e^{3/2} + 2\sqrt[3]{2}c^4\sqrt{e}\sqrt[3]{e^2} + 2c^2e^{7/2} + 2\sqrt[3]{2}c^2e^{5/2}\sqrt[3]{e^2} + c^2e^{3/2} + \\
&\quad 2ce^2\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) + c^3\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) + e^{7/2}}) \\
\mathbf{V}_{43_{\text{numerator}}} &= 2^{2/3}c^4e^2\sqrt[3]{e} + 2^{2/3}c^2e^4\sqrt[3]{e} + 2^{2/3}ce^{5/2}\sqrt[3]{e}\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) + \\
&\quad 2^{2/3}c^3\sqrt{e}\sqrt[3]{e}\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} \\
\mathbf{V}_{43_{\text{denominator}}} &= -c^4e^{5/2} - c^2e^{9/2} - e^{3/2}(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) - \\
&\quad 2ce^3\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e) - c^3e\sqrt{(c^2 + e^2)(2\sqrt[3]{2}c^2\sqrt[3]{e^2} + c^2e + e)} \\
\mathbf{V}_{44} &= 1
\end{aligned} \tag{A39}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{\text{numerator}}}}{\mathbf{V}_{41_{\text{denominator}}}} \\ \frac{\mathbf{V}_{42_{\text{numerator}}}}{\mathbf{V}_{42_{\text{denominator}}}} \\ \frac{\mathbf{V}_{43_{\text{numerator}}}}{\mathbf{V}_{43_{\text{denominator}}}} \\ \mathbf{V}_{44} \end{bmatrix} \tag{A40}$$

Appendix E Closed Form Equations for Lemma 2 Condition Five

The Eigenvalues:

$$\lambda_1 = 0 \tag{A41}$$

$$\lambda_2 = 0 \tag{A42}$$

$$\lambda_3 = \frac{1}{2}(-\sqrt{2}\sqrt{b^2 - 3b + 2} - \sqrt{-b}) \tag{A43}$$

$$\lambda_4 = \frac{1}{2}(\sqrt{2}\sqrt{b^2 - 3b + 2} - \sqrt{-b}) \tag{A44}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -2\sqrt{-b} \\ 1 \\ 0 \\ 1 \end{bmatrix} \tag{A45}$$

$$\mathbf{V}_2 = \begin{bmatrix} b - 1 \\ \sqrt{-b} \\ 1 \\ 0 \end{bmatrix} \tag{A46}$$

$$\mathbf{V}_{31_{\text{numerator}}} = b^2 + \sqrt{2}\sqrt{-b}\sqrt{b^2 - 3b + 2b} - \sqrt{2}\sqrt{-b}\sqrt{b^2 - 3b + 2} - 4b + 4$$

$$\mathbf{V}_{31_{\text{denominator}}} = (b^2 - 2)(\sqrt{2}\sqrt{b^2 - 3b + 2} + \sqrt{-b})$$

$$\mathbf{V}_{32_{\text{numerator}}} = 3\sqrt{-bb^2} - \sqrt{2}\sqrt{b^2 - 3b + 2b^2} + 2\sqrt{2}\sqrt{b^2 - 3b + 2b} - 2\sqrt{2}\sqrt{b^2 - 3b + 2} - 8\sqrt{-bb} + 6\sqrt{-b}$$

$$\begin{aligned}
\mathbf{V}_{32_{denominator}} &= (b^2 - 2) \left(\sqrt{2} \sqrt{b^2 - 3b + 2} + \sqrt{-b} \right) \\
\mathbf{V}_{33_{numerator}} &= 2b^3 - 3b^2 + \sqrt{2} \sqrt{-b} \sqrt{b^2 - 3b + 2} - 2b + 4 \\
\mathbf{V}_{33_{denominator}} &= (b^2 - 2) \left(\sqrt{2} \sqrt{b^2 - 3b + 2} + \sqrt{-b} \right) \\
\mathbf{V}_{34} &= 1
\end{aligned} \tag{A47}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{numerator}}}{\mathbf{V}_{31_{denominator}}} \\ \frac{\mathbf{V}_{32_{numerator}}}{\mathbf{V}_{32_{denominator}}} \\ \frac{\mathbf{V}_{33_{numerator}}}{\mathbf{V}_{33_{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix} \tag{A48}$$

$$\begin{aligned}
\mathbf{V}_{41_{numerator}} &= b^2 - \sqrt{2} \sqrt{-b} \sqrt{b^2 - 3b + 2} + \sqrt{2} \sqrt{-b} \sqrt{b^2 - 3b + 2} - 4b + 4 \\
\mathbf{V}_{41_{denominator}} &= (b^2 - 2) \left(\sqrt{-b} - \sqrt{2} \sqrt{b^2 - 3b + 2} \right) \\
\mathbf{V}_{42_{numerator}} &= 3\sqrt{-b}b^2 + \sqrt{2} \sqrt{b^2 - 3b + 2}b^2 - 2\sqrt{2} \sqrt{b^2 - 3b + 2} + 2\sqrt{2} \sqrt{b^2 - 3b + 2} - 8\sqrt{-b}b + 6\sqrt{-b} \\
\mathbf{V}_{42_{denominator}} &= (b^2 - 2) \left(\sqrt{-b} - \sqrt{2} \sqrt{b^2 - 3b + 2} \right) \\
\mathbf{V}_{43_{numerator}} &= 2b^3 - 3b^2 - \sqrt{2} \sqrt{-b} \sqrt{b^2 - 3b + 2} - 2b + 4 \\
\mathbf{V}_{43_{denominator}} &= (b^2 - 2) \left(\sqrt{-b} - \sqrt{2} \sqrt{b^2 - 3b + 2} \right) \\
\mathbf{V}_{44} &= 1
\end{aligned} \tag{A49}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix} \tag{A50}$$

Appendix F Closed Form Equations for Lemma 2 Condition Six

The Eigenvalues:

$$\lambda_1 = 0 \tag{A51}$$

$$\lambda_2 = 0 \tag{A52}$$

$$\lambda_3 = -\frac{\sqrt{b^2 - 3b + 2}}{\sqrt{2}} - \frac{\sqrt{-b}}{2} \tag{A53}$$

$$\lambda_4 = \frac{\sqrt{b^2 - 3b + 2}}{\sqrt{2}} - \frac{\sqrt{-b}}{2} \tag{A54}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -2\sqrt{-b} \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A55})$$

$$\mathbf{V}_2 = \begin{bmatrix} b-1 \\ \sqrt{-b} \\ 1 \\ 0 \end{bmatrix} \quad (\text{A56})$$

$$\begin{aligned} \mathbf{V}_{31_{\text{numerator}}} &= -b^2 + \sqrt{2}(-b)^{3/2}\sqrt{b^2-3b+2} + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} + 4b - 4 \\ \mathbf{V}_{31_{\text{denominator}}} &= (b^2 - 2) \left(\sqrt{2}\sqrt{b^2-3b+2} + \sqrt{-b} \right) \\ \mathbf{V}_{32_{\text{numerator}}} &= \sqrt{2}b^2\sqrt{b^2-3b+2} - 2\sqrt{2}b\sqrt{b^2-3b+2} + 2\sqrt{2}\sqrt{b^2-3b+2} - 3(-b)^{5/2} + 8b\sqrt{-b} - 6\sqrt{-b} \\ \mathbf{V}_{32_{\text{denominator}}} &= (b^2 - 2) \left(\sqrt{2}\sqrt{b^2-3b+2} + \sqrt{-b} \right) \\ \mathbf{V}_{33_{\text{numerator}}} &= -2b^3 + 3b^2 - \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} + 2b - 4 \\ \mathbf{V}_{33_{\text{denominator}}} &= (b^2 - 2) \left(\sqrt{2}\sqrt{b^2-3b+2} + \sqrt{-b} \right) \\ \mathbf{V}_{34} &= 1 \end{aligned} \quad (\text{A57})$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{\text{numerator}}}}{\mathbf{V}_{31_{\text{denominator}}}} \\ \frac{\mathbf{V}_{32_{\text{numerator}}}}{\mathbf{V}_{32_{\text{denominator}}}} \\ \frac{\mathbf{V}_{33_{\text{numerator}}}}{\mathbf{V}_{33_{\text{denominator}}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A58})$$

$$\begin{aligned} \mathbf{V}_{41_{\text{numerator}}} &= b^2 + \sqrt{2}(-b)^{3/2}\sqrt{b^2-3b+2} + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} - 4b + 4 \\ \mathbf{V}_{41_{\text{denominator}}} &= (b^2 - 2) \left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2} \right) \\ \mathbf{V}_{42_{\text{numerator}}} &= \sqrt{2}b^2\sqrt{b^2-3b+2} - 2\sqrt{2}b\sqrt{b^2-3b+2} + 2\sqrt{2}\sqrt{b^2-3b+2} + 3(-b)^{5/2} + 8(-b)^{3/2} + 6\sqrt{-b} \\ \mathbf{V}_{42_{\text{denominator}}} &= (b^2 - 2) \left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2} \right) \\ \mathbf{V}_{43_{\text{numerator}}} &= -2b^3 + 3b^2 + \sqrt{2}\sqrt{-b}\sqrt{b^2-3b+2} + 2b - 4 \\ \mathbf{V}_{43_{\text{denominator}}} &= (b^2 - 2) \left(\sqrt{-b} - \sqrt{2}\sqrt{b^2-3b+2} \right) \\ \mathbf{V}_{44} &= 1 \end{aligned} \quad (\text{A59})$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{\text{numerator}}}}{\mathbf{V}_{41_{\text{denominator}}}} \\ \frac{\mathbf{V}_{42_{\text{numerator}}}}{\mathbf{V}_{42_{\text{denominator}}}} \\ \frac{\mathbf{V}_{43_{\text{numerator}}}}{\mathbf{V}_{43_{\text{denominator}}}} \\ \mathbf{V}_{44} \end{bmatrix} \quad (\text{A60})$$

Appendix G Closed Form Equations for Lemma 2 Condition Seven

The Eigenvalues:

$$\lambda_1 = 0 \quad (\text{A61})$$

$$\lambda_2 = 0 \quad (\text{A62})$$

$$\lambda_3 = -\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} + \frac{bd}{\sqrt{-b}} + (-b)^{5/2} - (-b)^{3/2} - \sqrt{-b} \quad (\text{A63})$$

$$\lambda_4 = \sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} + \frac{bd}{\sqrt{-b}} + (-b)^{5/2} - (-b)^{3/2} - \sqrt{-b} \quad (\text{A64})$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} \frac{d}{\sqrt{-b}} - 2(-b)^{3/2} \\ b \\ 0 \\ 1 \end{bmatrix} \quad (\text{A65})$$

$$\mathbf{V}_2 = \begin{bmatrix} 0 \\ -\sqrt{-b} \\ 1 \\ 0 \end{bmatrix} \quad (\text{A66})$$

$$\mathbf{V}_{31_{\text{numerator}}} = (b^2 - b + 1) \left(\sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - \frac{bd}{\sqrt{-b}} - (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b} \right)$$

$$\begin{aligned} \mathbf{V}_{31_{\text{denominator}}} &= 2(b^2 - b + 1)d^2 + \frac{d}{\sqrt{-b}} \left(2b^2 \sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - \right. \\ &\quad \left. b \sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} + \sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - \right. \\ &\quad \left. 4(-b)^{9/2} - 3(-b)^{7/2} - 2(-b)^{5/2} + 2(-b)^{3/2} + \sqrt{-b} \right) - \\ &\quad b \left(-2b^5 + 2b^4 - 2b^3 + b^2 - 2(-b)^{5/2} \sqrt{(b^2 - b + 1)\left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1\right)} - b + 1 \right) \end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{32_{\text{numerator}}} &= 2b^4 - 4b^3 - \frac{(b^2 - b + 1)d^2}{b} + 5b^2 + \\
&\frac{bd \left(-\sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b} \right)}{\sqrt{-b}} + \\
&2(-b)^{3/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \\
&2\sqrt{-b} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - 3b + 1 \\
\mathbf{V}_{32_{\text{denominator}}} &= 2(b^2 - b + 1)d^2 + \frac{d}{\sqrt{-b}} \left(2b^2 \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \right. \\
&b \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \\
&4(-b)^{9/2} - 3(-b)^{7/2} - 2(-b)^{5/2} + 2(-b)^{3/2} + \sqrt{-b} \left. \right) - \\
&b \left(-2b^5 + 2b^4 - 2b^3 + b^2 - 2(-b)^{5/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - b + 1 \right) \\
\mathbf{V}_{33_{\text{numerator}}} &= \sqrt{-b} \left(2b^4 - 4b^3 - \frac{(b^2 - b + 1)d^2}{b} + 5b^2 + \right. \\
&\frac{bd \left(-\sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b} \right)}{\sqrt{-b}} + \\
&2(-b)^{3/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \\
&\left. 2\sqrt{-b} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - 3b + 1 \right) \\
\mathbf{V}_{33_{\text{denominator}}} &= 2(b^2 - b + 1)d^2 + \frac{d}{\sqrt{-b}} \left(2b^2 \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \right. \\
&b \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \\
&4(-b)^{9/2} - 3(-b)^{7/2} - 2(-b)^{5/2} + 2(-b)^{3/2} + \sqrt{-b} \left. \right) - \\
&b \left(-2b^5 + 2b^4 - 2b^3 + b^2 - 2(-b)^{5/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - b + 1 \right)
\end{aligned}$$

$$\mathbf{V}_{34} = 1 \tag{A67}$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{\text{numerator}}}}{\mathbf{V}_{31_{\text{denominator}}}} \\ \frac{\mathbf{V}_{32_{\text{numerator}}}}{\mathbf{V}_{32_{\text{denominator}}}} \\ \frac{\mathbf{V}_{33_{\text{numerator}}}}{\mathbf{V}_{33_{\text{denominator}}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A68})$$

$$\mathbf{V}_{41_{\text{numerator}}} = (b^2 - b + 1) \left(\sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \frac{bd}{\sqrt{-b}} + (-b)^{5/2} - (-b)^{3/2} - \sqrt{-b} \right)$$

$$\begin{aligned} \mathbf{V}_{41_{\text{denominator}}} &= 2(b^2 - b + 1)d^2 - \frac{d}{\sqrt{-b}} \left(2b^2 \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \right. \\ &\quad \left. b \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \right. \\ &\quad \left. 4(-b)^{9/2} + 3(-b)^{7/2} + 2(-b)^{5/2} - 2(-b)^{3/2} - \sqrt{-b} \right) - \\ &\quad b \left(-2b^5 + 2b^4 - 2b^3 + b^2 + 2(-b)^{5/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - b + 1 \right) \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{42_{\text{numerator}}} &= 2b^4 - 4b^3 - \frac{(b^2 - b + 1)d^2}{b} + \\ &\quad 5b^2 + \frac{bd \left(\sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b} \right)}{\sqrt{-b}} - \\ &\quad 2(-b)^{3/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \\ &\quad 2\sqrt{-b} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - 3b + 1 \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{42_{\text{denominator}}} &= 2(b^2 - b + 1)d^2 - \frac{d}{\sqrt{-b}} \left(2b^2 \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \right. \\ &\quad \left. b \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \right. \\ &\quad \left. 4(-b)^{9/2} + 3(-b)^{7/2} + 2(-b)^{5/2} - 2(-b)^{3/2} - \sqrt{-b} \right) - \\ &\quad b \left(-2b^5 + 2b^4 - 2b^3 + b^2 + 2(-b)^{5/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - b + 1 \right) \end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{43_{\text{numerator}}} &= \sqrt{-b} \left(2b^4 - 4b^3 - \frac{(b^2 - b + 1)d^2}{b} + 5b^2 + \right. \\
&\quad \left. \frac{bd \left(\sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + (-b)^{5/2} + (-b)^{3/2} + \sqrt{-b} \right)}{\sqrt{-b}} \right. \\
&\quad \left. 2(-b)^{3/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \right. \\
&\quad \left. 2\sqrt{-b} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - 3b + 1 \right) \\
\mathbf{V}_{43_{\text{denominator}}} &= 2(b^2 - b + 1)d^2 - \frac{d}{\sqrt{-b}} \left(2b^2 \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - \right. \\
&\quad \left. b \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} + \right. \\
&\quad \left. 4(-b)^{9/2} + 3(-b)^{7/2} + 2(-b)^{5/2} - 2(-b)^{3/2} - \sqrt{-b} \right) - \\
&\quad \left. b \left(-2b^5 + 2b^4 - 2b^3 + b^2 + 2(-b)^{5/2} \sqrt{(b^2 - b + 1) \left(-b^3 + b^2 - \frac{d^2}{b} + 2bd - b + 1 \right)} - b + 1 \right) \right) \\
\mathbf{V}_{44} &= 1
\end{aligned} \tag{A69}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{\text{numerator}}}}{\mathbf{V}_{41_{\text{denominator}}}} \\ \frac{\mathbf{V}_{42_{\text{numerator}}}}{\mathbf{V}_{42_{\text{denominator}}}} \\ \frac{\mathbf{V}_{43_{\text{numerator}}}}{\mathbf{V}_{43_{\text{denominator}}}} \\ \mathbf{V}_{44} \end{bmatrix} \tag{A70}$$

Appendix H Closed Form Equations for Lemma 2 Condition Eight

The Eigenvalues:

$$\lambda_1 = 0 \tag{A71}$$

$$\lambda_2 = 0 \tag{A72}$$

$$\lambda_3 = \frac{-\frac{e^3}{d^{3/2}} - \sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d \left(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2 \right) e^2 + \frac{8(d+\sqrt{d}+1)e^4}{d} + 4\sqrt{d}(d-\sqrt{d}+1)e}}{8d^{3/2}} \tag{A73}$$

$$\lambda_4 = \frac{-\frac{e^3}{d^{3/2}} + \sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d \left(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2 \right) e^2 + \frac{8(d+\sqrt{d}+1)e^4}{d} + 4\sqrt{d}(d-\sqrt{d}+1)e}}{8d^{3/2}} \tag{A74}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} 0 \\ -\sqrt{d} \\ 0 \\ 1 \end{bmatrix} \tag{A75}$$

$$\mathbf{V}_2 = \begin{bmatrix} -\frac{e^2}{2d^2} - \sqrt{d} \\ \frac{e}{2d} \\ 1 \\ 0 \end{bmatrix} \quad (\text{A76})$$

$$\begin{aligned} \mathbf{V}_{31_{\text{numerator}}} &= \left(4(d^2 + d) + \frac{e^2}{d}\right) \left(-\frac{e^3}{d^{3/2}} - \sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}} + 4\sqrt{d}(d - \sqrt{d} + 1)e\right) \\ \mathbf{V}_{31_{\text{denominator}}} &= 2d \left(\frac{(2d + \sqrt{d} + 2)e^4}{d^2} + 16d^2(d+1)^2 + 4(d^{3/2} + 2d^2 + 5d + \sqrt{d} + 3)e^2 - \frac{(2d - \sqrt{d} + 2)e\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}}}{\sqrt{d}}\right) \\ \mathbf{V}_{32} &= \frac{1}{\sqrt{d}} \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{33_{\text{numerator}}} &= -\frac{e^7}{d^{7/2}} - \frac{4(\sqrt{d} + 1)^2 e^5}{d^{3/2}} - 8\sqrt{d}(3d^{3/2} + 2d + 3\sqrt{d} + 1)e^3 + 32d^{5/2}(d^{5/2} + 2d^{3/2} - 2d^2 - 3d + \sqrt{d} - 1)e - \frac{e^4\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}}}{d^2} \\ &\quad - 4\sqrt{d}e^2\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}} - 8d^{5/2}(d+1)\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}} \\ \mathbf{V}_{33_{\text{denominator}}} &= 4d^2 \left(\frac{(2d + \sqrt{d} + 2)e^4}{d^2} + 16d^2(d+1)^2 + 4(d^{3/2} + 2d^2 + 5d + \sqrt{d} + 3)e^2 - \frac{(2d - \sqrt{d} + 2)e\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}}}{\sqrt{d}}\right) \\ \mathbf{V}_{34} &= 1 \end{aligned} \quad (\text{A77})$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \frac{\mathbf{V}_{31_{\text{numerator}}}}{\mathbf{V}_{31_{\text{denominator}}}} \\ \mathbf{V}_{32} \\ \frac{\mathbf{V}_{33_{\text{numerator}}}}{\mathbf{V}_{33_{\text{denominator}}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A78})$$

$$\begin{aligned}
\mathbf{V}_{41_{\text{numerator}}} &= \left(4(d^2 + d) + \frac{e^2}{d}\right) \left(-\frac{e^3}{d^{3/2}} + \sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}} + 4\sqrt{d}(d - \sqrt{d} + 1)e\right) \\
\mathbf{V}_{41_{\text{denominator}}} &= 2d \left(\frac{(2d + \sqrt{d} + 2)e^4}{d^2} + 16d^2(d+1)^2 + 4(d^{3/2} + 2d^2 + 5d + \sqrt{d} + 3)e^2 - \frac{(-2d + \sqrt{d} - 2)e\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}}}{\sqrt{d}}\right) \\
\mathbf{V}_{42} &= \frac{1}{\sqrt{d}} \\
\mathbf{V}_{43_{\text{numerator}}} &= -\frac{e^7}{d^{7/2}} - \frac{4(\sqrt{d} + 1)^2 e^5}{d^{3/2}} - 8\sqrt{d}(3d^{3/2} + 2d + 3\sqrt{d} + 1)e^3 + 32d^{5/2}(d^{5/2} + 2d^{3/2} - 2d^2 - 3d + \sqrt{d} - 1)e + \frac{e^4\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}}}{d^2} + 4\sqrt{d}e^2\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}} + 8d^{5/2}(d+1)\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}} \\
\mathbf{V}_{43_{\text{denominator}}} &= 4d^2 \left(\frac{(2d + \sqrt{d} + 2)e^4}{d^2} + 16d^2(d+1)^2 + 4(d^{3/2} + 2d^2 + 5d + \sqrt{d} + 3)e^2 - \frac{(-2d + \sqrt{d} - 2)e\sqrt{\frac{e^6}{d^3} + 64d^3(d+1)^2 + 16d(2d^{3/2} + d^2 + 3d + 2\sqrt{d} + 2)e^2 + \frac{8(d + \sqrt{d} + 1)e^4}{d}}}{\sqrt{d}}\right) \\
\mathbf{V}_{44} &= 1
\end{aligned} \tag{A79}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{\text{numerator}}}}{\mathbf{V}_{41_{\text{denominator}}}} \\ \mathbf{V}_{42} \\ \frac{\mathbf{V}_{43_{\text{numerator}}}}{\mathbf{V}_{43_{\text{denominator}}}} \\ \mathbf{V}_{44} \end{bmatrix} \tag{A80}$$

Appendix I Closed Form Equations for Lemma 3 Condition One

The Eigenvalues:

$$\lambda_1 = 0 \tag{A81}$$

$$\lambda_2 = 0 \tag{A82}$$

$$\lambda_3 = \frac{1}{2} \left(-\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1} + e - 2^{2/3}\sqrt[3]{e} + 1 \right) \tag{A83}$$

$$\lambda_4 = \frac{1}{2} \left(\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e - 2^{2/3}\sqrt[3]{e} + 1} \right) \quad (\text{A84})$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -\sqrt{e} \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{A85})$$

$$\mathbf{V}_2 = \begin{bmatrix} -\frac{\sqrt[3]{e}}{\sqrt[3]{2}} \\ -\frac{\sqrt{e}}{2^{2/3}\sqrt[3]{e}} \\ 1 \\ 0 \end{bmatrix} \quad (\text{A86})$$

$$\mathbf{V}_{31} = \frac{1}{\sqrt{e}}$$

$$\mathbf{V}_{32_{\text{numerator}}} = 4 \left(\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) \sqrt[3]{e^2} + 2^{2/3}e \left(\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 5} \right) + 6\sqrt[3]{2}e\sqrt[3]{e}$$

$$\mathbf{V}_{32_{\text{denominator}}} = e\sqrt[3]{e} \left(\sqrt[3]{2} \left(-\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) - 2\sqrt[3]{e} \right)$$

$$\mathbf{V}_{33_{\text{numerator}}} = 2 \left(\sqrt[3]{2} \left(\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) \sqrt[3]{e} + (e+3)\sqrt[3]{e^2} + 2^{2/3}e \right)$$

$$\mathbf{V}_{33_{\text{denominator}}} = \sqrt{e}\sqrt[3]{e} \left(\sqrt[3]{2} \left(-\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) - 2\sqrt[3]{e} \right)$$

$$\mathbf{V}_{34} = 1 \quad (\text{A87})$$

Where

$$\mathbf{V}_3 = \begin{bmatrix} \mathbf{V}_{31} \\ \frac{\mathbf{V}_{32_{\text{numerator}}}}{\mathbf{V}_{32_{\text{denominator}}}} \\ \frac{\mathbf{V}_{33_{\text{numerator}}}}{\mathbf{V}_{33_{\text{denominator}}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A88})$$

$$\mathbf{V}_{41_{\text{numerator}}} = \frac{1}{\sqrt{e}}$$

$$\mathbf{V}_{42_{\text{numerator}}} = \left(-4\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + 4e + 4} \right) \sqrt[3]{e^2} + 2^{2/3}e \left(-\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 5} \right) + 6\sqrt[3]{2}e\sqrt[3]{e}$$

$$\mathbf{V}_{42_{\text{denominator}}} = e\sqrt[3]{e} \left(\sqrt[3]{2} \left(\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) - 2\sqrt[3]{e} \right)$$

$$\mathbf{V}_{43_{\text{numerator}}} = 2 \left(\sqrt[3]{2} \left(-\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 2 \cdot 2^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) \sqrt[3]{e} + (e+3)\sqrt[3]{e^2} + 2^{2/3}e \right)$$

$$\mathbf{V}_{43_{denominator}} = \sqrt{e} \sqrt[3]{e} \left(\sqrt[3]{2} \left(\sqrt{e^2 + 6e + 2\sqrt[3]{2}(e+2)\sqrt[3]{e^2} + 22^{2/3}(e+1)\sqrt[3]{e} + 1 + e + 1} \right) - 2\sqrt[3]{e} \right)$$

$$\mathbf{V}_{44} = 1 \quad (\text{A89})$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \mathbf{V}_{41} \\ \frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ \frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix} \quad (\text{A90})$$

Appendix J Closed Form Equations for Lemma 3 Condition Two

The Eigenvalues:

$$\lambda_1 = 0 \quad (\text{A91})$$

$$\lambda_2 = 0 \quad (\text{A92})$$

$$\lambda_3 = \left(\frac{2}{5}\right)^{2/3} e^{2/3} - \frac{1}{4} \sqrt{-\frac{64}{5} \left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + \frac{e}{2} + \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + \frac{1}{2}} \quad (\text{A93})$$

$$\lambda_4 = \frac{1}{4} \left(4 \left(\frac{2}{5}\right)^{2/3} e^{2/3} + \sqrt{-\frac{64}{5} \left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2e + 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 2} \right) \quad (\text{A94})$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -\frac{2\sqrt{e}}{\sqrt{5}} \\ \frac{\sqrt[3]{e}}{2^{2/3} \sqrt[3]{5}} \\ 0 \\ 1 \end{bmatrix} \quad (\text{A95})$$

$$\mathbf{V}_2 = \begin{bmatrix} \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} \\ \frac{\sqrt[6]{e}}{\sqrt[3]{2} \sqrt[3]{5}} \\ 1 \\ 0 \end{bmatrix} \quad (\text{A96})$$

$$\begin{aligned}
\mathbf{V}_{31_{\text{numerator}}} &= \sqrt{5} \left(2 \left(\frac{2}{5} \right)^{2/3} e^{5/3} + \frac{8}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} - \frac{8e^2}{5} + \right. \\
&\quad \frac{2}{5} \left(2 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 3} \right) e - \\
&\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 2} \right) e^{2/3} + \right. \\
&\quad \left. \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} - 2} \right) \right) \\
\mathbf{V}_{31_{\text{denominator}}} &= \sqrt{e} \left(-\frac{22}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 4e^2 + \right. \\
&\quad \frac{2}{5} \left(4 - 5 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4} \right) e - \\
&\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 2} \right) e^{2/3} - \right. \\
&\quad \left. 2 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 8 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4} \right) \right) \\
\mathbf{V}_{32_{\text{numerator}}} &= 2 \sqrt[3]{\frac{2}{5}} \left(\sqrt[3]{\frac{2}{5}} \sqrt[3]{e} - 1 \right) \sqrt[3]{e} \left(4 \left(\frac{2}{5} \right)^{2/3} e^{2/3} - \right. \\
&\quad \left. \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2e + 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 2} \right) \\
\mathbf{V}_{32_{\text{denominator}}} &= -\frac{22}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 4e^2 + \right. \\
&\quad \frac{2}{5} \left(4 - 5 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4} \right) e - \\
&\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 2} \right) e^{2/3} - \right. \\
&\quad \left. 2 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 8 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4} \right) \right) \\
\mathbf{V}_{33_{\text{numerator}}} &= \sqrt[3]{2} \sqrt[3]{5} \left(-\frac{8}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{8}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + \frac{4e^2}{5} - \right. \\
&\quad \frac{2}{5} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 5} \right) e - \\
&\quad \left. 2 \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 2} \right) e^{2/3} - \right. \\
&\quad \left. \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 2} \right) \right) \\
\mathbf{V}_{33_{\text{denominator}}} &= \sqrt[3]{e} \left(-\frac{22}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 4e^2 + \right. \\
&\quad \frac{2}{5} \left(4 - 5 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4} \right) e - \\
&\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 2} \right) e^{2/3} - \right. \\
&\quad \left. 2 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 8 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4} \right) \right)
\end{aligned}$$

$$\mathbf{V}_{34} = 1$$

(A97)

Where

$$\mathbf{V}_3 = \begin{bmatrix} -\mathbf{V}_{31_{\text{numerator}}} \\ \mathbf{V}_{31_{\text{denominator}}} \\ -\mathbf{V}_{32_{\text{numerator}}} \\ \mathbf{V}_{32_{\text{denominator}}} \\ -\mathbf{V}_{33_{\text{numerator}}} \\ \mathbf{V}_{33_{\text{denominator}}} \\ \mathbf{V}_{34} \end{bmatrix} \quad (\text{A98})$$

$$\begin{aligned} \mathbf{V}_{41_{\text{numerator}}} &= \sqrt{5} \left(-2 \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{8}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + \frac{8e^2}{5} + \right. \\ &\quad \left. \frac{2}{5} \left(2 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 3} \right) e + \right. \\ &\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2} \right) e^{2/3} + \right. \\ &\quad \left. \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 2} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{41_{\text{denominator}}} &= \sqrt{e} \left(-\frac{22}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 4e^2 + \right. \\ &\quad \left. \frac{2}{5} \left(5 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 4} \right) e + \right. \\ &\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2} \right) e^{2/3} + \right. \\ &\quad \left. 2 \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2} \right) - 8 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{42_{\text{numerator}}} &= 2 \sqrt[3]{\frac{2}{5}} \left(\sqrt[3]{\frac{2}{5}} \sqrt[3]{e} - 1 \right) \sqrt[3]{e} \left(4 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + \right. \\ &\quad \left. \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2e + 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 2} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{42_{\text{denominator}}} &= -\frac{22}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 4e^2 + \right. \\ &\quad \left. \frac{2}{5} \left(5 \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 4} \right) e + \right. \\ &\quad \left. \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2} \right) e^{2/3} + \right. \\ &\quad \left. 2 \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2} \right) - 8 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} \right) \end{aligned}$$

$$\begin{aligned} \mathbf{V}_{43_{\text{numerator}}} &= \sqrt[3]{2} \sqrt[3]{5} \left(-\frac{8}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{8}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + \frac{4e^2}{5} + \right. \\ &\quad \left. \frac{2}{5} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 5} \right) e + \right. \\ &\quad \left. 2 \left(\frac{2}{5} \right)^{2/3} \left(\sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2} \right) e^{2/3} + \right. \\ &\quad \left. \sqrt{-\frac{64}{5} \left(\frac{2}{5} \right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5} \right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 - 4 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 2} \right) \end{aligned}$$

$$\begin{aligned}
\mathbf{V}_{43_{denominator}} &= \sqrt[6]{e} \left(-\frac{22}{5} \left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 4e^2 + \right. \\
&\quad \left. \frac{2}{5} \left(5\sqrt{-\frac{64}{5}} \left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 4 \right) e + \right. \\
&\quad \left. \left(\frac{2}{5}\right)^{2/3} \left(\sqrt{-\frac{64}{5}} \left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2 \right) e^{2/3} + \right. \\
&\quad \left. 2 \left(\sqrt{-\frac{64}{5}} \left(\frac{2}{5}\right)^{2/3} e^{5/3} - \frac{16}{5} \sqrt[3]{\frac{2}{5}} e^{4/3} + 16 \left(\frac{2}{5}\right)^{2/3} e^{2/3} + 4e^2 + \frac{24e}{5} - 16 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} + 4 + 2 \right) - 8 \sqrt[3]{\frac{2}{5}} \sqrt[3]{e} \right) \\
\mathbf{V}_{44} &= 1 \tag{A99}
\end{aligned}$$

Where

$$\mathbf{V}_4 = \begin{bmatrix} \frac{\mathbf{V}_{41_{numerator}}}{\mathbf{V}_{41_{denominator}}} \\ -\frac{\mathbf{V}_{42_{numerator}}}{\mathbf{V}_{42_{denominator}}} \\ -\frac{\mathbf{V}_{43_{numerator}}}{\mathbf{V}_{43_{denominator}}} \\ \mathbf{V}_{44} \end{bmatrix} \tag{A100}$$

Appendix K Closed Form Equations for Lemma 3 Condition Three

The Eigenvalues:

$$\lambda_1 = 0 \tag{A101}$$

$$\lambda_2 = 0 \tag{A102}$$

$$\lambda_3 = a + 2\sqrt{c} \tag{A103}$$

$$\lambda_4 = 1 + c + b^2 \tag{A104}$$

Corresponding Eigenvectors:

$$\mathbf{V}_1 = \begin{bmatrix} -b \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{A105}$$

$$\mathbf{V}_2 = \begin{bmatrix} \sqrt{c} \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{A106}$$

$$\mathbf{V}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{A107}$$

$$\mathbf{V}_4 = \begin{bmatrix} \frac{1}{b} \\ 0 \\ -\frac{\sqrt{c}}{b} \\ 1 \end{bmatrix} \tag{A108}$$

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