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An Independence Result on Riemann Hypothesis: A Gödel Metalogic Approach

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Abstract

The present work studies the Riemann hypothesis from metalogical perspectives. It argues that Riemann hypothesis is independent of the current Riemann analytic continuation. Consequently, as a corollary, if the Riemann hypothesis held, its predicting power on the prime density would be incomplete. This argument is based on the modifications of Gödel's independent result (1931). This paper shows integrations of Riemann hypothesis and the Gödel structure. On one hand, Riemann hypothesis is construed into the Gödel structure by making a number of modifications. On the other hand, the Gödel structure is applied to disclose the metalogic behind the Riemann hypothesis.

Keywords: Riemann hypothesis; Gödel structure; Riemann monad; independent result; Gödel coding; expressibility; self-reference; prediction; prime density; semantics

1. Introduction

1.1 The Riemann Hypothesis is a long-standing open conjecture in mathematics. This mathematical conjecture was proposed by Riemann in 1859. The Riemann hypothesis concerns with the Riemann ζ function, which is as follows: for any complex number s,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, Re(s) > 1 \qquad (1.1)$$

The Riemann analytic continuation of the Riemann ζ function is as follows,

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{-\infty}^{\infty} \frac{(-z)^s}{e^z - 1} \frac{dz}{z} \qquad (1.2)$$

where the Γ function, Γ(s), is the analytic continuation of the factorial function on the complex plane.

Riemann Hypothesis: All the non-trivial zeroes of $\zeta(s)$ are on the line $\sigma = \frac{1}{2}$ (called the critical line); i.e., $\mathrm{Re}(s) = \frac{1}{2}$, when $\zeta(s) = 0$.

The Riemann hypothesis is important because it rather accurately predicts the distribution of prime numbers (Mazur and Stein, 2016).

1.2. The Riemann monad

Definition 1. Let Z be the category that contains all those complex numbers s as objects that satisfy Riemann hypothesis; i.e., $\zeta(s) = 0$ and $\mathrm{Re}(s) = \frac{1}{2}$. Plus, we introduce a monad A, called the Riemann monad, over the category Z which stands for Riemann hypothesis.

The above definition is a brief version, which demands some explanation (Hongbin Wang, personal communication, July 29, 2025). Let z denote the set of complex numbers s that satisfies the Riemann hypothesis; that is, $\zeta(s) = 0$ and $\mathrm{Re}(s) = \frac{1}{2}$. From this set, we construct a free (strict) symmetric monoidal category Z, using the tensor product ⊗ and a unit object I. Within Z, there

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exists an object X such that $X \cong X \otimes X$, indicating that X is a monoid object in Z. This allows us to define an endofunctor $\theta_{RH} = X \otimes - : Z \to$ Z, which satisfies the monad axioms. We refer to θ_{RH} as the *Riemann monad*.

2. Gödel Formula and Coding

Gödel (1931) constructed a mother-statement as follows:

$$\mathrm{P}(x) = \forall y \neg \mathrm{G}(x, y) \quad (2.1)$$

where x is a free variable. This formula was based on the first-order theory. As an analytic continuation, we allow both variables, x and y, to range over the complex numbers. This formula is assigned a Gödel code as follows,

$$\mathrm{g}[\mathrm{P}(x)] = \theta_{RH} \quad (2.2)$$

We then construct a Self-Referential Daughter Formula,

$$\mathrm{S} = \mathrm{P}(\theta_{RH}/x) \quad (2.3)$$

where free variable x is replaced by the Riemann monad. This daughter formula is assigned the Gödel code by the density function of primes, D_n, for any given large N; we have:

$$\mathrm{g}[(\mathrm{Pre}(\mathrm{S})] = D_n \quad (2.4)$$

Definition 2. *The Gödel Relation is defined as follows:*

$$\mathbb{G} = \mathbb{G}(\theta_{RH},\ D_n) \quad (2.5)$$

Definition 3. *The expressibility is defined by the following two conditions:*

$$\vDash_m \mathbb{G}(\theta_{RH},\ D_n) \Longrightarrow\ \vdash_L \mathrm{G}(\boldsymbol{\theta_{RH}},\ \boldsymbol{D_n}) \quad (2.6)$$

$$\nvDash_m \mathbb{G}(\theta_{RH},\ D_n) \Longrightarrow\ \vdash_L \neg\mathrm{G}(\boldsymbol{\theta_{RH}},\ \boldsymbol{D_n}) \quad (2.7)$$

3. The Theorem of Independence

Theorem 1 (Independence).

$$\nvdash \mathrm{S} \text{ and } \nvdash \neg\mathrm{S} \quad (2.1)$$

Proof for $\nvdash$ S. Assume for contradiction that $\vdash$ S. Then by $\vDash_m \mathbb{G}(\theta_{RH},\ D_n)$ and expressibility we have $\vdash_L$ G $(\boldsymbol{\theta_{RH}},\ \boldsymbol{D_n})$. which is contradicts with formula (2.1). Thus, $\nvdash$ S. (QED)

Proof for $\nvdash \neg$S. Assume for contradiction that $\vdash \neg$S, which implies that S doesn't hold. Then, S has no predicting power for D_n. In other words, we have:

$$\nvDash_m \mathbb{G}\ (\theta_{RH},\ D_n)$$

Hence, by the expressibility, it implies:

$$\vdash_L \neg\mathrm{G}\ (\boldsymbol{\theta_{RH}},\ \boldsymbol{D_n})$$

By the formula (2.1), this means that S holds, which contradicts to the assumption $\vdash \neg$S. Hence, $\nvdash \neg$S must hold. (QED)

Theorem 1 shows that S is independent of the current framework of analytic number theory. It also shows the following

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Corollary 1 (Incompleteness). By Theorem 1, it is easy to see by speculation that if the Riemann hypothesis held, its predicting power on the prime density would be incomplete (Marharis, 1967).

4. The Gödel Structure

The Gödel structure is based on the first order theory (Marharis, 1967), which is an integration of the first order logic and Peano Arithmetic. Note that the first order theory involves re-formalization process that is not necessary to our present work here. We only make a simple difference between logic and mathematics when it is necessary. We denote L as logical and M as mathematical, respectively. The Gödel structure created four technics which are listed below.

Technic 1. For the natural numbers n, write it in bold **n** when it is used in logic (e.g., the first-order theory as a formal system). which is inductively defined by the set-theoretic method, named as enumerers that constructed from the empty set ∅ and the successor function. We use nonbold n when it is used in mathematics, which is named intuitive numbers. This technic is a regular treatment. This distinction was used in early sections.

Technic 2. The Gödel numbering

Gödel numbering is one of the key techniques used in Gödel's incompleteness theorem and Tarski's indefinability theorem (Marharis, 1967). Below we explain the Gödel numbering method. Mathematical language always deals with symbols, formulas, and derivations. For a mathematical framework, even though its base domains (such as real or complex fields) are uncountable infinities (i.e., the continuum), the number of symbols used to denote variables, functions, operators, etc., is infinite but countably many. Thus, we can have an effective procedure to mechanically assign a unique odd number to each and every symbol in order, called *Gödel number*. For a given symbol e, its Gödel number is written as $g(e)$, which can be seen as a function or an odd number. A formula is a finite string of symbols, written as:

$$L = e_1e_2 \dots\dots e_n \tag{4.1}$$

The Gödel number of a formula can be calculated by:

$$g(L) = \quad {q_1}^{g(e_1)}{q_2}^{g(e_2)}\dots{q_n}^{g(e_n)} \tag{4.2}$$

where q_i is the first i prime numbers in its natural order, and $g(e_i)$ is the Gödel number of the ith symbol in the formula L. A derivation is a finite sequence of formulas, written as:

$$Der(L) \equiv < u_1, u_2, \dots, u_m > \tag{4.3}$$

The Gödel number of a derivation can be calculated by:

$$g(L) = \quad {q_1}^{g(u_1)}{q_2}^{g(u_2)}\dots\ {q_m}^{g(u_m)} \tag{4.4}$$

where $g(u_i)$ is the Gödel number of the ith formula in the derivation sequence. The Gödel number of any given formula or derivation is always an even number, which is also a composite number.

The above method is called *Gödel numbering* [4]. The beauty and power of Gödel numbering is that, based on the so-called first theorem of arithmetic (i.e., Pair forming LCM), from a given Gödel number we can uniquely recapture the original derivation, the original formula, or the original symbol used in the context.

Technic 3. Self-referential statement. Let us recall the mother formula (2.1)

$$\mathrm{P}(x) = \forall y \neg \mathrm{G}(x, y)$$

It has the Gödel number $\mathrm{g}[\mathrm{P}(x)] = i$. To substitute x by I, we obtain the daughter formula:

$$\mathrm{S} = \mathrm{P}(\boldsymbol{i}/x) = \forall y \neg \mathrm{G}(\boldsymbol{i}, y) \tag{4.5}$$

Here, S is the so-called self-referential formula. This is the major technic created by Gödel. Assume S is provable, its proof would be a sequence of formulas, denoted as Bew(S), which has the Gödel number $\mathrm{g}(Bew(S)) = j$.

Technic 4. Expressibilities

Let i be the Gödel number of a mother formula and j be the Gödel number of the proof of the self-referential daughter formula, Gödel relation is defined as $\mathbb{G}(i,\ j)$. Further, the Gödel expressibilities are defined as follows:

$$\vDash_m \mathbb{G}(i,\ j) \Longrightarrow\ \vdash_L \mathrm{G}(\boldsymbol{i},\ \boldsymbol{j}) \tag{4.6}$$

$$\nvDash_m \mathbb{G}(i,\ j) \Longrightarrow\ \vdash_L \neg\mathrm{G}(\boldsymbol{i},\ \boldsymbol{j}) \tag{4.7}$$

5. The Gödel-Riemann Semantics

The above work is an integration of Riemann hypotheses and the Gödel method. Certain modifications we made deserve explanations. Indeed, it demands a new semantics, which we refer to as The Gödel-Riemann Semantics. The three major modifications are explained below.

Modification 1. The Riemann analytic continuation goes beyond the Peano algebra, so it goes beyond the scope of the Gödel numbering. This is an advantage rather than disadvantage, as it leaves a room to construe the Riemann hypothesis into Gödel's mother formula (2.1)

$$\mathrm{P}(x) = \forall y \neg \mathrm{G}(x, y)$$

Instead of using Gödel number i, we use the Riemann monad as Gödel-Riemann coding in formula (2.2)

$$\mathrm{g}[\mathrm{P}(x)] = \theta_{RH}$$

By doing so, the Gödel's mother formula is related with Riemann hypothesis. As shown in the self-referential daughter formula (2.3)

$$\mathrm{S} = \mathrm{P}(\theta_{RH}/x)$$

where S becomes Riemann hypothesis oriented.

Modification 2. Riemann hypothesis is important due to the expectation that it can well-predict prime behavior. This is a prediction rather than a formal proof. This prediction is denoted as Pre(s), and the corresponding density function is denoted as D_n, for any given large N. Hence, instead of using the Gödel number of Bew(S), we use the Gödel-Riemann prediction code in the formula (2.4)

$$\mathrm{g}[\mathrm{Pre(S)}] = D_n$$

Modification 3. By Modifications 1 and 2, we modify the Gödel relation $\mathbb{G}(i, j)$ by the Gödel-Riemann relation $\mathbb{G}(\theta_{RH},\ D_n)$. Further, reconsider the Gödel's expressibility formulas (4.6) and (4.7).

$$\vDash_m \mathbb{G}(i,\ j) \Longrightarrow\ \vdash_L \mathrm{G}(\boldsymbol{i},\ \boldsymbol{j})$$

$$\nvDash_m \mathbb{G}(i,\ j) \Longrightarrow\ \vdash_L \neg\mathrm{G}(\boldsymbol{i},\ \boldsymbol{j})$$

By the Modifications 1 and 2, we can naturally modify (4.6) and (4.7) as the Gödel-Riemann expressibilties as (2.6) and (2.7).

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$$\vDash_m \mathbb{G}(\theta_{RH},\ D_n) \Longrightarrow\ \vdash_L \mathrm{G}(\boldsymbol{\theta_{RH}},\ \boldsymbol{D_n})$$

$$\nvDash_m \mathbb{G}(\theta_{RH},\ D_n) \Longrightarrow\ \vdash_L \neg\mathrm{G}(\boldsymbol{\theta_{RH}},\ \boldsymbol{D_n})$$

Modification 4. By the above modifications, the proof of the independent result in the present work, Theorem 1, follows the argument structure used in Gödel's proof of his well-known independent result for the first-order theory with slight further modifications (Marharis, 1967).

6. Concluding Remarks

Remark 1. The present work studies the Riemann hypothesis from metalogical perspectives. It argues that Riemann hypothesis is independent of the current Riemann analytic continuation. This argument is based on the modifications of Gödel's independent result (1931).

Remark 2. This paper shows integrations of Riemann hypothesis and Gödel structure. On one hand, Riemann hypothesis is construed into the Gödel structure by making a number of modifications. On the other hand, the Gödel structure is applied to disclose the metalogic behind the Riemann hypothesis. the Gödel incompleteness theorem is well-known. However, in addition to philosophical speculations, only few technical applications were found (e.g., the halting problem in computer science). The present work provides a general method to utilize as well as to modify the Gödel structure from metalogical perspectives.

Remark 3. The independent result obtained from this work opens a new path to investigate Riemann hypothesis. By assume as our working hypothesis that it is not analytically soluble, Riemann hypothesis still serves as an empirical testing bed of prime distribution (Mazur and Stein, 2016). We found that the Riemann hypothesis can be characterized by the Riemann sphere of two-state system (Penrose, 2004), which allows us to address the measurement problem in terms of the Riemann ζ function and the possible prime distribution (Yang, 2025).

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