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Ulam–Hyers Stabilization of New Fuzzy Fractional Partial Differential Symmetry Coupled Systems involving Caputo–Katugampola Generalized Hukuhara Type Differentiability

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Abstract

In this paper, we study existence and stability of solutions for a class of new symmetry coupled systems of fuzzy fractional partial differential equations involving Caputo–Katugampola and generalized Hukuhara (gH-) type derivatives, which provides a solid theoretical foundation and effective analytical tools for scientific research and engineering practice to address highly complex, uncertain, and memory-dependent interacting systems. Under some suitable assumptions that break through the limitations of traditional Lipschitz conditions, existence of classical solutions of the symmetry coupled systems is strictly proved by innovatively applying Schauder fixed point theorem. Furthermore, the existence of two kinds of gH-type weak solutions is confirmed by constructing typical examples. Further, stability of the fuzzy fractional partial differential symmetry coupled systems is analyzed based on Ulam–Hyers stability theory.

Keywords: existence theorem; Ulam–Hyers stabilization; fuzzy fractional partial differential symmetry coupled system; Caputo–Katugampola and generalized Hukuhara type derivative; Schauder fixed point theorem

MSC: 47H10; 35A01; 35D30; 35R11; 26A33

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1. Introduction

Based on Caputo–Katugampola (C-K) fractional derivative approach due to Katugampola [1], which expand the theoretical framework of fractional differential via unifying Riemann–Liouville and Hadamard fractional derivatives, Singh et al. [2] investigated the following prey-predator fractional-order biological population model with carrying capacity and understanding their interactivity:

$$\begin{cases} {}^{\mathrm{KC}}_{a}D^{\mu,\rho}_{t}X(t) = X\left(\delta_1 - \dfrac{\delta_1 X}{P_1}\right) - \sigma_1 XY, \\ {}^{\mathrm{KC}}_{a}D^{\mu,\rho}_{t}Y(t) = Y(-\delta_2 + \sigma_2 X), \\ X(0) = X_0 = \alpha_1 > 0, \\ Y(0) = Y_0 = \alpha_2 > 0, \end{cases} \tag{1}$$

which contributes significantly to ecological community, where $X(t)$ and $Y(t)$ denote the population densities of the prey and predator, respectively, δ_1 represents the growth rate of the prey, P_1 the carrying capacity, σ_1 and σ_2 the competitive interaction rates, and δ_2 the growth rate of the predator; all these coefficients are positive constants.

We remark that in recent years, C-K fractional derivative has been widely adopted owing to its capability to capture local differential and integral characteristics and provided a framework for handling systems with fractional exponents. It is especially applicable to modeling and analyzing systems exhibiting fractional dynamic behaviors [3,4], which laid an important foundation for the follow-up research. Since then, scholars have carried out systematic research on C-K derivative. In fact, with the in-depth study of complex systems, the traditional integer-order derivative model shows limitations in describing processes with memory, heredity or nonlocal characteristics, which promotes the development of fractional partial differential equations (FPDEs) [5], which can describe the dynamic evolution of natural phenomena [6] and model the evolution process of physical quantities with space and time in multivariable systems [7]. At the same time, the successful application of this mathematical tool in noise suppression [8], biological engineering [9], physical science [10] and other fields further promotes the deep integration of the theoretical development of FPDEs and practical problems.

On the other hand, biodiversity constitutes the most fundamental attribute of an ecosystem. Nevertheless, prior investigations predominantly centered on the survival and proliferation of individual species, thereby overlooking the competitive dynamics engendered by the coexistence of multiple species. Such mutual interrelations are defined as "coupling" [11] when two or more entities engage in reciprocal interaction and influence. As noted by Ding et al. [12], coupling mechanisms are capable of effectively characterizing the interaction dynamics between two competing species in ecological systems. A representative instance of such coupled structures is the following elliptic system [13]:

$$\begin{cases} \Delta u = \varphi_1(x,u,v) & \text{in } \Omega, \\ \Delta v = \varphi_2(x,u,v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\varphi_1, \varphi_1 \in C(\overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0,+\infty)$ and $\Omega \subset \mathbb{R}^n$ $(n > 3)$ is a smooth bounded domain, this system possesses significant capability for capturing the intrinsic dynamics of ecosystems. Furthermore, Zhang et al. [14] applied the fuzzy fractional coupled partial differential equation of Caputo derivative to the initial value problem, and established existence theory of solutions under the gH-type derivative framework. Muatjetjeja et al. [15] conducted a comprehensive Noether symmetry analysis of a generalized coupled Lane-Emden-Klein-Gordon-Fock system with central symmetry.

It is well known that practical engineering systems often have to face issues such as parameter uncertainties, measurement noise, or model inaccuracies. The deterministic framework of traditional fractional-order models is difficult to fully accommodate these challenges, while the introduction of fuzzy theory provides an effective approach to address such uncertainty problems [16,17]. Osman [18] proposed the fuzzy Adomian decomposition method and the modified Laplace decomposition method, successfully solved the fuzzy fractional Navier-Stokes equations, and developed the fuzzy Elzaki transform to deal with the linear-nonlinear Schrodinger equation. Pandey et al. [19] realized efficient numerical approximation of variable-order fuzzy partial differential equations (PDEs) by using Bernstein spectral technique. Mazandarani [20] solved the numerical solution problem of fuzzy fractional initial value problem by improving the fractional Euler method. Furthermore, employing Caputo's definition of fractional derivatives and gH-difference sets, Singh et al. [21] described fuzzy differential equations and discussed a numerical solution method for fuzzy fractional differential equations with fuzzy fractional counterparts using power series approximation and Taylor's theorem. It is worth mentioning that Pythagorean fuzzy fractional calculus provides a new paradigm for complex system analysis by virtue of its strong uncertainty modeling ability. For example, Akram et al. [22] successfully analyzed the fractional-order fuzzy wave equation by means of multivariate Pythagorean fuzzy Fourier transform. Baleanu et al. [23] analyzed stability of differential equations under such derivatives by combining Adomian polynomials and fractional Taylor series. Hoa et al. [24] constructed analytical solutions of C-K fuzzy fractional differential equations by the solution of fuzzy integer order differential equation, and verified existence and uniqueness of the solution under the generalized Lipschitz condition. These results significantly enrich the theory and application boundary of fractional calculus.

Since then, researchers have investgated the symmetry coupled systems from various aspects. In the fuzzy fractional population dynamics model, stability condition reveals the long-term evolution trend of species number under fractal habitat and fuzzy environmental carrying capacity, and guides the protection strategy of endangered species. Wang et al. [25] constructed a fractional predator-prey model, and characterized the fractal characteristics of habitat by Caputo fractional derivative. We note that the stability analysis shows that the change of fractional order will significantly affect the stability of population equilibrium point. For different equations, the stability of the solution is not the same. By using Banach contraction principle and Krasnoselskii fixed point theorem, Ali et al. [26] obtained Ulam's stability of solutions for symmetry coupled systems of fractal fractional differential equations.

$$\begin{cases} {}^{FD}D^{\eta}r(t) = \xi t^{\xi-1}f(t,r(t),p(t)), \quad t \in I = [0,T], \\ {}^{FD}D^{\eta}p(t) = \xi t^{\xi-1}g(t,r(t),p(t)), \\ r(0) = r_0 + \phi(r), \\ p(0) = p_0 + \psi(p), \end{cases}$$

where ${}^{FD}D^{\eta}$ is the Caputo fractional derivative and $r_0, p_0 \in \mathbb{R}$, $\eta, \xi \in (0,1]$, $f, g : I \times \mathbb{R}^2 \to \mathbb{R}$ and $\phi, \psi : I = [0,T] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. András et al. [27] studied Ulam–Hyers stability (U-HS) of a class of elliptic PDEs,

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

which are defined on a bounded domain with Lipschitz boundary by using the direct technique and the abstract method of Picard operator.

Moreover, in the study of calculus theory, Lipschitz condition has long been the core premise of classical results. However, in practical problems, due to complex nonlinear and non-smooth characteristics of the systems, its applicability is limited [28]. Long et al. [29] introduced Schauder-type nonlinear substitution technique to deal with fuzzy-valued continuous functions that do not satisfy Lipschitz condition. By bypassing the traditional dependence on the local smoothness of the function and using the compactness condition of the topological fixed point theory, the second existence result of two kinds of gH-weak solutions for special coupled systems was proved, which expands the scope of application of the theory and provides a more flexible tool for mathematical modeling of complex systems. On this basis, Zhang et al. [14] proved existence of two kinds of gH-weak solutions of coupled fractional equations by the same method, which promoted the development of multi-scale symmetry coupled system theory. In fact, the strict requirements of Lipschitz conditions on the smoothness of functions make it difficult to cover a large number of non-ideal situations in practice. Thus, we consider the related problems without Lipschitz conditions.

Inspired by the previous work [2], [14], [27], to enhance the ability of system (2) to capture the intrinsic dynamic characteristics of ecological systems, particularly slow diffusion behavior and historical uncertainty effects, we make improvements, introducing C-K gH-type differentiable operator to the left-hand side of (2), more specifically defining the right-hand side, and imposing fuzzy initial conditions. Let M is a fuzzy number space, M_c is a space of fuzzy number $\vartheta \in M$, which has the property that the function $\varrho \to [\vartheta]^{\varrho}$ is continuous with respect to Hausdorff metric on [0, 1], where

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$[\vartheta]^\varrho = \{x \in \mathbb{R} | \vartheta(x) \geq \varrho\}$ is ϱ-level set of ϑ. Thus, we will consider the following symmetry coupled system of fuzzy fractional partial differential equations (FFPDEs) with C-K gH-type differentiability:

$$\begin{cases} {}^{CK}_{gh}D_k^{\alpha,p} h_1(x,y) = \hat{f}_1(x,y,h_1(x,y),h_2(x,y)), \\ {}^{CK}_{gh}D_k^{\beta,p} h_2(x,y) = \hat{f}_2(x,y,h_1(x,y),h_2(x,y)), \\ h_1(x,0) = \zeta_1(x), \quad h_2(x,0) = \eta_1(x), \\ h_1(0,y) = \zeta_2(y), \quad h_2(0,y) = \eta_2(y), \end{cases} \tag{3}$$

for any $(x,y) \in \varpi = [0,a] \times [0,b]$ and $h_1, h_2 \in M$, $k \in \{1,2\}$, ${}^{CK}_{gh}D_k^{\gamma,p}$ is a C-K gH-type derivative operator with the fractional order $\gamma = (\gamma_1, \gamma_2) \in (0,1] \times (0,1]$ and $\zeta_1, \eta_1 : [0,a] \to M_c$ and $\zeta_2, \eta_2 : [0,b] \to M_c$ are continuous, $p > 0$ is a real number. What is noteworthy is that ${}^{CK}_{gh}D_k^{\alpha,p}$ and ${}^{CK}_{gh}D_k^{\beta,p}$ of (3) suppose the existence of Hukuhara (H-) difference and gH-type difference.

Remark 1. *There are the following points to note:*

- *When $\hat{f}_2(x,y,h_1,h_2) \equiv \hat{g}(x,y,h_1)$ and $p = 1$ in ${}^{CK}_{gh}D_k^{\gamma,p}$, the form of (3) reduces to the fuzzy fractional coupled PDEs considered by Zhang et al. [13].*
- *One can easily observe that the right-hand side of the second equation for (3) is a function of $h_1(x,y)$ or $h_2(x,y)$. This feature broadens the scope of solvable problems in comparison with the work in [13] and the references therein.*
- *Notably, the C-K gH-type derivative operators in (3) generalize the Caputo gH-type derivative operators in [13].*

Consequently, the system (3) are entirely novel and merit in-depth study.

The rest of this paper is as follows: In Section 2, some necessary concepts and other necessary conditions are given. By using Schauder fixed point theorem, existence of two kinds of gH-weak solutions of equation (3) are proved in Section 3. In Section 4, a numerical example is presented. U-HS of the solutions of the symmetry coupled system (3) is proposed in Section 5. Finally, some conclusions and future work are discussed.

2. Preliminaries

In this section, we will define the fractional integral and C-K gH-derivative for fuzzy-valued multivariate functions, and introduce the theory of relative compactness in fuzzy number space. It should be noted that some of these concepts have been more thoroughly explored in [2,12,30].

Definition 1 ([14])**.** *Denote* M *as the space of fuzzy number on* $\mathbb{R}$*, which is a mapping* $\vartheta : \mathbb{R} \to [0,1]$ *satisfying normal, fuzzy convex, upper semi-continuous, and compactly supported properties. The* ϱ*-level set of fuzzy number* ϑ *are defined by:*

$$[\vartheta]^\varrho = \begin{cases} \{x \in \mathbb{R} \mid \vartheta(x) \geq \varrho\}, & \text{if } 0 < \varrho \leq 1, \\ \mathrm{cl}(\mathrm{supp}\,\vartheta), & \text{if } \varrho = 0, \end{cases} \tag{4}$$

where cl *denotes the closure of the sets and* supp ϑ *is the support of* ϑ*.*

It is evident that the ϱ-level set of the fuzzy number ϑ is a closed and bounded interval $[\vartheta_\varrho^-, \vartheta_\varrho^+]$, where ϑ_ϱ^- denotes the left-hand endpoint of $[\vartheta]^\varrho$, and ϑ_ϱ^+ denotes the right-hand endpoint. The diameter of the ϱ-level set of ϑ is defined as $len[\vartheta]^\varrho = \vartheta_\varrho^+ - \vartheta_\varrho^-$. The highest measure d_∞ on M is expressed as

$$d_\infty(\vartheta, \theta) = \sup_{0 \leq \varrho \leq 1} d_H([\vartheta]^\varrho, [\theta]^\varrho)$$

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$$= \sup_{0\leq\varrho\leq1} \max\Big\{|\vartheta_\varrho^- - \theta_\varrho^-|, |\vartheta_\varrho^+ - \theta_\varrho^+|\Big\}, \ \forall\, \vartheta, \theta \in M, \tag{5}$$

where $[\vartheta]^\varrho = [\vartheta_\varrho^-, \vartheta_\varrho^+]$. In $C(\varpi, M)$, the supremum metric D is taken into account

$$D(h_1, h_2) = \sup_{(x,y)\in\varpi} d_\infty(h_1(x,y), h_2(x,y)), \tag{6}$$

Thus, (M, d_∞) and (M, D) are complete metric spaces. For all $\vartheta, \theta \in M$, and any $\varrho \in [0,1]$, by [31], we know that

$$\begin{cases} \text{(i)} \quad [\vartheta + \theta]^\varrho = [\vartheta]^\varrho + [\theta]^\varrho, \\ \text{(ii)} \quad [k\vartheta]^\varrho = k[\vartheta]^\varrho, \quad k \in \mathbb{R}\backslash\{0\}, \\ \text{(iii)} \quad [\vartheta \ominus \theta]^\varrho = [\vartheta_\varrho^- - \theta_\varrho^-, \vartheta_\varrho^+ - \theta_\varrho^+], \end{cases} \tag{7}$$

where $\vartheta \ominus \theta$ is H-difference of fuzzy numbers ϑ and θ. We suppose that the H-difference always exists. For $i = 1,2$, the space M_c^i is defined as the collection of fuzzy numbers $\vartheta \in M$ that are continuous with respect to the Hausdorff metric (abbreviated as H_m-continuous). According to [32,33], the fuzzy number spaces M and M_c^1 are semilinear spaces possessing the cancellation property. Equipped with metric d_∞, both M and M_c^i $(i = 1,2)$ form complete metric semilinear spaces. Consequently, the set of fuzzy-valued continuous functions $C(\varpi, M_c^i)$ $(i = 1,2)$ inherits completeness, thereby constituting a Banach semilinear space with the cancellation property. Meanwhile, $\widehat{L}(\varpi, M_c^1)$ is defined as the Lebesgue integrable space for fuzzy-valued continuous functions.

For any positive real number r, the closed sphere $B(\hat{0}, r)$ in the metric space $\big(C(\varpi, M_c^1), D\big)$ consists of all fuzzy numbers ϑ satisfying $D(\vartheta, \hat{0}) \leq r$. Here, the metric D is defined by (6), and $\hat{\theta}$ is given by $\hat{0} = \vartheta \ominus \vartheta$ for all $\vartheta \in M_c^1$, where $\hat{0}(x) = 1$ if $x = 0$, and $\hat{0}(x) = 0$ otherwise.

Lemma 1 ([2]). *For all $\mu, v, \omega, \lambda \in M_c^1$, the following properties hold:*

(i) $d_\infty(\mu + v, \omega + \lambda) \leq d_\infty(\mu + \omega) + d_\infty(v + \lambda)$.
(ii) *If $\mu \ominus v$ and $\omega \ominus \lambda$ hold, then $d_\infty(\mu \ominus v, \omega \ominus \lambda) \leq d_\infty(\mu, \omega) + d_\infty(v, \lambda)$.*
(iii) *If $\mu \ominus (v + \omega)$ exist, then $\mu \ominus v \ominus \omega$ hold and $\mu \ominus (v + \omega) = \mu \ominus v \ominus \omega$.*
(iv) *If $\mu \ominus v$ and $v \ominus \omega$ are defined, then $\mu \ominus (v \ominus \omega)$ is defined and satisfies $\mu \ominus (v \ominus \omega) = (\mu \ominus v) + \omega$.*
(v) *If $\mu \ominus v$ exists, then so does $(-1)\mu \ominus (-1)v$, and we have $(-1)(\mu \ominus v) = (-1)\mu \ominus (-1)v$.*

Definition 2 ([34]). *Let $\iota \in \mathbb{L}$ and $f \in C(\varpi, M_c^1)$ be a fuzzy-valued mapping. Then f is said to be ι^{th} order gH-type differentiable with respect to x at $(x_0, y_0) \in \varpi$ if the following conditions hold:*

(i) *f is gH-type differentiable of all orders from 1 to $\iota - 1$ at (x_0, y_0).*
(ii) *There exists an element $\dfrac{\partial^\iota f(x_0, y_0)}{\partial x^\iota} \in M_c^1$ such that for all sufficiently small $\hbar > 0$ with $(x_0 + \hbar, y_0) \in \varpi$, the gH-difference $f^{\iota-1}(x_0 + \hbar, y_0) \ominus_{gH} f^{\iota-1}(x_0, y_0)$ exists, and the following limit holds*

$$\lim_{\hbar\to0} \frac{f^{\iota-1}(x_0 + \hbar, y_0) \ominus_{gH} f^{\iota-1}(x_0, y_0)}{\delta} = \frac{\partial^\iota f(x_0, y_0)}{\partial x^\iota}, \tag{8}$$

where the gH-type difference $f_1 \ominus_{gH} f_2$, as defined in ([35])*, satisfies*

$$f_1 \ominus_{gH} f_2 = \xi \Longleftrightarrow \begin{cases} (\dagger) f_1 = f_2 + \xi, \\ (\ddagger) f_2 = f_1 + (-1)\xi. \end{cases} \tag{9}$$

In this case, $\dfrac{\partial^\iota f(x_0, y_0)}{\partial x^\iota}$ is called the ι-order gH-type derivative of f with respect to x at (x_0, y_0).

Remark 2. *By* Definition 2, *the higher-order fuzzy gH-type partial derivatives with respect to y are similarly defined. When $\iota = 1$, the equation (8) simplifies to*

$$\lim_{\hbar\to 0} \frac{f(x_0+\hbar, y_0) \ominus_{gH} f(x_0,y_0)}{\delta} = \frac{\partial f(x_0,y_0)}{\partial x}, \tag{10}$$

representing the first-order partial derivative of f at (x_0, y_0) with respect to x.

Definition 3 ([36]). *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (0,1] \times (0,1]$, $p > 0$, and $f(x,y) \in C(\varpi, M_c) \cap \widehat{L}(\varpi, M_c)$. For $(x,y) \in \varpi$ and $\varrho_1 \in [0,1]$, let $[f(x,y)^{\varrho_1}] = [f^-_{\varrho_1}(x,y), f^+_{\varrho_1}(x,y)]$, then the mixed Riemann–Liouville fractional integral of orders $\boldsymbol{\alpha}$ for fuzzy-valued multivariable function $f(x,y)$ is defined as:*

$${}^{RL}_{F}\mathcal{I}^{\boldsymbol{\alpha},p}_{0^+} f(x,y) = \frac{p^{2-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x^p - s^p)^{\alpha_1-1}(y^p - t^p)^{\alpha_2-1} s^{p-1} t^{p-1} f(s,t)dtds. \tag{11}$$

Definition 4 ([13]). *the mappings f: $\varpi \times C(\varpi, M^1_c) \times C(\varpi, M^2_c) \to M^1_c$ and g: $\varpi \times C(\varpi, M^1_c) \times C(\varpi, M^2_c) \to M^2_c$ are said to be jointly continuous at the point $(x_0, y_0, \Psi, \Phi) \in \varpi \times C(\varpi, M^1_c) \times C(\varpi, M^2_c)$ if for every $\varepsilon > 0$, there exists $\hat{\delta} > 0$ such that whenever $|x - x_0| + |y - y_0| + d_\infty(h_1, \Psi) + d_\infty(h_2, \Phi) < \hat{\delta}$, the following inequalities hold: $d_\infty(f(x,y,h_1,h_2),$
$f(x_0,y_0,\Psi,\Phi)) < \varepsilon$, $d_\infty(g(x,y,h_1,h_2), g(x_0,y_0,\Psi,\Phi)) < \varepsilon$.*

For all $(x,y) \in \varpi = [0,a] \times [0,b]$, define

$$\begin{aligned} \Psi(x,y) &= \zeta_2(y) + [\zeta_1(x) \ominus \zeta_1(0)], \\ \Phi(x,y) &= \eta_2(y) + [\eta_1(x) \ominus \eta_1(0)], \end{aligned} \tag{12}$$

where $\zeta_1 \in C([0,a], M^1_c)$, $\eta_1 \in C([0,a], M^2_c)$, $\zeta_2 \in C([0,b], M^1_c)$ and $\eta_2 \in C([0,b], M^2_c)$ are given functions such that $\zeta_2(y) \ominus \zeta_1(0)$ and $\eta_2(y) \ominus \eta_1(0)$ exist. Then define the function spaces:

$$\hat{C}^{\hat{f}_1}_{\Psi}(\varpi, M^1_c) = \Big\{ h_1 \in C(\varpi, M^1_c) |\, \Psi(x,y) \ominus (-1)\,{}^{RL}_{F}\mathcal{I}^{\boldsymbol{\alpha},p}_{0^+} \hat{f}_1(x,y,h_1(x,y),h_2(x,y)) \text{ exist}, \forall (x,y) \in \varpi \Big\}, \tag{13}$$

$$\hat{C}^{\hat{f}_2}_{\Phi}(\varpi, M^2_c) = \Big\{ h_2 \in C(\varpi, M^2_c) |\, \Phi(x,y) \ominus (-1)\,{}^{RL}_{F}\mathcal{I}^{\boldsymbol{\beta},p}_{0^+} \hat{f}_2(x,y,h_1(x,y),h_2(x,y)) \text{ exist}, \forall (x,y) \in \varpi \Big\}, \tag{14}$$

where Ψ and Φ are defined by (12)

For m, n = 0,1,2, denote by $C^{m,n}_{gH}(\varpi, M^i_c)$ the collection of all functions $\tau : \varpi \subset \mathbb{R}^2 \to M^i_c$, i=1,2 that possess partial gH-type derivatives up to order m with respect to x and up to n with respect to y in the domain ϖ.

Definition 5 ([36]). *Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in (0,1] \times (0,1]$, and $f(x,y) \in C^{2,2}_{gH}(\varpi, M_c)$. Then the C-K gH-type derivative of order α with respect to x and y for the function f is defined by*

$$\begin{aligned} {}^{CK}_{gH}\mathcal{D}^{\boldsymbol{\alpha},p}_{k} h(x,y) &= {}^{RL}_{F}\mathcal{I}^{1-\boldsymbol{\alpha},p}_{0^+} \left(\frac{\partial^2 h(x,y)}{\partial x \partial y} \right) \\ &= \frac{p^{\alpha_1+\alpha_2}}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)} \int_0^x \int_0^y (x^p - s^p)^{-\alpha_1}(y^p - t^p)^{-\alpha_2} \frac{\partial^2 h(s,t)}{\partial s \partial t} dtds, \end{aligned} \tag{15}$$

where the right-hand expression is required to be well-defined, with $1 - \boldsymbol{\alpha} = (1-\alpha_1, 1-\alpha_2) \in [0,1) \times [0,1)$.

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Particularly, we need to distinguish two cases corresponding to $(\dagger)$ and $(\ddagger)$ in (9) for any $h_1 \in M_c^1$, as follows:

(i) A function h_1 satisfies the condition of C-K gH-type differentiability of order $\boldsymbol{\alpha}$ concerning x and y if $\frac{\partial^2 h_1(\cdot,\cdot)}{\partial x \partial y}$ acts as a gH-type derivative of type $(\dagger)$ (i.e., with $k = 1$ in (3)) at the point (x, y). This property is denoted by ${}^{CK}_{gH}\mathcal{D}_1^{\boldsymbol{\alpha}} h_1(x, y)$.

(ii) h_1 is $(\ddagger)$-C-K gH-type differentiable of order $\boldsymbol{\alpha}$ with respect to x and y when $\frac{\partial^2 h_1(\cdot,\cdot)}{\partial x \partial y}$ serves as a gH-type derivative of type $(\ddagger)$ (i.e., where $k = 2$ in (3)) at (x, y), For this, the notation ${}^{CK}_{gH}\mathcal{D}_2^{\boldsymbol{\alpha}} h_1(x, y)$ is used.

Definition 6 ([30]). *For a subset $S \subset M_c$, S is equicontinuous at w_0 if the following holds: for all $\varepsilon > 0$, there exists $\delta > 0$ such that for each $w \in M_c$ and $T \in S$, $d_\infty(w, w_0) < \delta$ implies $d_\infty(T(w), T(w_0)) < \varepsilon$. We state that S is equicontinuous if S is equicontinuous at every $\omega_0 \in M_c$.*

Definition 7 ([30]). *A subset $S \subset M_c$ is said to be compactly supported if for every fuzzy number $w \in S$, there exists a compact set $K \subseteq \mathbb{R}$ such that the support of w, denoted $[w]^0$, is contained in K.*

Definition 8 ([30]). *A subset $S \subseteq M_c$ is defined as level-equicontinuous at $\tau_0 \in [0, 1]$ when the following holds: for all $\varepsilon > 0$, there exists $\delta > 0$, such that $|\tau - \tau_0| < \delta$ implies Hausdorff metric $d_H([w]^\tau, [w]^{\tau_0}) < \varepsilon$ for each $w \in S$. Additionally, S is termed level-equicontinuous on $[0, 1]$ if S is level-equicontinuous at every $\tau \in [0, 1]$.*

Definition 9 ([13]). *Let U_c and V_c be H_m-continuous fuzzy number spaces. A continuous mapping $\Lambda : U_c \to V_c$ is called a compact operator if it maps every bounded subset $S_c \subseteq U_c$ to a relatively compact set in V_c, i.e., the closure $\overline{\Lambda(S_c)}$ forms a compact subset of V_c.*

Lemma 2 ([30]). *For a subset S in M_c, S is a compact-supported if and only if S is a relatively compact subset of (M_c, d_∞) and S is level-equicontinuous on $[0, 1]$.*

Lemma 3. *Let Ψ and Φ be defined as in (12), let $\hat{f}_1$ and $\hat{f}_2$ be jointly continuous according to* Definition 4*, and let $h_1 \in C^{2,2}_{gH}(\varpi, M_c^1)$ and $h_2 \in C^{2,2}_{gH}(\varpi, M_c^2)$ be fuzzy-valued functions. Then (3) is equivalent to the following nonlinear fractional Volterra integro-differential symmetry coupled system For all $(x, y) \in \varpi$:*

$$\begin{cases} h_1(x, y) = \Psi(x, y) + {}^{RL}_{F}\mathcal{I}^{\boldsymbol{\alpha},p}_{0^+} \hat{f}_1(x, y, h_1(x, y), h_2(x, y)), \\ h_2(x, y) = \Phi(x, y) + {}^{RL}_{F}\mathcal{I}^{\boldsymbol{\beta},p}_{0^+} \hat{f}_2(x, y, h_1(x, y), h_2(x, y)), \end{cases} \text{when } k = 1 \tag{16}$$

or when $k = 2$,

$$\begin{cases} h_1(x, y) = \Psi(x, y) \ominus (-1){}^{RL}_{F}\mathcal{I}^{\boldsymbol{\alpha},p}_{0^+} \hat{f}_1(x, y, h_1(x, y), h_2(x, y)), \\ h_2(x, y) = \Phi(x, y) \ominus (-1){}^{RL}_{F}\mathcal{I}^{\boldsymbol{\beta},p}_{0^+} \hat{f}_2(x, y, h_1(x, y), h_2(x, y)). \end{cases} \tag{17}$$

Proof. The proof process of this equivalence is similar to the proof of (Lemma 3 in [37]), so it is omitted here. □

3. Main Result

In this section, a novel proof approach is developed, distinct from previous methods. Specifically, the Schauder fixed point theorem is applied in Banach semilinear spaces without requiring Lipschitz conditions on $\hat{f}_1$ and $\hat{f}_2$, thereby establishing the existence of both $(\dagger)$-weak and $(\ddagger)$-weak solutions for the general symmetry coupled system (3).

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Lemma 4. *Suppose there exists a constant $\ell > 0$ such that $\hat{f}_i : \varpi \times B(\hat{0}, \ell) \times B(\hat{0}, \ell) \to M_c^i$ $(i = 1, 2)$ are compact operators and $\Psi, \Phi \in C(\varpi, B(\hat{0}, \frac{\ell}{2}))$. Then there exist $\hat{a} \in [0, a]$ and $\hat{b} \in [0, b]$ such that the operator $\widetilde{\aleph}_1 : \bar{N} \to \bar{N}$, here*

$$\bar{N} := \Big\{(\vartheta, \theta) \in C\Big(\widehat{\varpi}, M_c^1\Big) \times C(\widehat{\varpi}, M_c^2) | \max\{D(\vartheta, \hat{0}), D(\theta, \hat{0})\} \leq \ell\Big\},$$

where $\widehat{\varpi} := [0, \hat{a}] \times [0, \hat{b}]$, by

$$\widetilde{\aleph}_1(h_1, h_2) = \begin{pmatrix} \widetilde{W}_1(h_1, h_2) \\ \widetilde{Q}_1(h_1, h_2) \end{pmatrix}, \quad \forall (h_1, h_2) \in \bar{N}, \tag{18}$$

is continuous, where $\widetilde{W} : \bar{N} \to C(\widehat{\varpi}, M_c^1)$, $\widetilde{Q}_1 : \bar{N} \to C(\widehat{\varpi}, M_c^2)$ are respectively determined by

$$\begin{aligned} \widetilde{W}_1(h_1, h_2) &= \Psi(x, y) + {}_F^{RL}\mathcal{I}_{0^+}^{\alpha, p} \hat{f}_1(x, y, h_1(x, y), h_2(x, y)), \\ \widetilde{Q}_1(h_1, h_2) &= \Phi(x, y) + {}_F^{RL}\mathcal{I}_{0^+}^{\beta, p} \hat{f}_2(x, y, h_1(x, y), h_2(x, y)). \end{aligned} \tag{19}$$

Proof. For any two pairs of functions $\begin{pmatrix} h_{11}(x, y) \\ h_{21}(x, y) \end{pmatrix}, \begin{pmatrix} h_{12}(x, y) \\ h_{22}(x, y) \end{pmatrix} \in C(\varpi, M_c^1) \times C(\varpi, M_c^2)$, we have

$$\begin{aligned} &D(\widetilde{\aleph}_1(h_{11}, h_{21}), \widetilde{\aleph}_1(h_{12}, h_{22})) \\ &= \max\Big\{D(\widetilde{W}_1(h_{11}, h_{21}), \widetilde{W}_1(h_{12}, h_{22})), D(\widetilde{Q}_1(h_{11}, h_{21}), \widetilde{Q}_1(h_{12}, h_{22}))\Big\} \\ &= \max\Bigg\{\sup_{(x,y)\in\varpi} d_\infty\Big(\widetilde{W}_1(h_{11}(x, y), h_{21}(x, y)), \widetilde{W}_1(h_{12}(x, y), h_{22}(x, y))\Big), \\ &\qquad \sup_{(x,y)\in\varpi} d_\infty\Big(\widetilde{Q}_1(h_{11}(x, y), h_{21}(x, y)), \widetilde{Q}_1(h_{12}(x, y), h_{22}(x, y))\Big)\Bigg\}. \end{aligned} \tag{20}$$

The compactness of $\hat{f}_1$ and $\hat{f}_2$ implies their boundedness. Let $L_i = \sup_{((x,y),\vartheta,\theta)\in\varpi\times B(\hat{0},\ell)\times B(\hat{0},\ell)} D((\hat{f}_i(x, y, \vartheta, \theta), \hat{0})$, for $i = 1, 2$. Then for any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2) \in (0, 1] \times (0, 1]$, there exist $a_1, a_2, b_1, b_2 > 0$ such that $\frac{p^{-(\alpha_1+\alpha_2)} a_1^{\alpha_1 p} b_1^{\alpha_2 p} L_1}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \leq \frac{\ell}{2}$ and $\frac{p^{-(\beta_1+\beta_2)} a_2^{\beta_1 p} b_2^{\beta_2 p} L_2}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)} \leq \frac{\ell}{2}$, (since $a^{ip} b^{jp}$ are positive power polynomials for any $a, b \in \mathbb{R}$ and $i, j \in (0, 1]$). Taking $\hat{a} = \min\{a, a_1, a_2\}$ and $\hat{b} = \min\{b, b_1, b_2\}$ and denoting $\widehat{\varpi} = [0, \hat{a}] \times [0, \hat{b}] \subset \varpi$, we obtain

$$\frac{p^{-(\alpha_1+\alpha_2)} \hat{a}^{\alpha_1 p} \hat{b}^{\alpha_2 p} L_1}{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1)} \leq \frac{\ell}{2}, \quad \frac{p^{-(\beta_1+\beta_2)} \hat{a}^{\beta_1 p} \hat{b}^{\beta_2 p} L_2}{\Gamma(\beta_1 + 1)\Gamma(\beta_2 + 1)} \leq \frac{\ell}{2}. \tag{21}$$

We first show that $\widetilde{\aleph}_1(h_1, h_2)$ is a self-mapping on $\bar{N}$, i.e., $\widetilde{\aleph}_1(\bar{N}) \subset \bar{N}$. By (20), for any $(h_1, h_2) \in \bar{N}$, we have

$$d_\infty\Big(\widetilde{\aleph}_1(h_1, h_2), \hat{0}\Big) = \max\Big\{d_\infty\Big(\widetilde{W}_1(h_1, h_2), \hat{0}\Big), d_\infty\Big(\widetilde{Q}_1(h_1, h_2), \hat{0}\Big)\Big\}. \tag{22}$$

By Lemma 1 (i), one gets

$$\begin{aligned} &d_\infty\Big(\widetilde{W}_1(h_1(x, y), h_2(x, y), \hat{0}\Big) \\ &\leq d_\infty(\Psi(x, y), \hat{0}) + \frac{p^{2-(\alpha_1+\alpha_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x^p - s^p)^{\alpha_1 - 1}(y^p - t^p)^{\alpha_2 - 1} \\ &\qquad \cdot s^{p-1} t^{p-1} d_\infty\Big(\hat{f}_1(s, t, h_1(s, t), h_2(s, t)), \hat{0}\Big) dt ds \end{aligned}$$

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$$\leq D(\Psi,\hat{0}) + \frac{p^{-(\alpha_1+\alpha_2)}\hat{a}^{\alpha_1 p}\hat{b}^{\alpha_2 p}L_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}. \quad (23)$$

From $\Psi \in C(\varpi \times B(\hat{0}, \frac{\ell}{2}))$, we obtain $D(\Psi,\hat{0}) \leq \frac{\ell}{2}$. Based on $D(\Psi,\hat{0}) \leq \frac{\ell}{2}$. Substituting into (21) and (23) yields

$$d_\infty\left(\widetilde{W}_1(h_1,h_2),\hat{0}\right) \leq \frac{\ell}{2} + \frac{\ell}{2} = \ell. \quad (24)$$

Similarly,

$$d_\infty\left(\widetilde{Q}_1(h_1,h_2),\hat{0}\right) \leq \ell. \quad (25)$$

Combining (24), (25) and (22), gives $H(\tilde{\aleph}_1(h_1,h_2),\hat{0}) \leq \ell$, hence $\tilde{\aleph}_1(h_1,h_2) \in \bar{N}$.

We now prove the continuity of $\tilde{\aleph}_1$. Let (h_{1n},h_{2n}) tends to (h_1,h_2) in $\bar{N}$. By Lemma 1 (i), we have

$$\begin{aligned} & d_\infty\left(\widetilde{W}_1(h_{1n},h_{2n}),\widetilde{W}_1(h_1,h_2)\right) \\ \leq & 0 + \frac{p^{2-(\alpha_1+\alpha_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x^p-s^p)^{\alpha_1-1}(y^p-t^p)^{\alpha_2-1}s^{p-1}t^{p-1} \\ & \quad \times d_\infty\left(\hat{f}_1(s,t,h_{1n}(s,t),h_{2n}(s,t)),\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\right)dtds \\ \leq & \frac{p^{-(\alpha_1+\alpha_2)}\hat{a}^{\alpha_1 p}\hat{b}^{\alpha_2 p}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \\ & \times \sup_{(s,t)\in\hat{\varpi}} d_\infty\left(\hat{f}_1(s,t,h_{1n}(s,t),h_{2n}(s,t)),\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\right). \end{aligned} \quad (26)$$

The compactness and continuity of $\hat{f}_1$ imply the continuity of $\widetilde{W}_1$. Similarly,

$$\begin{aligned} & d_\infty\left(\widetilde{Q}_1(h_{1n},h_{2n}),\widetilde{Q}_1(h_1,h_2)\right) \\ \leq & \frac{p^{-(\beta_1+\beta_2)}\hat{a}^{\beta_1 p}\hat{b}^{\beta_2 p}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} \\ & \times \sup_{(s,t)\in\hat{\varpi}} d_\infty\left(\hat{f}_2(s,t,h_{1n}(s,t),h_{2n}(s,t)),\hat{f}_2(s,t,h_1(s,t),h_2(s,t))\right), \end{aligned} \quad (27)$$

hence $\widetilde{Q}_1$ is also continuous. Combining (26), (27) and (20), yields $d_\infty\Big(\tilde{\aleph}_1(h_{1n},h_{2n}),$ $\tilde{\aleph}_1(h_1,h_2)\Big) \to 0$. This completes the proof. □

Lemma 5. *Under the assumptions of* Lemma 4, *if* Ψ *and* Φ *are compactly supported, Then* $\tilde{\aleph}_1(\bar{N})$ *is relatively compact in* $C(\hat{\varpi}, M_c^1) \times C(\hat{\varpi}, M_c^2)$.

Proof. The proof proceeds in two steps.
Step 1: We first show that $\tilde{\aleph}_1(\bar{N})$ is equicontinuous in $C(\hat{\varpi}, M_c^1) \times C(\hat{\varpi}, M_c^2)$. For any $(x_1,y_1),(x_2,y_2) \in \hat{\varpi}$ with $x_1 < x_2, y_1 < y_2$, and each $(h_1,h_2) \in \bar{N}$, let $s_1 = x_1^p - s^p, s_2 = x_2^p - s^p, t_1 = y_1^p - t^p, t_2 = y_2^p - t^p$ then we have

$$\begin{aligned} & d_\infty\Bigg(\int_0^{x_1}\int_0^{y_1} s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds, \\ & \quad \int_0^{x_1}\int_0^{y_1} s_1^{\alpha_1-1}t_1^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds\Bigg) \\ & = d_\infty\Bigg(\int_0^{x_1}\int_0^{y_1}(s_2^{\alpha_1-1}t_2^{\alpha_2-1} - s_1^{\alpha_1-1}t_1^{\alpha_2-1})s^{p-1}t^{p-1}\cdot\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds, \end{aligned}$$

$$\int_0^{x_1}\int_0^{y_1}(s_2^{\alpha_1-1}t_2^{\alpha_2-1}-s_1^{\alpha_1-1}t_1^{\alpha_2-1})s^{p-1}t^{p-1}\cdot\hat{0}\,dt\,ds\Big)$$
$$\leq L_1\cdot\int_0^{x_1}\int_0^{y_1}(s_2^{\alpha_1-1}t_2^{\alpha_2-1}-s_1^{\alpha_1-1}t_1^{\alpha_2-1})s^{p-1}t^{p-1}\,dt\,ds, \tag{28}$$

$$d_\infty\left(\int_{x_1}^{x_2}\int_0^{y_1}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\hat{0}\right)$$
$$\leq L_1\cdot\int_{x_1}^{x_2}\int_0^{y_1}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\,dt\,ds, \tag{29}$$

$$d_\infty\left(\int_0^{x_1}\int_{y_1}^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\hat{0}\right)$$
$$\leq L_1\cdot\int_0^{x_1}\int_{y_1}^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\,dt\,ds, \tag{30}$$

and

$$d_\infty\left(\int_{x_1}^{x_2}\int_{y_1}^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\hat{0}\right)$$
$$\leq L_1\cdot\int_{x_1}^{x_2}\int_{y_1}^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\,dt\,ds. \tag{31}$$

Then, by (28)-(31), one obtains

$$\begin{aligned}
&d_\infty\Bigg(\int_0^{x_2}\int_0^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\\
&\qquad\int_0^{x_1}\int_0^{y_1}s_1^{\alpha_1-1}t_1^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds\Bigg)\\
\leq& d_\infty\Bigg(\int_0^{x_1}\int_0^{y_1}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\\
&\qquad\int_0^{x_1}\int_0^{y_1}s_1^{\alpha_1-1}t_1^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds\Bigg)\\
&+d_\infty\left(\int_{x_1}^{x_2}\int_0^{y_1}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\hat{0}\right)\\
&+d_\infty\left(\int_0^{x_1}\int_{y_1}^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\hat{0}\right)\\
&+d_\infty\left(\int_{x_1}^{x_2}\int_{y_1}^{y_2}s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,\hat{0}\right)\\
\leq&\frac{L_1}{\alpha_1\alpha_2p^2}\left(x_2^{\alpha_1p}y_2^{\alpha_2p}-x_1^{\alpha_1p}y_1^{\alpha_2p}\right),
\end{aligned}$$

by virtue of Lemma 1 (i), we obtain:

$$\begin{aligned}
&d_\infty\left(\widetilde{W}_1(h_1(x_2,y_2),h_2(x_2,y_2)),\widetilde{W}_1(h_1(x_1,y_1),h_2(x_1,y_1))\right)\\
\leq& d_\infty(\Psi((x_2,y_2),\Psi)(x_1,y_1))+\frac{p^{2-(\alpha_1+\alpha_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}
\end{aligned}$$

$$\times\, d_\infty\Bigg(\int_0^{x_2}\int_0^{y_2} s_2^{\alpha_1-1}t_2^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds,$$
$$\int_0^{x_1}\int_0^{y_1} s_1^{\alpha_1-1}t_1^{\alpha_2-1}s^{p-1}t^{p-1}\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds\Bigg)$$
$$\le d_\infty(\Psi((x_2,y_2),\Psi(x_1,y_1))) + \frac{p^{-(\alpha_1+\alpha_2)}L_1}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}\left(x_2^{\alpha_1 p}y_2^{\alpha_2 p} - x_1^{\alpha_1 p}y_1^{\alpha_2 p}\right).$$

The continuity of Ψ yields

$$\lim_{(x_1,y_1)\to(x_2,y_2)} d_\infty\left(\widetilde{W}_1(h_1(x_2,y_2),h_2(x_2,y_2)),\ \widetilde{W}_1(h_1(x_1,y_1),h_2(x_1,y_1))\right) = 0. \tag{32}$$

Similarly, for $(x_1,y_1)\to(x_2,y_2)$,

$$d_\infty\left(\widetilde{Q}_1(h_1(x_2,y_2),h_2(x_2,y_2)),\widetilde{Q}_1(h_1(x_1,y_1),h_2(x_1,y_1))\right)$$
$$\le d_\infty(\Phi(x_2,y_2),\Phi(x_1,y_1)) + \frac{p^{-(\beta_1+\beta_2)}L_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\left(x_2^{\beta_1 p}y_2^{\beta_2 p} - x_1^{\beta_1 p}y_1^{\beta_2 p}\right),$$

and

$$\lim_{(x_1,y_1)\to(x_2,y_2)} d_\infty(\Phi(x_2,y_2),\Phi(x_1,y_1)) + \frac{p^{-(\beta_1+\beta_2)}L_2}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}\left(x_2^{\beta_1 p}y_2^{\beta_2 p} - x_1^{\beta_1 p}y_1^{\beta_2 p}\right) = 0, \tag{33}$$

which follows from the continuity of Ψ. Combining (6), (20), (32) and (33) gives $d_\infty(\widetilde{\aleph}_1(h_1(x_2,y_2),h_2(x_2,y_2)),$ $\widetilde{\aleph}_1(h_1(x_1,y_1),h_2(x_1,y_1)))\to 0$. Hence, $\widetilde{\aleph}_1(\bar{N})$ is equicontinuous on $C(\widehat{\varpi},M_c^1)\times C(\widehat{\varpi},M_c^2)$.

Step 2: We show that $\widetilde{\aleph}_1(\bar{N})$ is relatively compact. By Lemma 2, it suffices to verify: (C_1) $\widetilde{\aleph}_1(\bar{N})$ is level-equicontinuous; (C_2) $\widetilde{\aleph}_1(\bar{N})$ is a compact-supported subset of $M_c^1\times M_c^2$.

(i) Verify (C_1). For any fixed $(x,y)\in\widehat{\varpi}$, $\widetilde{\aleph}_1(\bar{N})\in C(\widehat{\varpi},M_c^1)\times C(\widehat{\varpi},M_c^2)$. If $(\hbar_1,\hbar_2)^T\in\widetilde{\aleph}_1(\bar{N})$, then there exists $(h_1,h_2)^T\in\bar{N}$ such that

$$\hbar_1(x,y) = \Psi(x,y) + {}_F^{RL}\mathcal{I}_{0^+}^{\alpha,p}\hat{f}_1(x,y,h_1(x,y),h_2(x,y)),$$
$$\hbar_2(x,y) = \Phi(x,y) + {}_F^{RL}\mathcal{I}_{0^+}^{\beta,p}\hat{f}_2(x,y,h_1(x,y),h_2(x,y)).$$

Let $\hat{f}_1$, $\hat{f}_2$ be compact operators with $\hat{f}_1$ relatively compact on M_c^1 and $\hat{f}_2(\widehat{\varpi}\times\bar{N}_*)$ relatively compact in M_c^2, where $\bar{N}_* := \{\imath\in C(\widehat{\varpi},M_c^2)\,|\,H(\imath,\hat{0})\le\ell\}$. By Lemma 2, $\hat{f}_1(\widehat{\varpi}\times\bar{N})$ and $\hat{f}_2(\widehat{\varpi}\times\bar{N}_*)$ are level-equicontinuous. Thus for any $\varepsilon>0$, there exists $\delta>0$ such that for all $(x,y)\in\widehat{\varpi}$ and $(h_1,h_2)\in\bar{N}$, when $|n_1-n_2|<\delta$,

$$d_H\left(\left[\hat{f}_1(x,y,h_1(x,y),h_2(x,y))\right]^{n_1},\left[\hat{f}_1(x,y,h_1(x,y),h_2(x,y))\right]^{n_2}\right)$$
$$< \frac{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}{p^{-(\alpha_1+\alpha_2)}a^{\alpha_1 p}b^{\alpha_2 p}}\cdot\frac{\varepsilon}{2},$$
$$d_H\left(\left[\hat{f}_2(x,y,h_1(x,y),h_2(x,y))\right]^{n_1},\left[\hat{f}_2(x,y,h_1(x,y),h_2(x,y))\right]^{n_2}\right)$$
$$< \frac{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}{p^{-(\beta_1+\beta_2)}a^{\beta_1 p}b^{\beta_2 p}}\cdot\frac{\varepsilon}{2},$$

and

$$d_H([\Psi(x,y)]^{n_1},[\Psi(x,y)]^{n_2}) \le \frac{\varepsilon}{2} d_H([\Phi(x,y)]^{n_1},[\Phi(x,y)]^{n_2}) \le \frac{\varepsilon}{2}.$$

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Since

$$\begin{aligned} &d_H([\hbar_1]^{n_1}, [\hbar_2]^{n_1}) = d_H\Big([\widetilde{W}_1(h_1,h_2)]^{n_1}, [\widetilde{W}_1(h_1,h_2)]^{n_2}\Big) \\ &\leq d_H([\Psi(x,y)]^{n_1}, [\Psi(x,y)]^{n_2}) + \frac{p^{2-(\alpha_1+\alpha_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \\ &\quad \cdot d_H\Bigg(\bigg[\int_0^x \int_0^y (x^p - s^p)^{\alpha_1 - 1}(y^p - t^p)^{\alpha_2 - 1} s^{p-1} t^{p-1} \\ &\quad \cdot \hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds\bigg]^{n_1}, \bigg[\int_0^x \int_0^y (x^p - s^p)^{\alpha_1 - 1}(y^p - t^p)^{\alpha_2 - 1} \\ &\quad \cdot s^{p-1} t^{p-1} \hat{f}_1(s,t,h_1(s,t),h_2(s,t))\,dt\,ds\bigg]^{n_2}\Bigg) \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned} \tag{34}$$

and

$$d_H([\hbar_2]^{n_1}, [\hbar_2]^{n_2}) = d_H\Big([\widetilde{Q}_1(h_1,h_2)]^{n_1}, [\widetilde{Q}_1(h_1,h_2)]^{n_2}\Big) < \varepsilon \tag{35}$$

hold for $|n_1 - n_2| < \delta$, it follows from (34) and (35) that

$$\begin{aligned} &d_H\Big(\Big[[\widetilde{\aleph}_1(h_1,h_2)]^{n_1}, [\widetilde{\aleph}_1(h_1,h_2)\Big]^{n_2}\Big) \\ &:= \max\Big\{d_H\Big(\Big[\widetilde{W}_1(h_1,h_2)\Big]^{n_1}, \Big[\widetilde{W}_1(h_1,h_2)\Big]^{n_2}\Big), d_H\Big(\Big[\widetilde{Q}_1(h_1,h_2)\Big]^{n_1}, \Big[\widetilde{Q}_1(h_1,h_2)\Big]^{n_2}\Big)\Big\} \\ &< \varepsilon. \end{aligned} \tag{36}$$

This implies $\widetilde{\aleph}_1(\bar{N})$ is level-equicontinuous on $M_c^1 \times M_c^2$.

(ii) To verify condition (C_2). Given the relative compactness of $\hat{f}_1(\widehat{\varpi} \times \bar{N})$ and $\hat{f}_2(\widehat{\varpi} \times \bar{N}_*)$, Lemma 2 implies that $\hat{f}_1(\widehat{\varpi} \times \bar{N})$ and $\hat{f}_2(\widehat{\varpi} \times \bar{N}_*)$ possess compact supports and are level-equicontinuous on [0,1]. By Definition 7, there exist compact sets $K_{11}, K_{21} \subseteq \mathbb{R}$ such that $(x,y,\Psi,\Phi) \in \widehat{\varpi} \times \bar{N}$, $[\hat{f}_1(x,y,h_1(x,y),h_2(x,y))]^0 \subseteq K_{11}$, and $[\hat{f}_2(x,y,h_1(x,y),h_2(x,y))]^0 \subseteq K_{21}$ for all $(x,y,\Psi,\Phi) \in \widehat{\varpi} \times \bar{N}_*$.

Furthermore, the compact supports of Ψ and Φ guarantee the existence of compact sets $K_{12}, K_{22} \subseteq \mathbb{R}$ satisfying: $[\Psi(x,y)]^0 \in K_{12}$, $[\Phi(x,y)]^0 \in K_{22}$. we obtain the inclusion relation:

$$\begin{aligned} &\Big[\widetilde{W}_1(h_1,h_2)\Big]^0 \\ &= [\Psi(x,y)]^0 + \frac{p^{2-(\alpha_1+\alpha_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^x \int_0^y (x^p - s^p)^{\alpha_1-1}(y^p - t^p)^{\alpha_2-1} s^{p-1} t^{p-1} \\ &\qquad \cdot \Big[\hat{f}_1(s,t,h_1(s,t),h_2(s,t))\Big]^0\,dt\,ds \\ &\subseteq K_{12} + \frac{p^{-(\alpha_1+\alpha_2)} x^{\alpha_1 p} y^{\alpha_2 p}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} K_{11}. \end{aligned} \tag{37}$$

Since $x^{\alpha_1 p} y^{\alpha_2 p}$ is bounded on $\widehat{\varpi}$, there exists compact $K_1 \in \mathbb{R}$ such that $\Big[\widetilde{W}_1(h_1,h_2)\Big]^0 \subseteq K_1$, establishing the compact support of $\widetilde{W}_1(\bar{N})$. Similarly, $\Big[\widetilde{Q}_1(h_1,h_2)\Big]^0 \subseteq K_{22} + \frac{p^{-(\beta_1+\beta_2)} x^{\beta_1 p} y^{\beta_2 p}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)} K_{21}$ for some compact $K_2 \subseteq \mathbb{R}$, proving $\widetilde{Q}_1(\bar{N})$ has compact supported. From (18), we obtain

$$\Big[\widetilde{\aleph}_1(h_1,h_2)\Big]^0 \subseteq \{K_i \subseteq \mathbb{R} | \, K_i \subseteq K_j, i,j = 1,2\}, \tag{38}$$

confirming $\widetilde{\aleph}_1(h_1,h_2)$ is a compactly supported.

Thus $\widetilde{\aleph}_1(\bar{N})$ is relatively compact on $M_c^1 \times M_c^2$, and by Ascoli-Arzelá theorem, also on $C(\widehat{\varpi}, M_c^1) \times C(\widehat{\varpi}, M_c^2)$. □

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Lemma 6 ([38]). *Let S be a nonempty, bounded, closed and convex subset of a Banach semilinear space $C(\varpi, M_c)$ endowed with the cancellation property. If h: $S \to S$ is a compact operator, then h admits at least one fixed point in S.*

Theorem 1. *Assume there exists $\ell > 0$ such that $\hat{f}_i : \varpi \times B(\hat{0},\ell) \times B(\hat{0},\ell) \to M_c^i$ $(i = 1,2)$ acts as compact operator and $\Psi, \Phi \in C(B(\hat{0}, \frac{\ell}{2}))$ are compact-supported. Then there exist $\hat{a} \in (0, a]$ and $\hat{b} \in (0, b]$ such that the equation (3) admits at least one $(\dagger)$-weak solution on $C(\widehat{\varpi}, M_c^1) \times C(\widehat{\varpi}, M_c^2)$, where $\widehat{\varpi} = [0, \hat{a}] \times [0, \hat{b}] \subset \varpi$.*

Proof. Define the operator $\widetilde{\aleph}_1 : \bar{N} \to C(\widehat{\varpi}, M_c^1) \times C(\widehat{\varpi}, M_c^2)$ as in (18) and operators $\widetilde{W}_1 : \bar{N} \to C(\widehat{\varpi}, M_c^1)$, $\widetilde{Q}_1 : \bar{N} \to C(\widehat{\varpi}, M_c^2)$ as in (19). One readily verifies that $\widetilde{\aleph}_1$ is well-defined. By Lemma 4, $\widetilde{\aleph}_1 : \bar{N} \to \bar{N}$ is a continuous. Lemma 5 combined with the Ascoli-Arzelá theorem implies that $\widetilde{\aleph}_1(\bar{N})$ is relatively compact. Hence $\widetilde{\aleph}_1$ is a compact operator by Definition 9. Lemma 6 guarantees that $\widetilde{\aleph}_1$ admits at least one fixed point in $\bar{N}$, which constitutes a $(\dagger)$-weak solution of (3). □

We now establish the existence of $(\ddagger)$-weak solution for (3) under the following hypotheses:

$(\mathbf{J_1})$ $\widehat{C}_{\Psi}^{\hat{f}_1}(\varpi, M_c^1) \neq \varnothing$, $\widehat{C}_{\Phi}^{\hat{f}_2}(\varpi, M_c^2) \neq \varnothing$.

$(\mathbf{J_2})$ If $h_1 \in \widehat{C}_{\Psi}^{\hat{f}_1}(\varpi, M_c^1)$, then for all $(x,y) \in \varpi = [0,a] \times [0,b]$ and each $h_2 \in \widehat{C}_{\Phi}^{\hat{f}_2}(\varpi, M_c^2)$, the following inclusions hold:

$$H_1(x,y) = \Psi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\alpha,p} \hat{f}_1(x,y,h_1(x,y),h_2(x,y)) \in \widehat{C}_{\Psi}^{\hat{f}_1}(\varpi, M_c^1),$$
$$H_2(x,y) = \Phi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\beta,p} \hat{f}_2(x,y,h_1(x,y),h_2(x,y)) \in \widehat{C}_{\Phi}^{\hat{f}_2}(\varpi, M_c^2).$$

Lemma 7. *Assume hypotheses $(\mathbf{J_1})$ and $(\mathbf{J_2})$ hold, and there exists $\ell > 0$ such that*
(i) For $i = 1,2$, $\hat{f}_i : \varpi \times B(\hat{0},\ell) \times B(\hat{0},\ell) \to M_c^i$ is compact operator.
(ii) $\Psi, \Phi \in C(B(\hat{0}, \frac{\ell}{2}))$.
Then there exist $\hat{a} \in [0, a]$ and $\hat{b} \in [0, b]$ such that the operator $\widetilde{\aleph}_2(h_1,h_2) = \begin{pmatrix} \widetilde{W}_2(h_1,h_2) \\ \widetilde{Q}_2(h_1,h_2) \end{pmatrix}$, constitutes a continuous operator from $\bar{N}$ to itself, where for $(x,y) \in \widehat{\varpi} = [0,\hat{a}] \times [0,\hat{b}] \subset \varpi$:

$$\widetilde{W}_2(h_1,h_2) := \Psi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\alpha,p} \hat{f}_1(x,y,h_1(x,y),h_2(x,y)),$$
$$\widetilde{Q}_2(h_1,h_2) := \Phi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\beta,p} \hat{f}_2(x,y,h_1(x,y),h_2(x,y)),$$

with $\bar{N} := \Big\{(h_1,h_2) \in \widehat{C}_{\Psi}^{\hat{f}_1}(\widehat{\varpi}, M_c^1) \times \widehat{C}_{\Phi}^{\hat{f}_2}(\widehat{\varpi}, M_c^2) \mid \max\{H(h_1,\hat{0}), H(h_2,\hat{0})\} \leq \ell\}$.

Proof. Since hypotheses $(\mathbf{J_1})$ and $(\mathbf{J_2})$ hold for all $(x,y) \in \varpi$ and every $\begin{pmatrix} h_{11}(x,y) \\ h_{21}(x,y) \end{pmatrix}$, $\begin{pmatrix} h_{12}(x,y) \\ h_{22}(x,y) \end{pmatrix} \in \widehat{C}_{\Psi}^{\hat{f}_1}(\varpi, M_c^1) \times \widehat{C}_{\Phi}^{\hat{f}_2}(\varpi, M_c^2)$, we obtain

$$\begin{aligned} &H(\widetilde{\aleph}_2(h_{11},h_{21}), \widetilde{\aleph}_2(h_{12},h_{22})) \\ &= \max\Big\{ \sup_{(x,y)\in\varpi} d_\infty\Big(\widetilde{W}_2(h_{11}(x,y),h_{21}(x,y)), \widetilde{W}_2(h_{12}(x,y),h_{22}(x,y))\Big), \\ &\qquad \sup_{(x,y)\in\varpi} d_\infty\Big(\widetilde{Q}_2(h_{11}(x,y),h_{21}(x,y)), \widetilde{Q}_2(h_{12}(x,y),h_{22}(x,y))\Big)\Big\}. \end{aligned}$$

As the proof follows identical reasoning to Lemma 4, the detailed derivation is omitted here. □

Following an analogous proof to Lemma 5, we obtain the following result.

Lemma 8 ([14]). *Under all assumptions of* Lemma 7, *if* Ψ *and* Φ *possess compact supports, then* $\widetilde{\aleph}_2$ *is relatively compact on* $\widehat{C}^{\hat{f}_1}_{\Psi}(\varpi, M^1_c) \times \widehat{C}^{\hat{f}_2}_{\Phi}(\varpi, M^2_c)$.

Theorem 2. *If all conditions of* Theorem 1 *hold and* $(\mathbf{J_1})$ *and* $(\mathbf{J_2})$ *are satisfied, then there exist* $\hat{a} \in (0, a]$, $\hat{b} \in (0, b]$ *such that* (3) *admits at least one* $(\ddagger)$*-weak solution on* $\widehat{C}^{\hat{f}_1}_{\Psi}(\varpi, M^1_c) \times \widehat{C}^{\hat{f}_2}_{\Phi}(\varpi, M^2_c)$, *where* $\widehat{\varpi}$ *is as defined in* Lemma 7.

Proof. The proof follows identical reasoning to Theorem 1 and is therefore omitted. □

Remark 3. *Building upon the methodological framework established in* [29], *we implement* Lemmas 6-8 *and Ascoli-Arzelá theorem within Banach semilinear spaces without imposing Lipschitz conditions. This approach yields a fundamentally distinct proof methodology for the existence of* $(\ddagger)$*-weak solutions compared to* [37, Theorem 2].

4. Numerical Example with Potential Applications

In the sequel, we present the following numerical example with potential applications to verify our main results: For each $(x, y) \in \varpi = [0, a] \times [0, a]$ and $k = 1, 2$,

$$\begin{cases} {}^{CK}_{gh}D^{\frac{1}{2},2}_{k} h_1(x,y) = \dfrac{[\Gamma(\frac{1}{4})]^2}{8[\Gamma(\frac{3}{4})]^2} x^{-1}y^{-1}h_1(x,y) + \dfrac{P[\Gamma(\frac{1}{4})]^2}{4[\Gamma(\frac{3}{4})]^2} - P\pi \\ \qquad + \dfrac{[\Gamma(\frac{1}{4})]^2}{2[\Gamma(\frac{3}{4})]^2} x^{-\frac{5}{2}}y^{-\frac{5}{2}}h_2(x,y) + \dfrac{5P[\Gamma(\frac{1}{4})]^2}{2[\Gamma(\frac{3}{4})]^2} x^{-\frac{5}{2}}y^{-\frac{5}{2}} \\ {}^{CK}_{gh}D^{\frac{1}{2},2}_{k} h_2(x,y) = \dfrac{9[\Gamma(\frac{3}{4})]^2}{2[\Gamma(\frac{1}{4})]^2} h_1(x,y) - 18x^{-\frac{1}{2}}y^{-\frac{1}{2}}h_2(x,y) - 90Px^{-\frac{1}{2}}y^{-\frac{1}{2}} \\ \qquad - \dfrac{9P[\Gamma(\frac{3}{4})]^2}{[\Gamma(\frac{1}{4})]^2}, \\ h_1(x,0) = h_1(0,y) = h_1(0,0) = 2P, \\ h_2(x,0) = h_2(0,y) = h_2(0,0) = -5P, \end{cases} \tag{39}$$

where $h_1(x,y)$ and $h_2(x,y)$ are fuzzy-valued functions and P is a fuzzy number. Corresponding to the system (3), it is readily verified that the functions $\hat{f}_1(x,y,h_1(x,y),h_2(x,y)) := \frac{[\Gamma(\frac{1}{4})]^2}{8[\Gamma(\frac{3}{4})]^2} \cdot x^{-1}y^{-1}h_1(x,y) + \frac{[\Gamma(\frac{1}{4})]^2}{2[\Gamma(\frac{3}{4})]^2} x^{-\frac{5}{2}}y^{-\frac{5}{2}}h_2(x,y) + \frac{5P[\Gamma(\frac{1}{4})]^2}{2[\Gamma(\frac{3}{4})]^2} x^{-\frac{5}{2}}y^{-\frac{5}{2}} + \frac{P[\Gamma(\frac{1}{4})]^2}{4[\Gamma(\frac{3}{4})]^2} - P\pi$ and $\hat{f}_2(x,y,h_1(x,y),h_2(x,y)) := \frac{9[\Gamma(\frac{3}{4})]^2}{2[\Gamma(\frac{1}{4})]^2} h_1(x,y) - 18x^{-\frac{1}{2}}y^{-\frac{1}{2}}h_2(x,y) - 90Px^{-\frac{1}{2}}y^{-\frac{1}{2}} - \frac{9P[\Gamma(\frac{3}{4})]^2}{[\Gamma(\frac{1}{4})]^2}$ are compact operators in (39). Furthermore, from (12), we immediately obtain $\Psi(x,y) = 2P$ and $\Phi(x,y) = -5P$.

We note that in [2], stated that the prey-predator system (1) is related to ecological models by virtue of their connection with memory and fractal which are distinctive characteristics of these ecological models. Now, we extend model (1) to FFPDE (i.e., (39)), where the right-hand side of the equations is further generalized to represent a multi-species biological population model under uncertain environments, and consider the symmetry coupled system (39) to verify the existence of solutions to (3).

Let $P = (3,4,5)$ be a triangular fuzzy number. From [15], its ϱ-level set is given by:

$$[P]^{\varrho_1} = [P^-_{\varrho_1}, P^+_{\varrho_1}] = [\varrho_1 + 3, 5 - \varrho_1], [P]^{\varrho_2} = [P^-_{\varrho_2}, P^+_{\varrho_2}] = [\varrho_2 + 3, 5 - \varrho_2].$$

Consequently, we obtain:

$$[\Psi(x,y)]^{\varrho_1} = [2P]^{\varrho_1} = 2[\varrho_1 + 3, 5 - \varrho_1] = [2\varrho_1 + 6, 10 - 2\varrho_1],$$

$$[\Phi(x,y)]^{\varrho_2} = [-5P]^{\varrho_2} = [-5\varrho_2 - 15, -25 + 5\varrho_2].$$

By Definition 7, one has

$$[\Psi(x,y)]^0 = 2[3,5] \subseteq [6,10], [\Phi(x,y)]^0 = (-5)[3,5] = [-25,-15].$$

Let $K_1 = [6,10]$ and $K_2 = [-25,-15]$ be compact sets, implying that $\Psi(x,y)$ and $\Phi(x,y)$ possess compact supports. Define $\ell_1 = 4D(P,\hat{0}), \ell_2 = 10D(P,\hat{0})$, and take $\ell = \max\{\ell_1,\ell_2\} = 10D(P,\hat{0})$. By metric properties: $D(\Psi(x,y),\hat{0}) = D(2P,\hat{0}) = 2D(P,\hat{0}) \leq \frac{\ell_1}{2}$, $H(\Phi(x,y),\hat{0}) = D(-5P,\hat{0}) = 5D(P,\hat{0}) \leq \frac{\ell_2}{2}$. Hence, it is established that $\Psi(x,y), \Phi(x,y) \in C\left(\varpi, B\left(\hat{0}, \frac{\ell}{2}\right)\right)$.

(Case I) For $k = 1$, applying the Buckley-Feuring (BF) fuzzification strategy along with [39, Definition 4.1] and combining Theorem 1 with the compact support and continuity results, the $(\dagger)$-weak solution of system (39) is obtained as:

$$\begin{cases} h_1(x,y) = 2P - 2Px^{\frac{1}{2}}y^{\frac{1}{2}} - 2Pxy, \\ h_2(x,y) = -5P - \frac{1}{2}Px^{\frac{3}{2}}y^{\frac{3}{2}}. \end{cases}$$

(Case II) For $k = 2$, based on Lemma 1 (iii)-(v) and (17), while adopting a strategy analogous to **(Case I)**, the BF solution of (39) is derived as

$$(h_1(x,y), h_2(x,y)) = (2P \ominus 2Px^{\frac{1}{2}}y^{\frac{1}{2}} \ominus 2Pxy, -5P \ominus \frac{1}{2}Px^{\frac{3}{2}}y^{\frac{3}{2}}).$$

By the continuity of the extension principle, the level sets of the fuzzy solutions to (39) are:

$$[\Delta(x,y,P)]^{\varrho_1} = \Big[2(\varrho_1+3) - 2(\varrho_1+3)x^{\frac{1}{2}}y^{\frac{1}{2}} - 2(\varrho_1+3)xy,$$
$$2(5-\varrho_1) - 2(5-\varrho_1)x^{\frac{1}{2}}y^{\frac{1}{2}} - 2(5-\varrho_1)xy\Big], \tag{40}$$

$$[\Theta(x,y,P)]^{\varrho_2} = \left[-5(\varrho_2+3) - \frac{1}{2}(\varrho_2+3)x^{\frac{3}{2}}y^{\frac{3}{2}},\ -5(5-\varrho_2) - \frac{1}{2}(5-\varrho_2)x^{\frac{3}{2}}y^{\frac{3}{2}}\right]. \tag{41}$$

Figure 1. presents simulation results of the level sets for the fuzzy solutions in (40) and (41). The left and right subfigures show seven level sets of $\Theta(x,y,P)$ and $\Delta(x,y,P)$ at seven fixed x values, respectively. Each surface group corresponds to a ϱ_1 or ϱ_2 level set, with inter-surface distances characterizing the fuzzy solutions. When six y values are fixed, the variations of level sets with x and ϱ are consistent with Figure 1. The curves in the ϱOy planes represent contour lines of $[\Theta(x,y,P)]^{\varrho_2}$ and $[\Delta(x,y,P)]^{\varrho_1}$, respectively.

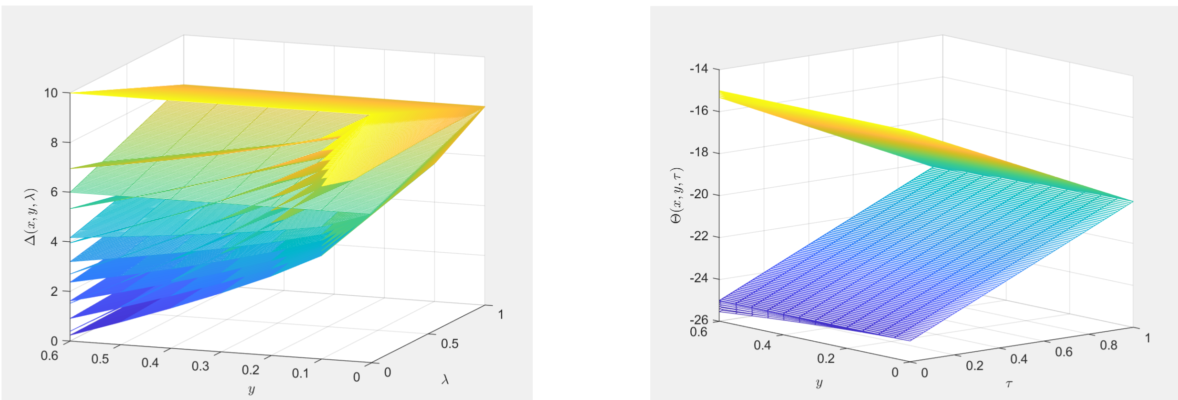

Figure 1. Numerical simulation for fuzzy solutions of (39)

To verify $(\mathbf{J_1})$–$(\mathbf{J_1})$, let $L_1 = \frac{[\Gamma(\frac{1}{4})]^2}{[\Gamma(\frac{3}{4})]^2}$, $L_2 = \frac{[\Gamma(\frac{3}{4})]^2}{[\Gamma(\frac{1}{4})]^2}$. Since

$$[h_1]^{\varrho_2} = \left[2P \ominus 2Px^{\frac{1}{2}}y^{\frac{1}{2}} \ominus 2Pxy\right]^{\varrho_2} = \left(2 - 2x^{\frac{1}{2}}y^{\frac{1}{2}} - 2xy\right)[\varrho_2 + 3, 5 - \varrho_2], \tag{42}$$

$$[h_2]^{\varrho_1} = \left[-5P \ominus \frac{1}{2}Px^{\frac{3}{2}}y^{\frac{3}{2}}\right]^{\varrho_1} = \left(-5 - \frac{1}{2}x^{\frac{3}{2}}y^{\frac{3}{2}}\right)[\varrho_1 + 3, 5 - \varrho_1], \tag{43}$$

we get

$$len[h_1]^{\varrho_2} = (2 - 2\varrho_2)\left(2 - 2x^{\frac{1}{2}}y^{\frac{1}{2}} - 2xy\right),$$
$$len[h_2]^{\varrho_1} = (2 - 2\varrho_1)\left(-5 - \frac{1}{2}x^{\frac{3}{2}}y^{\frac{3}{2}}\right),$$

and by (7), one has

$$\begin{aligned}
&len[(-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_2(x,y,h_1(x,y),h_2(x,y))]^{\varrho_2}\\
&= L_2(-72x^{\frac{5}{2}}y^{\frac{5}{2}} + \frac{18}{\pi}x^2y^2 + \frac{18}{\pi}x^{\frac{3}{2}}y^{\frac{3}{2}}) \cdot len[A]^{\varrho_2}\\
&\geq -0.414(2 - 2\varrho_2),\\
&len[\Phi(x,y)]^{\varrho_2} = -5(2 - 2\varrho_2).
\end{aligned}$$

Hence,

$$len[\Phi(x,y)]^{\varrho_2} \leq len[(-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_2(x,y,h_1(x,y),h_2(x,y))]^{\varrho_2}.$$

By Proposition 21 (b) of [31], the H-difference $\Phi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_2(x,y,h_1(x,y), h_2(x,y))$ exists. Taking

$$H_1(x,y) = \Psi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_1(x,y,h_1(x,y),h_2(x,y)),$$

its level set is:

$$\begin{aligned}
[H_1(x,y)]^{\varrho_2} = \Bigg(&2 - \frac{\pi(\Gamma(\frac{1}{4}))^2}{8(\Gamma(\frac{3}{4}))^2}x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{(\pi^2-4)(\Gamma(\frac{1}{4}))^2 + 16\pi(\Gamma(\frac{3}{4}))^2}{8\pi(\Gamma(\frac{3}{4}))^2}xy\\
&- \frac{(1-\pi)(\Gamma(\frac{1}{4}))^2}{8(\Gamma(\frac{3}{4}))^2}\Bigg) \cdot [\varrho_2 + 3, 5 - \varrho_2],
\end{aligned}$$

with interval length:

$$\begin{aligned}
len[H_1(x,y)]^{\varrho_2} = \Bigg(&2 - \frac{\pi(\Gamma(\frac{1}{4}))^2}{8(\Gamma(\frac{3}{4}))^2}x^{\frac{1}{2}}y^{\frac{1}{2}} - \frac{(\pi^2-4)(\Gamma(\frac{1}{4}))^2 + 16\pi(\Gamma(\frac{3}{4}))^2}{8\pi(\Gamma(\frac{3}{4}))^2}xy\\
&- \frac{(1-\pi)(\Gamma(\frac{1}{4}))^2}{8(\Gamma(\frac{3}{4}))^2}\Bigg) \cdot (2 - 2\varrho_2).
\end{aligned}$$

Using an analogous computational approach, we obtain:

$$\left[(-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_2(x,y,H_1(x,y),h_2(x,y))\right]^{\varrho_2} = \frac{1}{2}x^{\frac{3}{2}}y^{\frac{3}{2}} \cdot len[P]^{\varrho_2},$$

$$len\left[(-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_2(x,y,H_1(x,y),h_2(x,y))\right]^{\varrho_2} \leq \frac{1}{2}(2 - 2\varrho_2).$$

Hence, the H-difference $\Phi(x,y) \ominus (-1)_F^{RL}\mathcal{I}_{0+}^{\frac{1}{2},2}\hat{f}_2(x,y,H_1(x,y),h_2(x,y))$ exists.

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Similarly, from (42) and (43):

$$\begin{aligned}&\left[(-1)_F^{RL}\mathcal{I}_{0^+}^{\frac{1}{2},2}\hat{f}_1(x,y,h_1(x,y),h_2(x,y))\right]^{\varrho_1}\\&=\left(\frac{\pi(\Gamma(\frac{1}{4}))^2}{8(\Gamma(\frac{3}{4}))^2}x^{\frac{1}{2}}y^{\frac{1}{2}}+\frac{(\pi^2-4)(\Gamma(\frac{1}{4}))^2+16\pi(\Gamma(\frac{3}{4}))^2}{8\pi(\Gamma(\frac{3}{4}))^2}xy+\frac{(1-\pi)(\Gamma(\frac{1}{4}))^2}{8(\Gamma(\frac{3}{4}))^2}\right)\cdot[P]^{\varrho_1}.\end{aligned}$$

Consequently:

$$\begin{aligned}&len[\Psi(x,y)]^{\varrho_1}=2(2-2\varrho_1),\\&len\left[(-1)_F^{RL}\mathcal{I}_{0^+}^{\frac{1}{2},2}\hat{f}_1(x,y,h_1(x,y),h_2(x,y))\right]^{\varrho_1}\leq 1.175(2-2\varrho_1).\end{aligned}$$

This implies

$$len[\Psi(x,y)]^{\varrho_1}\geq len\left[(-1)_F^{RL}\mathcal{I}_{0^+}^{\frac{1}{2},2}\hat{f}_1(x,y,h_1(x,y),h_2(x,y))\right]^{\varrho_1}.$$

By Proposition 21(a) of [31], the H-difference $[H_2(x,y)]^{\varrho_1}=\Phi(x,y)\ominus(-1)_F^{RL}\mathcal{I}_{0^+}^{\frac{1}{2},2}\hat{f}_2(x,y,h_1(x,y),h_2(x,y))$ exists. Using identical methodology:

$$len[H_2(x,y)]^{\varrho_1}=\left(-5+72L_2x^{\frac{5}{2}}y^{\frac{5}{2}}-\frac{18}{\pi}L_2x^2y^2-\frac{18}{\pi}L_2x^{\frac{3}{2}}y^{\frac{3}{2}}\right)\cdot(2-2\varrho_1),$$

$$len\left[(-1)_F^{RL}\mathcal{I}_{0^+}^{\frac{1}{2},2}\hat{f}_1(x,y,h_1(x,y),H_2(x,y))\right]^{\varrho_1}\leq 1.175(2-2\varrho_1).$$

The aforementioned procedure establishes the existence of the H-difference $\Psi(x,y)\ominus(-1)_F^{RL}\mathcal{I}_{0^+}^{\frac{1}{2},2}\hat{f}_1(x,y,h_1(x,y),H_2(x,y)$ Heretofore, these verify the assumptions $(\mathbf{J_1})$ and $(\mathbf{J_2})$ in Theorem 2. Given that $\Psi(x,y)$ and $\Phi(x,y)$ are compact-supported and $\Psi(x,y),\Phi(x,y)$ $\in C(\varpi,B(\hat{0},\frac{\ell}{2}))$, it follows from Theorem 2 that a $(\ddagger)$-weak solution to (39) exists on $\widehat{C}_{\Psi}^{\hat{f}_1}(\varpi,M_c^1)\times\widehat{C}_{\Phi}^{\hat{f}_2}(\varpi,M_c^2)$ and

$$\begin{cases}h_1(x,y)=2P\ominus 2Px^{\frac{1}{2}}y^{\frac{1}{2}}\ominus 2Pxy,\\h_2(x,y)=-5P\ominus\frac{1}{2}Px^{\frac{3}{2}}y^{\frac{3}{2}}.\end{cases}$$

5. U-HS Analysis

In this section, we mainly study U-HS of the solutions to the system (3). Specifically, all results presented in this section are established under the assumption of existence. We contend that U-HS idea is important for practical issues in analysis. The interesting feature of stability is that to search a U-HS of a system, who exact solution does not exist, which is typically challenging or time consuming. According to U-HS, there exists a close approximate solution of system to exact solution. As we know that mostly a mathematical models are non-linear in nature and some time its exact solution does not exist or difficult to be obtained. Therefore, we need to find best approximate solution for such problems.

Definition 10 ([27])**.** *The system* (3) *is said to be* U-HS *if there exists* $\epsilon=\max\{\epsilon_1,\epsilon_2\}>0$ *such that the inequality system*

$$\begin{cases}\left|{}^{CK}_{gh}D_k^{\alpha,p}h_1(x,y)-\hat{f}_1(x,y,h_1(x,y),h_2(x,y))\right|\leq\epsilon_1,\\\left|{}^{CK}_{gh}D_k^{\beta,p}h_2(x,y)-\hat{f}_2(x,y,h_1(x,y),h_2(x,y))\right|\leq\epsilon_2.\end{cases}\tag{44}$$

holds. For notational simplicity in subsequent proofs, denote $h_i(x,y) = h_i$ and $\overline{h}_i(x,y) = \overline{h}_i$ for $i = 1,2,3,4$. If there exists a constant $\bar{C} > 0$ such that for every solution (h_1,h_2) of (44), there exists a solution $(\overline{h}_1,\overline{h}_2)$ of (3) satisfying

$$\left|(h_1,h_2)) - (\overline{h}_1,\overline{h}_2)\right| \le \bar{C}\epsilon,$$

then system (3) is U-HS.

Remark 4. *(h_1,h_2) is a solution of (44) if and only if: for each $i = 1,2$ and ϵ_i as in Definition 10, there exists $g_i \in (\varpi, M_c^i)$ (depending on (h_1,h_2)) satisfying $|g_i(x,y)| \le \epsilon_i$ and the perturbed system*

$$\begin{cases} {}^{CK}_{gh}D_k^{\alpha,p} h_1 = \hat{f}_1(x,y,h_1,h_2) + g_1(x,y), \\ {}^{CK}_{gh}D_k^{\beta,p} h_2 = \hat{f}_2(x,y,h_1,h_2) + g_2(x,y). \end{cases}$$

holds.

The following proof is provided only for type (†) solutions, as the proof for type (‡) solutions follows an identical procedure and is therefore omitted.

Lemma 9. *Let (h_1,h_2) be a solution of (44). Then the following inequality hold:*

$$\begin{cases} h_1 - \left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\alpha,p}\hat{f}_1(x,y,h_1,h_2)\right) \le \delta_1\epsilon_1, \\ h_2 - \left(\Phi(x,y) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\beta,p}\hat{f}_2(x,y,h_1,h_2)\right) \le \delta_2\epsilon_2. \end{cases}$$

Proof. By Remark 4, the system can be written as:

$$\begin{cases} {}^{CK}_{gh}D_k^{\alpha,p} h_1 = \hat{f}_1(x,y,h_1,h_2) + g_1(x,y), \\ {}^{CK}_{gh}D_k^{\beta,p} h_2 = \hat{f}_2(x,y,h_1,h_2) + g_2(x,y), \\ h_1(x,0) = \zeta_1(x), h_2(x,0) = \eta_1(x), \\ h_1(0,y) = \zeta_2(y), h_2(0,y) = \eta_2(y). \end{cases}$$

From Lemma 3, we obtain:

$$\begin{cases} h_1 = \Psi(x,y) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\alpha,p}\hat{f}_1(x,y,h_1,h_2) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\alpha,p}g_1(x,y), \\ h_2 = \Phi(x,y) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\beta,p}\hat{f}_2(x,y,h_1,h_2) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\beta,p}g_2(x,y). \end{cases}$$

Thus:

$$\begin{aligned} &\left|h_1 - \left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\alpha,p}\hat{f}_1(x,y,h_1,h_2)\right)\right| = \left|{}^{RL}_{F}\mathcal{I}_{0^+}^{\alpha,p}g_1(x,y)\right| \\ &\le \left|\frac{p^{2-\alpha_1-\alpha_2}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^x\int_0^y (x^p - s^p)^{\alpha_1-1}(y^p - t^p)^{\alpha_2-1}s^{p-1}t^{p-1}\epsilon_1 dtds\right| \\ &\le \delta_1\epsilon_1, \end{aligned}$$

where $\delta_1 = \dfrac{p^{-(\alpha_1+\alpha_2)}x^{\alpha_1 p}y^{\alpha_2 p}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)}$. Similarly:

$$\left|h_2(x,y) - \left(\Phi(x,y) + {}^{RL}_{F}\mathcal{I}_{0^+}^{\beta,p}\hat{f}_2(x,y,h_1(x,y),h_2(x,y))\right)\right| \le \delta_2\epsilon_2,$$

where $\delta_2 = \dfrac{p^{-(\beta_1+\beta_2)}x^{\beta_1 p}y^{\beta_2 p}}{\Gamma(\beta_1+1)\Gamma(\beta_2+1)}$. □

Now we present the following two hypotheses which are helpful in building our main result:
$(\mathbf{A_1})$ Letting (h_1, h_2) and (h_3, h_4) are any solutions of (3), then there exist $L_{\hat{f}_1}$ and $L_{\hat{f}_2}$ such that

$$\left|\hat{f}_1(x,y,h_1,h_2) - \hat{f}_1(x,y,h_3,h_4)\right| \le L_{\hat{f}_1}(|h_1 - h_3| + |h_2 - h_4|),$$
$$\left|\hat{f}_2(x,y,h_1,h_2) - \hat{f}_2(x,y,h_3,h_4)\right| \le L_{\hat{f}_2}(|h_1 - h_3| + |h_2 - h_4|).$$

$(\mathbf{A_2})$ There exist some constants $c_i, d_i > 0$ (i=1, 2) and $c_3, d_3 \ge 0$ with

$$\left|\hat{f}_1(x,y,h_1,h_2)\right| \le c_1|h_1| + c_2|h_2| + c_3,$$
$$\left|\hat{f}_2(x,y,h_1,h_2)\right| \le d_1|h_1| + d_2|h_2| + d_3.$$

Theorem 3. *Under the conditions in Theorems 1 and 2 and assumptions* $(\mathbf{A_1})$ *and* $(\mathbf{A_2})$, *the symmetry coupled system* (3) *is* U-HS *if* $c_1c_2L_{\hat{f}_1}L_{\hat{f}_2} \neq 1$.

Proof. Since (h_1, h_2) and (h_3, h_4) are arbitrary solutions of (3), it follows from (16) that

$$\begin{aligned}
&|h_1 - h_3| \\
&= \left|\left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_1,h_2)\right) - \left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_3,h_4)\right)\right| \\
&= \left(\Psi(x,y) - \left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_1,h_2)\right)\right. \\
&\quad + \left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_1,h_2)\right) + {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_1,h_2) \\
&\quad \left. - \left(\Psi(x,y) + {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_3,h_4)\right)\right) \\
&\le \delta_1\epsilon_1 + \left|{}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_1,h_2) - {}^{RL}_{F}\mathcal{I}^{\alpha,p}_{0^+}\hat{f}_1(x,y,h_3,h_4)\right| \\
&\le \delta_1\epsilon_1 + \frac{p^{-(\alpha_1+\alpha_2)}x^{\alpha_1 p}y^{\alpha_2 p}}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \times \left|\hat{f}_1(x,y,h_1,h_2) - \hat{f}_1(x,y,h_3,h_4)\right| \\
&\le \delta_1\epsilon_1 + \delta_1 L_{\hat{f}_1}|h_2 - h_4| + \delta_1 L_{\hat{f}_1}|h_1 - h_3|.
\end{aligned}$$

Thus, we have

$$|h_1 - h_3| - \frac{\delta_1 L_{\hat{f}_1}}{1 - \delta_1 L_{\hat{f}_1}}|h_2 - h_4| \le \frac{\delta_1\epsilon_1}{1 - \delta_1 L_{\hat{f}_1}}. \tag{45}$$

Similarly, one can obtain

$$|h_2 - h_4| - \frac{\delta_2 L_{\hat{f}_2}}{1 - \delta_2 L_{\hat{f}_2}}|h_1 - h_3| \le \frac{\delta_2\epsilon_2}{1 - \delta_2 L_{\hat{f}_2}}. \tag{46}$$

According to (45) and (46), its matrix can be expressed as the following form:

$$\begin{pmatrix} 1 & -L_{\hat{f}_1}c_1 \\ -L_{\hat{f}_2}c_2 & 1 \end{pmatrix} \times \begin{pmatrix} |h_1 - h_3| \\ |h_2 - h_4| \end{pmatrix} \le \begin{pmatrix} c_1\epsilon_1 \\ c_2\epsilon_2 \end{pmatrix}, \tag{47}$$

where $c_1 = \dfrac{\delta_1}{1 - \delta_1 L_{\hat{f}_1}}$ and $c_2 = \dfrac{\delta_2}{1 - \delta_2 L_{\hat{f}_2}}$. Letting $c_1c_2L_{\hat{f}_1}L_{\hat{f}_2} \neq 1$, then we solve the matrix inequality (47) and have

$$|h_1 - h_3| \le \frac{c_1\epsilon_1}{1 - c_1c_2L_{\hat{f}_1}L_{\hat{f}_2}} + \frac{c_1c_2L_{\hat{f}_1}\epsilon_2}{1 - c_1c_2L_{\hat{f}_1}L_{\hat{f}_2}},$$
$$|h_2 - h_4| \le \frac{c_2\epsilon_2}{1 - c_1c_2L_{\hat{f}_1}L_{\hat{f}_2}} + \frac{c_1c_2L_{\hat{f}_2}\epsilon_1}{1 - c_1c_2L_{\hat{f}_1}L_{\hat{f}_2}},$$

which mean that

$$|(h_1, h_2) - (h_3, h_4)| \leq |h_1 - h_3| + |h_2 - h_4| \leq \bar{C}\epsilon,$$

where $\bar{C} = \dfrac{c_1 + c_2 + c_1 c_2 (L_{\hat{f}_1} + L_{\hat{f}_2})}{1 - c_1 c_2 L_{\hat{f}_1} L_{\hat{f}_2}}$. Thus, the new fuzzy fractional partial differential symmetry coupled system (3) is U-HS. □

6. Conclusions

In this paper, the existence and stability of solutions for a class of C-K FFPDEs symmetry coupled systems involving gH-type difference were studied. Its form is as follows

$$\begin{cases} {}^{CK}_{gh}D_k^{\alpha,p} h_1(x,y) = \hat{f}_1(x,y,h_1(x,y),h_2(x,y)), \\ {}^{CK}_{gh}D_k^{\beta,p} h_2(x,y) = \hat{f}_2(x,y,h_1(x,y),h_2(x,y)), \\ h_1(x,0) = \zeta_1(x), h_2(x,0) = \eta_1(x), \\ h_1(0,y) = \zeta_2(y), h_2(0,y) = \eta_2(y), \end{cases} \tag{48}$$

for any $(x,y) \in \varpi = [0,a] \times [0,a]$ and $k = 1,2$. By employing Schauder fixed point theorem (i.e., Lemma 6), the relevant results were obtained under more general situations without Lipschitz conditions, which is common and very important.

For the work of this paper, the innovation points are as follows:

- By incorporating concepts of relative compactness and utilizing Schauder fixed point theorem, the existence of two classes of gH-weak solutions for (48) was proved without Lipschitz condition. Compared with [36], the system is considered in a more general setting in this study, which enhances its practical significance.
- By constructing specific examples, the existence of two classes of gH-type weak solutions was verified. Based on the obtained two classes of weak solutions, numerical simulations were conducted for analysis. The results demonstrate that the existence of solutions to (48) was established.
- Within the theoretical framework of Ulam-Hyers stability, the stability analysis of (48) was proposed. However, investigations into the existence theory and stability analysis of solutions for symmetry coupled systems remain scarcely documented. Moreover, the stability conditions reveal the long-term evolutionary trends of species populations under fractal habitats and fuzzy environmental carrying capacity, providing guidance for endangered species conservation strategies. Thus, the symmetry coupled system (48) exhibits substantial research value.

Compared with the existing research, the type of fuzzy fractional derivative investigated in this paper is more extensive, and a new demonstration method was adopted. A series of new conclusions were obtained, and the stability proof of the symmetry coupled system was given. Although the current research focuses on the field of PDEs, the future work intends to apply the method system proposed in this paper to other mathematical structures and engineering problems, including fuzzy stochastic fractional differential equations, time-delay systems, neural networks, signal processing and other directions.

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Abbreviations

The following abbreviations are used in this manuscript:

gH-	Generalized Hukuhara
C-K	Caputo–Katugampola
PDEs	partial differential equations
FPDEs	fractional partial differential equations
FFPDEs	fuzzy fractional partial differential equations
H-	Hukuhara
H_m-	Hausdorff metric
U-HS	Ulam–Hyers stability
BF	Buckley–Feuring

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