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Article

Feynman Path Integral and Landau Density Matrix in Probability Representation of Quantum States*

Olga V. Man'ko 

P.N. Lebedev Physical Institute, Russian Academy of Sciences, Leninskii Prospect 53, Moscow 119991, Russia; mankoov@lebedev.ru

Abstract

Using construction of probability distributions describing density operators of quantum system states, the relation of Feynman path integral [1] with the time evolution of the density operator (Landau density matrix) [2] (as well as the state wave function) is found. The explicit expression of the probability in terms of the Green function of the Schrödinger equation is obtained. The equation for the Green function determined by arbitrary integral of motion is written. The examples of the probability distributions describing the evolution of the free particle states, as well as of the states of the systems with time dependent integrals of motion (like the oscillator) are discussed.

Keywords: tomogram; path integral; probability representation of states; quantizer; integrals of motion

1. Introduction

The quantum system states are discussed, using either the complex wave function $\psi(q, t)$, where q is the position of a particle satisfying the Schrödinger equation [3] or the density matrix introduced by Landau [2] or the density operator $\hat{\rho}$ [4] acting in a Hilbert space \mathcal{H} . The classical system states of the particle are described either by the position $q(t)$ and the velocity $\dot{q}(t)$ (or the momentum $p(t) = m\dot{q}(t)$) satisfying the Newton equation or by the joint probability representation $f(q(t), p(t))$ of the particle satisfying the evolution equation determined by the Hamiltonian

$$H(q, p, t) = \frac{p^2}{2} + V(q), \quad m = 1,$$

where the kinetic energy term depends on the momentum of a particle and potential energy term $V(q)$ determines the force acting onto the particle.

Recently the probability representation of quantum states was suggested in [5] and developed in [6–9]. Within the framework of the probability representation of quantum mechanics, the particle states are determined by probability distributions. These probability distributions determine the quantum density operators $\hat{\rho}$ both for mixed states and for pure states with the wave function $\psi(x) = \langle x | \psi \rangle$ and the density operator $\hat{\rho}_\psi = |\psi\rangle\langle\psi|$. Different representations of quantum mechanics, where the quantum states are determined in terms of different functions, which are quasi-probabilities were known before, for example, the Wigner function [10], the Husimi–Kano function [11,12], the Glauber–Sudarshan function [13,14], the Blokhintsev mixed density matrix [15], and the Wootters quantum mechanics without probability amplitudes [16].

The languages of classical mechanics and quantum mechanics are significantly different. In the classical mechanics, probability distribution functions are used for describing the states. The properties of classical probability distributions are described by conventional probability theory [17]. The different aspects of quantum system properties and methods of quantum–mechanical applications

*This paper is dedicated to Prof. Viktor V. Dodonov, Supervisor of my PhD These. I never forget his kind assistance and valuable advices, which helped me a lot.

to different areas of science within the framework of the probability theory are done in [18,19]. The entanglement phenomena were discussed, for example, in [20,21]. The entanglement phenomenon in quantum mechanics considered within the framework of the probability representation of quantum mechanics provides the possibility to introduce the notion of the entangled probability distributions in classical probability theory [22,23]. The different processes in quantum and nonlinear optics were discussed in [24–26] and within the framework of the symplectic and optical tomography schemes in [27–30]. Some mathematical aspects of tomography and the probability representation of quantum and classical states were considered in [31–34]. The wave function formalism was introduced for classical oscillator systems in [35,36]; in [37] the Hermitian operators were introduced in order to make the classical mechanics and quantum mechanics languages closer. The groupoid approach to the state evolution of the system containing quantum and classical parts was considered in [38] and dynamics properties of classical, quantum, and hybrid systems were discussed in [39]. The different aspects of cosmology theory were considered within the framework of the probability representation of quantum mechanics in [40,41]. Some applications of tomographic methods to different kinds of processes and experiments were discussed in [42–52]. The interesting analysis of the foundation of quantum–mechanics papers from the view point of teaching aspects was done in [53].

The aim of this article is to discuss properties of the probability representation of particle system states and the relation of this representation with known representation of the states by the wave functions [3] as well as the connection with the Feynman path integral method [1], where the evolution of the system state described by the Green function $\mathcal{G}(x, x', t)$ of the Schrödinger equation expressed in terms of the path integral, in view of the Lagrangian of the classical particle system, is used. The examples of the free particle and oscillator probability distribution constructions in terms of the path integral (or the Green function $\mathcal{G}(x, x', t)$) are found explicitly as well as the relation of the method with time-dependent integrals of motion [54–57] connected with the Green function $\mathcal{G}(x, x', t)$. The idea to consider the quantum states in terms of the probability distribution $G(X|\mu, \nu)$ is to construct the invertible map of the density operators $\hat{\rho}$ of the quantum states acting in a Hilbert spaced \mathcal{H} , where the density operator $\hat{\rho}$ and other operators like the position operator and the momentum operator act. This means that we consider maps shortly shown as $\hat{\rho} \rightarrow G_\rho(Y)$, $G_\rho(Y) \rightarrow \hat{\rho}$. Analogous maps were constructed for the Wigner and other mentioned quasi-probability distributions but were unknown for the case of probability distributions $G(Y)$. The general scheme is to find an invertible map of the operator \hat{A} onto the function $f_A(Y)$ for all the cases described, using a generic scheme of constructing two sets of operators called the quantizer $\hat{D}(Y)$ and dequantizer $\hat{U}(Y)$ pairs. Below we discuss this method and find the case of the pairs determining the functions $f_A(Y)$, which are probability distributions for density operators $\hat{A} = \hat{\rho}$.

2. Quantizer–Dequantizer Method

In order to introduce the notion of probability distributions describing quantum states, we discuss possibilities to describe density operators $\hat{\rho}$ of the states by functions f called symbols of the operators. It turned out [58,59] that there exist invertible maps of operators \hat{A} acting in a Hilbert space \mathcal{H} onto the functions $f_A(\vec{x})$, where $\vec{x} = x_1, x_2, \dots, x_N$ and x_i , $i = 1, \dots, N$ are parameters. Some of them can be continuous and some of them are discrete ones. The mentioned map has the form

$$f_A(\vec{x}) = \text{Tr} \hat{A} \hat{U}(\vec{x}), \quad (1)$$

where $\hat{U}(\vec{x})$ is called the dequantizer operator (see, for example, [60]) acting in the Hilbert space. The inverse transform has the form

$$\hat{A} = \int f_A(\vec{x}) \hat{D}(\vec{x}) d\vec{x}. \quad (2)$$

The operator $\hat{D}(\vec{x})$ is called the quantizer operator. In the case, where the dequantizer operator $\hat{U}(\vec{x})$ has the properties of density operators, i.e. $\hat{U}(\vec{x}) = \hat{U}^\dagger(\vec{x})$, $\text{Tr} \hat{U}(\vec{x}) = 1$ and diagonal elements of the dequantizer operators are nonnegative, symbols of the density operator $\hat{\rho}$ should have the

properties of probability distribution functions. Due to relations (1), (2), all information on the state operator $\hat{\rho}$ is available in the symbol of the operator $f_{\rho}(\vec{x})$.

For systems like one-mode harmonic oscillator or a free particle, it is known that the dequantizer operator $\hat{U}(\vec{x})$, where \vec{x} has components $x_1 = X$, $x_2 = \mu$, $x_3 = \nu$, can be taken as

$$\hat{U}(X, \mu, \nu)_{xx'} = \frac{1}{2\pi} \int [\exp(ik(X\hat{1} - \mu\hat{q} - \nu\hat{p})) dk]_{xx'} = \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})_{xx'}, \quad (3)$$

and the quantizer operator can be taken as

$$\hat{D}(X, \mu, \nu)_{xx'} = \frac{1}{2\pi} [\exp(i(X\hat{1} - \mu\hat{q} - \nu\hat{p}))]_{xx'}. \quad (4)$$

We use here matrix elements of the quantizer and dequantizer operators in the position representation. In this case, the symbol of the density operators $\hat{\rho}$ of the system states $G(X|\mu, \nu)$ has the form

$$G(X|\mu, \nu) = \text{Tr} \hat{\rho} \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}); \quad (5)$$

it is called the state tomogram. Also tomograms are conditional probability distributions of a random variable (the position X) with condition parameters μ and ν , such that for determining the axes of the position q and momentum p in the system phase space we have

$$X = \mu q + \nu p, \quad \mu = s \cos \theta, \quad \nu = s^{-1} \sin \theta. \quad (6)$$

The parameter θ determines the angle between different pairs of the position and for these axes in the system phase space, and s is scaling parameter used for these axes. The conditional probability $G(X|\mu, \nu)$ in the case of classical oscillator is determined by the Radon transform [61] of the probability distribution function $f(q, p)$. The symplectic tomogram of classical system was introduced in [6].

As it was found [62] for the pure state with the wave function $\psi(y)$, the state tomogram reads

$$G_{\psi}(X|\mu, \nu) = \frac{1}{2\pi|\nu|} \left| \int \psi(y) \exp\left(\frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu}\right) dy \right|^2. \quad (7)$$

For the ground state of the harmonic oscillator, one has the conditional probability distribution of the form

$$G_0(X|\mu, \nu) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}} \exp\left(-\frac{X^2}{\mu^2 + \nu^2}\right). \quad (8)$$

We point out that all known state descriptions like the Wigner function or the Husimi–Kano function are determined by quasiprobability distributions, which are connected with the tomogram (8) by integral transforms relating pairs of their quantizer and dequantizer operators.

3. Green Function and Feynman Path Integral

The Schrödinger equation for the wave function $\psi(y, t)$ reads

$$i \frac{\partial \psi(y, t)}{\partial t} = \left(\frac{\hat{p}^2}{2} + V(y) \right) \psi(y, t), \quad \hat{p} = -i \frac{\partial}{\partial y}, \quad (9)$$

where y is the position, \hat{p} is the momentum operator, and $V(y)$ is the potential energy in the position representation. It can be considered as the evolution for the initial wave function $\psi(y, t = 0)$ determined by the Green function $\mathcal{G}(y, y', t)$, which is the matrix element of the evolution operator $\langle y | \hat{u}(t) | y' \rangle = [\hat{u}(t)]_{yy'}$ in the position representation. Here, $\hat{u}(t)$ is the evolution operator. So, one has

$$\psi(y, t) = \int \mathcal{G}(y, y', t) \psi(y', t = 0) dy'. \quad (10)$$

This matrix element was expressed as the Feynman path integral [63] (see, also [1])

$$\mathcal{G}(y, y', t) = \langle q_f | e^{-i\hat{H}t} | q_I \rangle = \int_{y(t)=q_f, y(0)=q_I} Dq \exp \left[i \int_0^t dt' L(q, \dot{q}) \right], \quad (11)$$

where $L(q, \dot{q})$ is the classical system Lagrangian

$$L(q, \dot{q}) = \frac{\dot{q}^2}{2} - V(q).$$

We assume the mass $m = 1$ and the Planck's constant $\hbar = 1$. Also

$$Dq = \lim_{N \rightarrow \infty} \left(\frac{N}{2it\pi} \right)^{N/2} \prod_{n=1}^{N-1} dq_n.$$

The aim of our consideration is to connect the Green function properties with the probability distribution - tomogram $G(X|\mu, \nu, t)$, which provides the dependence of the wave function $\psi(x, t)$ of the form (10). Then formula (7) gives the tomographic probability representation of the state

$$G_\psi(X|\mu, \nu, t) = \frac{1}{2\pi|\nu|} \left| \int \psi(y, t) \exp \left(\frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu} \right) dy \right|^2. \quad (12)$$

Expressing $\psi(y, t)$ in terms of the system Green function, we obtain

$$G_\psi(X|\mu, \nu, t) = \frac{1}{2\pi|\nu|} \left| \int \int \mathcal{G}(y, y', t) \psi(y', 0) dy' \exp \left(\frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu} \right) dy \right|^2. \quad (13)$$

Then one has the tomogram of the form

$$G_\psi(X|\mu, \nu, t) = \frac{1}{2\pi|\nu|} \left| \int g(y', X, \mu, \nu, t) \psi(y', 0) dy' \right|^2, \quad (14)$$

where

$$g(y', X, \mu, \nu, t) = \int \mathcal{G}(y, y', t) \exp \left(\frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu} \right) dy. \quad (15)$$

We call the function $g(y', X, \mu, \nu, t)$ the tomographic propagator in the probability representation of quantum mechanics.

4. Example of the Evolution of Free Particle Tomogram

We consider the example of the probability representation of the Green function for a free particle, namely, for given ground states of the harmonic oscillator with wave function

$$\psi_0(y, 0) = \frac{1}{\pi^{1/4}} \exp \left(-\frac{y^2}{2} \right), \quad (16)$$

we study the tomogram of the state obtained from this function, if to cut the spring of the oscillator. The evolution of this state becomes a free evolution with the Green function

$$\mathcal{G}(y, y', t) = \frac{1}{\sqrt{2\pi it}} \exp \left(\frac{i(y - y')^2}{2t} \right). \quad (17)$$

Thus, the wave function $\psi_0(y, t)$ of the system at time moment t becomes

$$\psi_0(y, t) = \int \frac{1}{\pi^{1/4} \sqrt{2\pi it}} \exp \left(\frac{i(y - y')^2}{2t} - \frac{y'^2}{2} \right) dy'. \quad (18)$$

We have

$$\psi_0(y, t) = \frac{1}{\pi^{1/4} \sqrt{1+it}} \exp\left(-\frac{y^2}{2t(t-i)}\right). \quad (19)$$

Due to unitarity of the evolution operator,

$$\int |\psi_0(y, t)|^2 dy = 1. \quad (20)$$

The density operator of the state (16) is

$$\hat{\rho}_0(t) = |\psi_0(t)\rangle\langle\psi_0(t)|, \quad (21)$$

and the corresponding tomogram reads

$$G_0(X|\mu, \nu, t) = \text{Tr} \hat{\rho}_0(t) \delta(X\hat{1} - \mu\hat{q} - \nu\hat{p}). \quad (22)$$

This tomogram can be calculated by using the wave function $\psi_0(y, t)$. Since $(\hat{\rho}_0(t))_{yy'} = \psi_0(y, t)\psi_0^*(y', t)$ and according to the approach developed for systems with quadratic Hamiltonian like free motion particle, the calculation of this trace can be done by means of the integrals of the motion of the system [54–57]

$$\hat{x}_0(t) = \hat{q} - \hat{p}t$$

and

$$\hat{p}_0(t) = \hat{p}.$$

Then the Heisenberg operators are $\hat{x}_H(t) = \hat{q} + \hat{p}t$, $\hat{p}_H(t) = \hat{p}$. After calculating the trace (22), we arrive at

$$G_0(X|\mu, \nu, t) = \text{Tr} \hat{\rho}_0(t=0) \delta(X\hat{1} - \mu_H(t)\hat{q} - \nu_H(t)\hat{p}). \quad (23)$$

For $\hat{\rho}_0(t=0)$, the function $\psi_0(y, t=0)$ is determined by formula (16). Then, in view of (7), we obtain

$$G_0(X|\mu, \nu, t=0) = \frac{1}{\sqrt{\pi(\mu^2 + \nu^2)}} \exp\left(-\frac{X^2}{\mu^2 + \nu^2}\right). \quad (24)$$

This means that formula (24) provides the possibility to calculate the explicit expression for $G(X|\mu, \nu, t)$ by change of parameters $\mu \rightarrow \mu$ and $\nu \rightarrow \nu + \mu t$. One arrives at

$$G_0(X|\mu, \nu, t) = \frac{1}{\sqrt{\pi(\mu^2 + (\nu + \mu t)^2)}} \exp\left(-\frac{X^2}{\mu^2 + (\nu + \mu t)^2}\right). \quad (25)$$

Thus, we obtain this expression, using the calculated wave function $\psi_0(y, t)$, the formula for propagator in the tomographic representation $g(y', X, \mu, \nu, t)$, and the connection of the propagator with the initial tomogram.

5. Integrals of Motion and the Green Function (Path Integral)

For given Schrödinger equation

$$i\frac{\partial\psi(x, t)}{\partial t} = \hat{H}(t)\psi(x, t), \quad \hbar = \omega = m = 1, \quad (26)$$

the evolution operator $\hat{u}(t)$ such that

$$\psi(x, t) = \hat{u}(t)\psi(x, 0), \quad (27)$$

or in the Dirac notation for the state with $\psi(x, t)$,

$$\langle x|\psi(t)\rangle = \psi(x, t);$$

here $|\psi(t)\rangle = \hat{u}(t)|\psi(0)\rangle$. Since one has (27) and $\hat{u}(t=0) = \hat{1}$, there exist integrals of motion $\hat{I}(t)$, such that the mean value $\langle \hat{I}(t) \rangle$ has the form

$$\langle \psi(t)|\hat{I}(t)|\psi(t)\rangle = \langle \psi(0)|\hat{I}(0)|\psi(0)\rangle. \quad (28)$$

The integral of motion $\hat{I}(t)$ has the form of the relation of the operator $\hat{I}(0)$ and the evolution operator $\hat{u}(t)$, such that

$$\hat{I}(t) = \hat{u}(t)\hat{I}(0)\hat{u}^\dagger(t) \quad (29)$$

for the Hermitian Hamiltonians and unitary evolution operators $\hat{u}(t)$. In the position representation, the position operator $\hat{q}\psi(x) = x\psi(x)$ and the momentum operator $\hat{p}\psi(x, t) = -i\frac{\partial\psi(x, t)}{\partial x}$. Also the Green function $\mathcal{G}(x, x', t)$ of the Schrödinger equation acts as the matrix element of the evolution operator, namely

$$\mathcal{G}(x, x', t) = \langle x|\hat{u}(t)|x'\rangle = u_{xx'}(t) \quad (30)$$

and

$$\mathcal{G}(x, x', 0) = \delta(x - x'). \quad (31)$$

Operator $\hat{u}(t)$ is unitary operator, it means $\hat{u}^\dagger(t) = \hat{u}^{-1}(t)$. Equation (29) means that arbitrary integral of motion satisfies the condition

$$\hat{I}(t)_{(x)}\mathcal{G}(x, x', t) = [\hat{u}(t)\hat{I}(0)\hat{u}^\dagger(t)]_{(x)}\mathcal{G}(x, x', t), \quad (32)$$

and this means that the Green function satisfies the equation

$$\hat{I}(t)_{(x)}\mathcal{G}(x, x', t) = \hat{I}^{tr}(0)_{(x')}\mathcal{G}(x, x', t). \quad (33)$$

For example, the integral of motion

$$\hat{q}_0(t) = \hat{u}(t)\hat{q}\hat{u}^\dagger(t)$$

preserves the initial values of the particle position and provides the equation for the Green function

$$\hat{q}_0(t)_{(x)}\mathcal{G}(x, x', t) = x'\mathcal{G}(x, x', t). \quad (34)$$

So, there exists the relations for the Green function $\mathcal{G}(x, x', t)$, where $\hat{q}_0(t)$ acting on the Green function provides an equality for the action of the operator $(\hat{q}_0(t))^{tr}$ on the second variable of the Green function. The integral of motion $\hat{p}_0(t) = \hat{u}(t)\hat{p}\hat{u}^\dagger(t)$ preserves the initial value of the particle momentum and gives the equation for the Green function

$$\hat{p}_0(t)_{(x)}\mathcal{G}(x, x', t) = i\frac{\partial\mathcal{G}(x, x', t)}{\partial x'} \quad (35)$$

for arbitrary Hamiltonian systems. An analogous relation takes place for the Green function, in view of the action of the integral of motion $\hat{p}_0(t)$ onto the first and the second variables of the Green function.

For example, for the free particle motion, the momentum operator reads $\hat{p}_0(t) = \hat{p} = -i\frac{\partial}{\partial x}$ and the preserved initial position operator $\hat{q}_0(t) = \hat{q} - t\hat{p} = x + it\frac{\partial}{\partial x}$.

This means that the free motion Green function satisfies the equations

$$-i \frac{\partial}{\partial x} \mathcal{G}(x, x', t) = i \frac{\partial}{\partial x'} \mathcal{G}(x, x', t) \quad (36)$$

and

$$\left(x + it \frac{\partial}{\partial x} \right) \mathcal{G}(x, x', t) = -x' \mathcal{G}(x, x', t). \quad (37)$$

Also, $\mathcal{G}(x, x', 0)$ should be equal to the identity operator matrix element, i.e., $\mathcal{G}(x, x', 0) = \langle x | \hat{1} | x' \rangle = \delta(x - x')$. Then we obtain the explicit Gaussian form of the Green function of the free particle

$$\mathcal{G}(x, x', t) = \frac{1}{\sqrt{2\pi it}} \exp\left(\frac{i(x - x')^2}{2t}\right), \quad (38)$$

satisfying the formulated relations with the integrals of motion. In the probability representation of quantum mechanics, the propagator reads

$$g(y', X, \mu, \nu, t) = \int \frac{dy}{\sqrt{2\pi it}} \exp\left(i \frac{(y - y')^2}{2t} + \frac{i\mu y^2}{2\nu} - \frac{iXy}{\nu}\right),$$

and its explicit form is

$$g(y', X, \mu, \nu, t) = \frac{\sqrt{\nu}}{\sqrt{\mu + \nu t}} \exp\left(\frac{-i(y'\nu + Xt)^2}{2\nu t(\nu + \mu t)} + \frac{iy'^2}{2t}\right).$$

Analogously, one has the integrals of motion for harmonic oscillator

$$\hat{q}_0(t) = \hat{q} \cos t - \hat{p} \sin t,$$

$$\hat{p}_0(t) = \hat{q} \sin t + \hat{p} \cos t.$$

The harmonic-oscillator Green function satisfies the equations

$$x \cos t \mathcal{G}(x, x', t) + i \sin t \frac{d\mathcal{G}(x, x', t)}{dx} = x' \mathcal{G}(x, x', t), \quad (39)$$

$$-i \cos t \frac{d\mathcal{G}(x, x', t)}{dx} + x \sin t \mathcal{G}(x, x', t) = i \frac{d\mathcal{G}(x, x', t)}{dx'}; \quad (40)$$

it reads

$$\mathcal{G}(x, x', t) = \sqrt{\frac{1}{2\pi i \sin t}} \exp\left(\frac{i}{2 \sin t} \left[(x^2 + x'^2) \cos t - 2xx' \right]\right). \quad (41)$$

Also, both Green functions for free motion (38) and for harmonic oscillator (41) should satisfy the equation with Hamiltonians

$$i \frac{\partial \mathcal{G}(x, x', t)}{\partial t} = \hat{H} \mathcal{G}(x, x', t), \quad (42)$$

where, in the first case of free motion, the Hamiltonian reads

$$\hat{H} = \frac{\hat{p}^2}{2}$$

and, in the second case of the harmonic oscillator, the Hamiltonian is

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}.$$

6. Three Methods of Calculating Tomograms

The tomogram of the state with the wave function $\psi_0(t)$ obtained for the free motion of the initial oscillator state $\psi_0(x, t = 0) = \frac{1}{\pi^{1/4}} e^{-x^2/2}$ can be obtained by three different procedures. Namely, one can find the tomogram of the initial oscillator state, using the density matrix of the oscillator state $\rho_0(x, x', t = 0) = \psi_0(x, t = 0)\psi_0^*(x', t = 0)$. There exists the rule to calculate the trace of the product of several operators $\hat{a}_i, i = 1, \dots, n$, which means equality

$$\text{Tr}(f(\hat{a}_1\hat{a}_2 \dots \hat{a}_n)) = \text{Tr}(f(\hat{a}_2\hat{a}_3 \dots \hat{a}_n\hat{a}_1)). \quad (43)$$

Let us calculate the product of the density operator $\hat{\rho}(t)$ in the time moment t and the dequantizer operator $\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})$, this product is the tomogram of the state. We will use the connection of the density operator in the time moment t with the density operator in the initial moment of time $t = 0$, namely, $\hat{\rho}(t) = \hat{u}\hat{\rho}(0)\hat{u}^\dagger(t)$, where $\hat{u}(t)$ is the evolution operator. Then, we obtain, in view of (43), the relations $\text{Tr}(\rho(t)\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})) = \text{Tr}(\hat{u}(t)\hat{\rho}(t = 0)\hat{u}^\dagger(t)\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})) = \text{Tr}\hat{\rho}(t = 0)\hat{u}^\dagger(t)\delta(X\hat{1} - \mu\hat{q} - \nu\hat{p})\hat{u}(t) = \text{Tr}(\hat{\rho}(t = 0)\delta(\hat{u}^\dagger(t)X\hat{1}\hat{u}(t) - \mu\hat{u}^\dagger(t)\hat{q}\hat{u}(t) - \nu\hat{u}^\dagger(t)\hat{p}\hat{u}(t)) = \text{Tr}(\rho(t = 0)\delta(X\hat{1} - \mu\hat{q}_H - \nu\hat{p}_H))$. The \hat{q}_H and \hat{p}_H are the Heisenberg position and momentum operators expressed in terms of the integrals of motion $\hat{q}_0(t)$ and $\hat{p}_0(t)$. For a free particle, we get the possibility to take tomogram of the state $\psi(x, t = 0)$ and simply replace $\hat{q}_0(t)$ and $\hat{p}_0(t)$ by the Heisenberg operators in the tomogram. In this case, we obtain the rule how to replace the parameters $\mu \rightarrow \mu_H(t)$ and $\nu \rightarrow \nu_H(t)$ in the tomogram. For free particle it provides the rule $\mu \rightarrow \mu + \nu t$ and $\nu \rightarrow \nu$ for the tomogram of the ground state of the harmonic oscillator.

The second way of calculating tomogram is to use the wave function. The same result can be obtained, if we calculate the wave function of the free particle motion, which has started in the state of oscillator $\psi(x, t = 0)$. Then one can calculate the tomogram using formula (12) [62].

The third way of calculating the tomogram is to use the propagator in the probability representation $g(y', X, \mu, \nu, t)$ (15). Here, we make the calculation of the tomogram by two steps. First, we calculate the propagator $g(y', X, \mu, \nu, t)$, which we introduce in this paper; it acts onto the initial wave function. But it contains the result of the action of the evolution onto the system by taking into account the influence of parameters μ and ν . Then, the second step is to calculate the action of the propagator $g(y', X, \mu, \nu, t)$ onto the wave function $\psi(0)$. The combination of both actions using two steps provides the final result which is equivalent to the result of the first and the second ways.

They correspond to employment of the path integral method related to the Green function expressed in terms of the path integral. Thus, we have, for example, for free particle motion of the initial state in the form of oscillator ground state the wave function given by (19). The tomogram of this state has the integral form (7) and in explicit form it is presented by (25).

7. Conclusions

To conclude, we point out the main results presented in the paper.

We consider the Green function of the Schrödinger equation for the state wave function. Since the Green function is associated with the Feynmann path integral, we use this property to point out the probability representation of the quantum state wave function introduced in [6], to apply the tomographic probability distributions to the Feynmann path integral and to introduce the corresponding propagator (the tomographic propagator), for example, the free motion tomographic propagator (the Green function) given by (15). We discussed the explicit forms of new equations for the Green functions given in terms of the integrals of motion operators and used them on the examples of the oscillator system and the free motion of the particle. Other systems like an inverted oscillator and its description by the path integral will be considered in future publications.

We summarize the main results of our paper. We introduced new relation associated the Green function of Schrödinger equation $\mathcal{G}(x, x', t)$ and the symplectic tomogram $G(X|\mu, \nu, t)$ which is the conditional probability distribution of random position X . The parameters μ and ν determine the

conditions, which provide the information on scales and orientations of the axes giving the position q and momentum p for the system phase space. The tomogram being related with the Green function of the system is determined by the wave function of the system $\psi(y)$ or by the density operator $\hat{\rho}$ of the system state.

We constructed the new function (propagator $g(y', X, \mu, \nu, t)$) which is used to obtain the connection of the wave function $\psi(y, t)$ of a system with the probability representation $G_\psi(X|\mu, \nu, t)$ of the form

$$G_\psi(X|\mu, \nu, t) = \int g(y', X, \mu, \nu, t)\psi(y', 0) dy';$$

here, for arbitrary systems, the introduced propagator is determined by the Green function of the system $\mathcal{G}(y, y', t)$ (15) (similar to the used for free motion and oscillator). For an arbitrary integral of motion, the equation of the Green function and the Feynman path integral is written. The new probability distributions can be obtained determining the system states with Hamiltonians, which are nonquadratic forms of the position and momentum. We obtained new equations for Green functions $\mathcal{G}(y, y', t)$ using arbitrary integrals of motion $\hat{I}(t)$ and will consider these equations for examples with nonquadratic Hamiltonians in future publications. Also, the examples of two-dimensional systems with wave function $\psi(x_1, x_2, t)$ (like Landau levels problem, see [57]) will be considered in future publications.

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