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Posted Date: 11 September 2025

doi: 10.20944/preprints202509.0991.v1

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Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 September 2025 doi:10.20944/preprints202509.0991.v1

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Timelike Thin-Shell Evolution in Gravitational Collapse: Geometric and Thermodynamic Perspectives in Classical General Relativity

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Abstract

Thermodynamic ideas—linking geometry, entropy, and the negative heat capacity of gravitating systems—play an increasing role in black hole physics and late-time gravitational dynamics. Motivated by this perspective, we present a conservative, fully classical analysis of spherical collapse using only standard tools from general relativity (GR), yet admitting a clear thermodynamic reading. A timelike thin shell connects a regular constant-curvature (de Sitter) interior to a Schwarzschild or Schwarzschild–de Sitter (SdS) exterior. After a brief formation stage, we focus on a post-transient regime with negligible inflow and fixed exterior mass (ADM mass if Lambda+ = 0). Casting the Israel junction condition into an effective potential, the analysis yields: (i) a closed-form *sufficient* threshold for outward shell evolution, balancing interior and exterior forces; (ii) bounded scalar curvature invariants throughout the covered spacetime; and (iii) a simple, falsifiable redshift bound for near-shell spectral modes, scaled by mass. Although purely geometric in derivation, these results are consistent with classical thermodynamic intuition: entropy-like area growth, energy-driven expansion, and the role of negative specific heat. The model offers a regular, horizon-free scenario for late-time collapse in classical GR, free from curvature singularities and geodesic focusing.

Keywords: imelike thin shell; junction geometry; static patch; classical collapse endpoint; Schwarzschild–de Sitter; curvature invariants; negative heat capacity; quasi-trapped modes

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1. Introduction

Thermodynamic concepts—such as entropy, temperature, and energy flow—have deeply influenced how we understand gravity [1], especially in the context of black holes. Since the pioneering work of Bekenstein, Hawking, and others [2–4], geometric quantities like horizon area have been linked to entropy and information content. At the same time, it has long been appreciated that self-gravitating systems exhibit negative specific heat, a feature already visible in Newtonian theory [5,6]. These thermodynamic insights are often associated with equilibrium or stationary horizons—but their role in dynamical, collapsing systems remains less concretely understood.

This paper investigates whether classical general relativity (GR), without quantum input, already supports such thermodynamic behavior during the late stages of gravitational collapse. Specifically, we examine a simple but instructive model: a timelike thin shell separates a regular constant-curvature (de Sitter) interior from a Schwarzschild or Schwarzschild–de Sitter exterior. Thin-shell techniques provide an exact classical framework for this setup, based on Israel's junction condition and the Birkhoff–Schwarzschild geometry [7–9].

After a short formation stage, we focus on a post-transient regime with negligible inflow, where the exterior mass is constant. The junction condition can then be reformulated as an effective potential for the shell radius $R(\tau)$, allowing a transparent analysis of its dynamics. Within this framework, we derive three core results:

(i) a closed-form sufficient threshold for outward shell evolution,

(ii) the boundedness of scalar curvature invariants across the entire spacetime domain, and
(iii) a simple, falsifiable, mass-scaled frequency bound for near-shell spectral features observable at infinity.

Although purely classical, the setup naturally supports a thermodynamic interpretation: area growth, energy-driven expansion, and behavior consistent with negative heat capacity. This interpretation is discussed separately and used to connect the geometric findings to coarse-graining intuitions—without treating thermodynamics as a fundamental input.

1.1. *Scope*

The analysis is restricted to classical GR with spherical symmetry, timelike thin shells, and static patches with $f_\pm > 0$. The lightlike (null) limit [10] is not considered. Quantum-information aspects such as evaporation, Page time, or firewalls lie outside the present scope; see [11–13] for background.

1.2. *Organization*

Section 2 introduces the geometric framework and conservation laws, leading to the effective-potential form of the junction condition. Section 3 derives the sufficient threshold for outward shell evolution. Section 4 proves curvature boundedness. Section 5 introduces the mass-scaled redshift bound for near-shell frequencies. Section 6 discusses physical interpretation, observational falsifiability, and future directions.

2. Geometric Setup and Methods

This section establishes the classical geometric framework used to model late-time gravitational collapse in spherical symmetry. The setup consists of two vacuum regions—an interior and an exterior—joined across a timelike thin shell. The entire analysis remains within general relativity (GR), using exact solutions and matching conditions.

2.1. *Exterior (Vacuum, Spherical Symmetry)*

By Birkhoff's theorem, the vacuum exterior of a spherically symmetric source must be Schwarzschild or Schwarzschild–de Sitter (SdS):

$$ds_+^2 = -f_+(r)\,dt_+^2 + f_+^{-1}(r)\,dr^2 + r^2 d\Omega^2, \qquad f_+(r) = 1 - \frac{2M}{r} - \frac{\Lambda_+ r^2}{3}, \tag{1}$$

where M is the exterior mass parameter (the ADM mass if $\Lambda_+ = 0$). The analysis is confined to the static region where $f_+(r) > 0$, consistent with timelike shell evolution.

2.2. *Interior (Regular Constant Curvature)*

The interior is modeled as a regular de Sitter region with constant positive curvature:

$$ds_-^2 = -(1 - H^2 r^2)\,dt_-^2 + (1 - H^2 r^2)^{-1}\,dr^2 + r^2 d\Omega^2, \qquad H^2 = \Lambda_-/3 > 0, \tag{2}$$

so that all interior curvature invariants are finite, e.g., the Kretschmann scalar $K_- = 8\Lambda_-^2/3$ [4].

2.3. *Timelike Thin Shell and Junction Condition*

The two spacetime regions are joined across a timelike shell at areal radius $R(\tau)$, where τ is the proper time of observers comoving with the shell. The shell is endowed with surface stress–energy tensor $S^a{}_b = \mathrm{diag}(-\sigma, p, p)$, and its motion obeys the Israel junction condition [7,9]:

$$\sqrt{f_+(R) + \dot{R}^2} - \sqrt{f_-(R) + \dot{R}^2} = \kappa(R), \qquad \kappa(R) = 4\pi\sigma(R)\,R. \tag{3}$$

Additionally, the shell satisfies surface-energy conservation,

$$\frac{d(\sigma A)}{d\tau} + p\,\frac{dA}{d\tau} = \Phi\,A, \qquad A = 4\pi R^2, \tag{4}$$

where Φ is the net normal energy flux across the shell.

2.4. Post-Transient Zero-Inflow Regime

After an initial formation stage, the analysis focuses on a late-time regime in which energy inflow is negligible. This is enforced by the condition

$$T^r{}_t\big|_{\text{shell}} = 0 \quad\Longrightarrow\quad dM \simeq 0, \qquad R_S = 2M = \text{const}, \tag{5}$$

ensuring that the exterior mass parameter remains fixed and attention shifts to the shell dynamics governed by $R(\tau)$ and the interior scale Λ_-.

2.5. Domain and Conventions

All analysis is restricted to static patches where $f_\pm(R) > 0$, ensuring that the shell remains timelike. Natural units are used throughout: $G = c = \hbar = k_B = 1$, and the metric signature is $(-+++)$.

The Israel condition can be recast in an effective-potential form by solving for $\dot{R}$ in (3):

$$\dot{R}^2 + V(R) = 0, \qquad V(R) = f_-(R) - \frac{\left(f_+(R) - f_-(R) - \kappa(R)^2\right)^2}{4\,\kappa(R)^2}. \tag{6}$$

Turning points occur where $V(R_*) = 0$, and the sign of $V'(R_*)$ determines the local evolution tendency. To control sign behavior, a linear surface equation of state $p = w\sigma$ with $w > -1/2$ and vanishing flux $\Phi = 0$ is assumed. Then energy conservation (4) reduces to

$$\frac{d\sigma}{dR} = -\frac{2(\sigma+p)}{R} \qquad\Longrightarrow\qquad \kappa'(R) = \frac{d}{dR}\left(4\pi\sigma R\right) = 4\pi\sigma(-1-2w) < 0, \tag{7}$$

ensuring monotonic behavior of the effective potential in physically relevant cases.

3. A Sufficient Outward-Evolution Threshold

Balancing interior de Sitter acceleration ($\sim H^2 R$) against exterior attraction ($\sim M/R^2$) singles out

$$R_{\text{thr}} := \left(\frac{3M}{\Lambda_-}\right)^{1/3}. \tag{8}$$

Proposition 1 (Sufficient outward evolution at zero inflow). Assume $\Phi = 0$ (so that M is time-independent), finite surface stresses with $p = w\sigma$ and $w > -1/2$, and $f_\pm(R) > 0$ (static patches). If for some τ_0 the shell satisfies $R(\tau_0) \geq R_{\text{thr}}$, then $\dot{R}(\tau_0) \geq 0$ and the shell evolves outward on a finite interval beyond τ_0.

Proof. *(sketch).* Differentiate (6) and evaluate at a point with $H^2R^3 = M$ (for $\Lambda_+ = 0$). One has

$$\begin{aligned} f_+'(R) &= \frac{2M}{R^2}, & f_-'(R) &= -2H^2R, \\ \Delta f := f_+ - f_- &= -\frac{2M}{R} + H^2R^2, & \Delta f' &= \frac{2M}{R^2} + 2H^2R = 4H^2R > 0, \end{aligned}$$

using $H^2R^3 = M$. Writing $V' = f_-' - \frac{\partial}{\partial R}\left[\frac{(\Delta f - \kappa^2)^2}{4\kappa^2}\right]$ and using $\kappa'(R) < 0$ from (7), one finds $V'(R_{\text{thr}}) < 0$ within the static domain $f_\pm > 0$. Hence $-V$ increases for $R > R_{\text{thr}}$ in a neighborhood of R_{thr}, implying outward evolution on a finite interval when $\dot{R}(\tau_0) \geq 0$. □

Remark 1 (Dimensionless form and static-patch consistency). *Let* $\chi = \Lambda_- M^2$. *Then*

$$\frac{R_{\text{thr}}}{R_S} = \frac{1}{2}\left(\frac{3}{\chi}\right)^{1/3}.$$

Timelike consistency requires $R_{\text{thr}} > R_S$ *and* $HR_{\text{thr}} < 1$ *(and, if* $\Lambda_+ > 0$, R_{thr} *below the cosmological horizon), i.e.* $f_\pm(R) > 0$ *throughout [8].*

Beyond providing control over the shell's motion, the static patch framework also ensures that curvature remains bounded. This regularity is addressed next.

4. Boundedness of Curvature Scalars in the Covered Domain

A central question in collapse scenarios is whether curvature invariants diverge somewhere in the constructed spacetime, signaling a singularity. In this section, we show that the entire region covered by the thin-shell construction remains regular. This follows directly from the known properties of the chosen interior and exterior geometries, together with the well-controlled shell junction.

Proposition 2 (Bounded curvature). Let $R(\tau) \geq R_{\text{min}} > 0$, with $f_\pm(R) > 0$ (i.e., the shell remains within the static patches), and assume that the shell surface stresses $\sigma(\tau), p(\tau)$ remain finite. Then all scalar curvature invariants are finite throughout the spacetime region covered by the construction.

Proof. *(sketch).* The interior region is a regular constant-curvature spacetime, specifically de Sitter, in which all scalar invariants are finite by construction. For example, the Kretschmann scalar is

$$K_- = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{8}{3}\Lambda_-^2.$$

In the exterior, which is Schwarzschild–de Sitter (SdS), the curvature is also fully determined and remains finite as long as the radius stays above the Schwarzschild horizon. The Kretschmann scalar is

$$K_+(r) = \frac{48M^2}{r^6} + \frac{8}{3}\Lambda_+^2,$$

which is manifestly bounded for $r \geq R_{\text{min}} > 0$. ☐

At the shell, the metric is continuous, and any curvature singularities would arise only from distributional sources due to the jump in extrinsic curvature. However, as long as the shell energy density σ and pressure p remain finite, the distributional curvature remains controlled. This is a standard result from the Israel junction formalism [7,9].

Moreover, because we restrict to static patches with $f_\pm(R) > 0$, the shell does not cross into regions containing horizons or trapped surfaces. This ensures that the construction remains within a regular coordinate patch, and the entire covered spacetime is free of curvature singularities [8].

Beyond geometric regularity, classical timelike shell models can also constrain persistent late-time features. In particular, a mass-scaled bound emerges from combining shell locality with redshift effects.

5. A Mass-Scaled Frequency Bound for Near-Shell Modes

Late-time observations often probe the presence of quasi-stationary or slowly decaying features near compact objects. In the present context, such features may be modeled as near-shell modes, and their observability at infinity is governed by redshift and confinement effects.

A minimal localization criterion for such modes is $k_{\text{loc}}R \gtrsim \xi$, with $\xi = \mathcal{O}(1)$, ensuring that the mode is stored near the shell with meaningful structure. The Tolman redshift relation then yields the observed frequency at infinity:

$$\omega_\infty = \sqrt{f_+(R)}\,\omega_{\text{loc}},$$

where ω_{loc} is the local proper frequency near the shell [4]. A full derivation and interpretation of this relation in the thin-shell context is given in Appendix A.

Assuming $\omega_{\text{loc}} \simeq k_{\text{loc}}$ and defining the redshifted cyclic frequency $f_c = \omega_\infty/(2\pi)$, we obtain the general mass-scaled expression:

$$f_c\, R_S = \frac{\xi}{2\pi}\,\frac{\sqrt{1-\dfrac{2M}{R}-\dfrac{\Lambda_+ R^2}{3}}}{R/R_S}\,. \tag{9}$$

For a Schwarzschild exterior ($\Lambda_+ = 0$), the right-hand side is maximized at $R = \frac{3}{2}R_S$, yielding the compact upper bound:

$$\boxed{f_c\, R_S \;\leq\; \frac{\xi}{3\sqrt{3}\,\pi}}\,. \tag{10}$$

This inequality provides a falsifiable, dimensionless constraint on persistent near-shell spectral features in the absence of exterior inflow and under classical GR assumptions. A robust observational violation of Equation (10) would falsify the near-shell storage scenario as described.

6. Discussion and Outlook

6.1. Classical Content

Within classical GR and spherical symmetry, with a timelike thin shell evolving in static patches $f_\pm > 0$, a constant-curvature (de Sitter) interior, and a post-transient zero-inflow regime ($dM \simeq 0$), the analysis establishes three conservative statements. First, a closed-form *sufficient* threshold for outward evolution, Equation (8), emerges from balancing interior de Sitter acceleration against exterior attraction. Second, scalar curvature invariants remain bounded throughout the spacetime region covered by the construction (Proposition 2), ensuring regularity without intersecting exterior trapping surfaces. Third, a mass-scaled redshift bound limits persistent near-shell frequencies as observed at infinity, Equation (10). These results derive directly from the Israel junction geometry and standard shell conservation laws, requiring no inputs beyond classical GR.

6.2. Energy Conditions

The de Sitter interior saturates the null and weak energy conditions but violates the strong energy condition, consistent with vacuum energy properties. The thin shell, governed by a linear surface equation of state $p = w\sigma$ with $w > -1/2$, satisfies the null, weak, and dominant energy conditions for physically admissible configurations ($\sigma > 0$, $|w| \leq 1$). Under these assumptions, the standard singularity theorems [14] are not activated within the domain, as the model avoids the focusing conditions that typically drive geodesic incompleteness [8]. This highlights how controlled violations of energy conditions can yield regular classical outcomes in collapse scenarios [15].

6.3. Thermodynamic Interpretation

Thermodynamic language serves as an *interpretive* framework, distinct from the geometric derivations. In the static-patch, zero-inflow setup with fixed exterior mass M, the shell radius $R(\tau)$ naturally links to a geometric area functional $S[R] := \alpha R^2$ (where α is an arbitrary scale). Outward evolution implies $dS > 0$ without external inflow, mirroring the phenomenology of self-gravitating systems with negative specific heat capacity ($C < 0$). In this scenario, larger shell configurations correspond to higher "entropy" states, consistent with the idea that energy loss promotes expansion rather than collapse. Crucially, $C < 0$ allows for local entropy production via $dS \sim dR$ during the evolution, potentially reaching an entropic maximum before geometric horizon formation. No statistical entropy, temperature, or microstates are invoked; the analogy complements the horizon-free junction geometry, offering a classical lens on late-time stability [16].

doi:10.20944/preprints202509.0991.v1

6.4. Scope and Limitations

The results are confined to classical GR, spherical symmetry, timelike shells in static patches, finite surface stresses (with $p = w\sigma$, $w > -1/2$), and exclusion of the null limit [10]. The threshold in Equation (8) is *sufficient*, not necessary. These restrictions ensure analytic control but limit applicability to highly idealized situations. Generalizations involving strong time dependence, rotation, or anisotropies are left for future work.

6.5. Observational Handle and Falsifiability

The mass–scaled inequality in Equation (10) furnishes a null test: given an independent estimate of M, any long–lived near–shell spectral feature measured at infinity should respect the bound. A robust, systematic violation would falsify the near–shell storage picture under the stated assumptions (timelike shell in static patches, negligible inflow, time–independent exterior mass). The bound also provides a practical template for targeted searches formulated directly in the mass–scaled variable $f_c R_S$.

6.6. Outlook

Natural extensions could test the robustness of the classical findings while linking to observational strategies. Key directions include: (i) incorporating the null limit [10] to model early, ultrarelativistic collapse phases, and exploring how lightlike shell dynamics might affect the outward-evolution threshold (Equation (8)) and entropy production ($dS \sim dR$); (ii) studying perturbative stability against nonspherical fluctuations, particularly regarding their impact on the boundedness of curvature invariants; and (iii) allowing for time-dependent exterior mass $M(t)$ due to inflow or radiation, to explore how mass evolution modifies the frequency bound (Equation (10)) and potentially introduces observational signatures. These avenues could connect the junction framework to broader spacetime symmetries, astrophysical realism, and alternative gravity theories.

Funding: This research received no external funding.

Data Availability Statement: All derivations and analytical results in this work follow directly from the equations presented in the manuscript. No external datasets were used or generated.

Use of Artificial Intelligence: The author used the OpenAI ChatGPT service (ScholarGPT via GPT-4o) to assist with language editing, formatting, and structural refinement of the manuscript. All scientific content, derivations, interpretations, and conclusions were conceived, written, and verified by the author.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A. Stationary Thin Shell and Tolman Potential (Proposition 5)

In a static geometry with Killing field ξ^a and lapse $N = \sqrt{-\xi^2}$, the Tolman relation gives $TN = \text{const}$ and $\theta := \ln T = -\ln N + \text{const}$ [4,17]. In the Newtonian limit, $\nabla\theta$ reproduces the gravitational field. For a static thin shell, θ is continuous across the shell while its normal derivative jumps by a surface term obtained from a pillbox integration of $\nabla^2\theta = -(4\pi G/c^2)\rho$, reproducing the thin-sheet jump (potential continuous, normal field discontinuous). In full GR this coincides with the static/Newtonian limit of Israel's condition [7,9] and introduces no new dynamics.

Appendix B. Proof: Outward Evolution (Proposition 1)

Differentiate $V(R)$ in (6). At $H^2R^3 = M$ (for $\Lambda_+ = 0$),

$$f'_+ = \frac{2M}{R^2}, \qquad f'_- = -2H^2R,$$

$$\Delta f = f_+ - f_- = -\frac{2M}{R} + H^2R^2, \qquad \Delta f' = \frac{2M}{R^2} + 2H^2R = 4H^2R > 0.$$

Writing $V' = f'_- - \frac{\partial}{\partial R}\Big[\frac{(\Delta f - \kappa^2)^2}{4\kappa^2}\Big]$ and using $\kappa'(R) < 0$ from (7), one finds $V'(R_{\mathrm{thr}}) < 0$ throughout the static domain $f_\pm > 0$. Therefore $-V$ increases for $R > R_{\mathrm{thr}}$ in a neighborhood of R_{thr}, implying outward evolution on a finite interval once $\dot{R}(\tau_0) \geq 0$. Throughout this section, physically admissible shells with $\sigma > 0$ are assumed, so that for $w > -1/2$ one indeed has $\kappa'(R) < 0$.

Appendix C. Proof: Bounded Curvature (Proposition 2)

Interior: constant curvature implies finite invariants, e.g. $K_- = 8\Lambda_-^2/3$. Exterior: for SdS,

$$K_+(r) = \frac{48M^2}{r^6} + \frac{8}{3}\Lambda_+^2,$$

hence bounded for $r \geq R_{\mathrm{min}}$. Across the shell, the induced metric is continuous and distributional curvature is controlled by Israel's condition with finite σ, p [7,9]. Since $R(\tau) > R_S$ and we work in static patches, no exterior trapping surface intersects the covered spacetime [8].

Appendix D. Proof of Frequency Bound (Equation (10))

We assume units $G = c = \hbar = k_B = 1$. A minimal near–shell storage criterion for quasi–trapped modes is

$$k_{\mathrm{loc}}\, R \gtrsim \xi, \qquad \xi = \mathcal{O}(1). \tag{A1}$$

With $\omega_{\mathrm{loc}} \simeq k_{\mathrm{loc}}$ and Tolman redshift $\omega_\infty = \sqrt{f_+(R)}\,\omega_{\mathrm{loc}}$, the observed (cyclic) frequency $f_c = \omega_\infty/(2\pi)$ satisfies

$$f_c \;=\; \frac{1}{2\pi}\sqrt{f_+(R)}\,\omega_{\mathrm{loc}} \;\gtrsim\; \frac{1}{2\pi}\sqrt{f_+(R)}\,\frac{\xi}{R}. \tag{A2}$$

Multiplying by $R_S = 2M$ and writing the exterior lapse $f_+(R) = 1 - \frac{2M}{R} - \frac{\Lambda_+ R^2}{3}$, we obtain the general mass–scaled expression

$$f_c\, R_S \;\gtrsim\; \frac{\xi}{2\pi}\,\frac{\sqrt{1 - \frac{2M}{R} - \frac{\Lambda_+ R^2}{3}}}{R/R_S}. \tag{A3}$$

Appendix D.1. Schwarzschild Exterior (Lambda+=0)

Let $x := R/M > 2$ so that $R/R_S = x/2$. Define

$$g(x) := \frac{\sqrt{1 - \frac{2}{x}}}{x/2} \;=\; \frac{2}{x}\sqrt{1 - \frac{2}{x}}, \qquad x > 2. \tag{A4}$$

Then (A3) reads $f_c\, R_S \gtrsim \frac{\xi}{2\pi}\, g(x)$. A straightforward maximization gives $g'(x) = 0 \Leftrightarrow x = 3$, and

$$g(3) = \frac{2}{3}\sqrt{1 - \frac{2}{3}} = \frac{2}{3\sqrt{3}}.$$

Therefore,

$$f_c\, R_S \;\lesssim\; \frac{\xi}{2\pi}\, g(3) \;=\; \frac{\xi}{3\sqrt{3}\,\pi}, \tag{A5}$$

which is the bound (10) in the main text.

Appendix D.2. Including a Cosmological Term

For $\Lambda_+ > 0$, the same recipe applies on the static patch $f_+(R) > 0$. Since the extra $-\Lambda_+ R^2/3$ lowers $\sqrt{f_+(R)}$ at large R, the Schwarzschild value is an upper envelope; any $\Lambda_+ > 0$ tightens the bound.

Appendix D.3. Interpretation

Given an independent mass estimate M, any persistent late–time feature localized near the shell and measured at infinity must obey the mass–scaled inequality above. A robust violation would falsify the near–shell storage picture within the assumptions of the main text.

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