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Posted Date: 28 August 2025

doi: 10.20944/preprints202508.2052.v1

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Chaplygin and Polytropic Gases Teleparallel Robertson-Walker $F(T)$ Gravity Solutions

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Abstract

This paper investigates the Teleparallel Robertson-Walker (TRW) $F(T)$ gravity solutions for a Chaplygin gas, and then for any polytropic gas cosmological source. We use the TRW $F(T)$ gravity field equations (FEs) for each k-parameter value case and the relevant gas equation of state (EoS) to find the new teleparallel $F(T)$ solutions. For flat $k = 0$ cosmological case, we find analytical solutions valid for any cosmological scale factor. For curved $k = \pm 1$ cosmological cases, we find new approximated teleparallel $F(T)$ solutions for slow, linear, fast and very fast universe expansion cases summarizing by a double power-law function. All the new solutions will be relevant for future cosmological applications on dark matter, dark energy (DE) quintessence, phantom energy, Anti-deSitter (AdS) spacetimes and several other cosmological processes.

Keywords: teleparallel robertson-walker; chaplygin gas; polytropic gas; teleparallel $F(T)$-type solution; polytropic and chaplygin conservation laws; cosmological spacetimes; cosmological teleparallel solutions

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1. Introduction

The teleparallel $F(T)$ gravity is a frame-based and alternative theory to general relativity (GR) fundamentally defined in terms of the coframe $\mathbf{h}^a$ and the spin-connection $\omega^a{}_{bc}$ [1–7]. These two last quantities define the torsion tensor $T^a{}_{bc}$ and torsion scalar T. We remind that GR is defined by the metric $g_{\mu\nu}$ and the spacetime curvatures $R^a{}_{b\mu\nu}$, $R_{\mu\nu}$ and R. Under some considerations, we can determine the symmetries for any independent coframe/spin-connection pairs, and then spacetime curvature and torsion are defined as geometric objects [4–9]. Any geometry described by a such pair whose curvature and non-metricity are both zero ($R^a{}_{b\mu\nu} = 0$ and $Q_{a\mu\nu} = 0$ conditions) is a teleparallel gauge-invariant geometry (valid for any g_{ab}). The fundamental pairs must satisfy two Lie derivative-based relations and we use the Cartan–Karlhede algorithm to solve these two fundamental equations for any teleparallel geometry. For a pure teleparallel $F(T)$ gravity spin-connection solution, we also solve the null Riemann curvature condition leading to a Lorentz transformation-based definition of the spin-connection $\omega^a{}_{b\mu}$. There is a direct equivalent to GR in teleparallel gravity: the teleparallel equivalent to GR (TEGR) generalizing to teleparallel $F(T)$-type gravity [7,9–12]. All the previous considerations are also adapted for the new general relativity (NGR) (refs. [13–15] and refs. therein), the symmetric teleparallel $F(Q)$-type gravity (refs. [16–19] and refs. therein) and some extended theories like $F(T,Q)$-type, $F(R,Q)$-type, $F(R,T)$-type, and several other ones (refs. [20–29] and refs. therein). In the current paper we will restrict our study to the teleparallel $F(T)$ gravity framework.

There are a large number of research papers on spherically symmetric spacetimes and solutions in teleparallel $F(T)$ gravity using a large number of approaches, energy-momentum sources and made for various purposes [30–56]. But there are a special class of teleparallel spacetime which the field equations (FEs) are purely symmetric: the teleparallel Robertson-Walker (TRW) spacetime [55–57]. This type of spacetime is defined in terms of the k-parameter where $k = 0$ is a flat cosmological spacetime, $k = \pm 1$ cases are respectively positive and negative space cosmological curvature [58–61]. We have proven that a such teleparallel geometry is described by a G_6 Lie algebra group, where the 4th to 6th

Killing Vectors (KVs) are characteristic of this spacetime [56,57]. The main apparent consequence of additional KVs is the trivial antisymmetric parts of FEs. The same spacetime geometry structure also exists for some extensions such as teleparallel $F(T,B)$ gravity [62–66]. We have found teleparallel $F(T)$ and $F(T,B)$ solutions for perfect fluid (PF) and scalar field (SF) sources. In the last case, the teleparallel $F(T)$ and $F(T,B)$ solutions are scalar potential independent and only SF dependent [55,62]. But there are additional possible sources of energy-momentum which can lead to new further teleparallel solutions in $F(T)$-type, but also for extensions.

The Dark Energy (DE) states are usually studied by using the PF equivalent equation of state (EoS) $P_\phi = \alpha_Q \rho_\phi$ where α_Q is the DE index (or quintessence index in some refs.) [67–69]. The possible DE forms in terms of α_Q are:

1. **Quintessence** $-\mathbf{1} < \alpha_Q < -\frac{\mathbf{1}}{\mathbf{3}}$: This describes a controlled accelerating universe expansion where energy conditions are always satisfied, i.e., $P_\phi + \rho_\phi > 0$ [70–82]. This usual DE form has been significantly studied in the literature in recent decades for the fascination it provokes and the realism of the models.
2. **Phantom energy** $\alpha_Q < -\mathbf{1}$: This describes an uncontrolled universe expansion accelerating toward a Big Rip event (or singularity) [83–91]. The energy condition is violated, i.e., $P_\phi + \rho_\phi \not> 0$. But this DE form is fascinating because we can find new teleparallel solutions and physical models.
3. **Cosmological constant** $\alpha_Q = -\mathbf{1}$: This is an intermediate limit between the quintessence and phantom DE states, where $P_\phi + \rho_\phi = 0$. A constant SF source $\phi = \phi_0$ added by a positive scalar potential $V(\phi_0) > 0$ will directly lead to this primary DE state. Note that a negative scalar potential (i.e., $V(\phi_0) \leq 0$) will not lead to a positive cosmological constant and/or a DE solution.
4. **Quintom models**: This is a mixture of previous DE types, usually described by double SF models [92–97]. This type of model is more complex to study and solve in general. Several types of models are in principle possible and these physical processes need further studies in the future.

There are in the recent literature in teleparallel gravity and its extension a number of paper on possible solution in static radial-dependent, time-dependent and cosmological teleparallel $F(T)$ solutions [48–52,55,56,62]. The most of those are suitable for DE models, in particular for quintessence DE state. We use cosmological PF as SF approaches, and we find that the solution classes are similar between the both cases.

The quartessence model is an unified models of DE and Dark Matter (DM) arising from a Chaplygin cosmological gas. The DE and DM are two states of a single and simple quartessence dark cosmological fluid [98–102]. This non-linear cosmological fluid model can be considered as an alternative and unified explanation of DE and DM evolving in the universe influencing the cosmological processes. For polytropic gases, there are additional possible physical models arising from this class of EoS-based cosmological solutions. [103–110]. But the Chaplygin, polytropic gas and any superposition of those models in general can explain not only DE and DM mixed or separate models, but also any non-linear cosmological gas and fluid system at the limit [111]. Most of the previous works have been done in the GR (or $F(R)$-type) framework, but these last cases may also arise in teleparallel theories of gravity. Under this last consideration, the current investigation concerning the Chaplygin and polytropic fluids deserves to be achieved in teleparallel framework and will constitute the main aim of the paper, in particular for $F(T)$-type case. Ultimately, we want to build some pure teleparallel quartessence and polytropic-based cosmology, allowing to study more realistic universe models.

Ultimately we want to study in detail the quartessence suitable teleparallel cosmological solutions and its physical impacts. Therefore we need at the current stage to find the possible Chaplygin and polytropic cosmological gas teleparallel $F(T)$ gravity solutions in a Robertson-Walker spacetime (TRW). We had found the TRW geometry and we had solved the TRW FEs and conservation laws (CL) for PF and SF solutions in teleparallel $F(T)$ and $F(T,B)$ gravities [55–57,62]. But we can do further and aim to solve for polytropic and Chaplygin cosmological gases teleparallel $F(T)$ solutions as the next step of development. For satisfying this main aim, we will use the same TRW geometry, FEs and CLs to develop the teleparallel Chaplygin $F(T)$ solutions in section 3, the teleparallel polytropic $F(T)$

solutions in section 4. We will then compare and highlight the similarities and differences by using graphs in section 5 before concluding the current paper in section 6.

2. Summary of Teleparallel Gravity and Field Equations

2.1. Teleparallel $F(T)$-Gravity Theory Field Equations and Torsional Quantities

The teleparallel $F(T)$-type gravity action integral with any gravitational source is [2,3,5,7,47–52,55]:

$$S_{F(T)} = \int d^4 x \left[\frac{h}{2\kappa} F(T) + \mathcal{L}_{Source}\right], \tag{1}$$

where h is the coframe determinant, κ is the coupling constant and $\mathcal{L}_{Source}$ is the gravitational source term. We will apply the least-action principle on the Equation (1) to find the symmetric and antisymmetric parts of FEs as [47–52,55]:

$$\begin{aligned} \kappa\,\Theta_{(ab)} &= F_T(T)\overset{\circ}{G}_{ab} + F_{TT}(T)\,S_{(ab)}^{\;\;\;\mu}\,\partial_\mu T + \frac{g_{ab}}{2}\,[F(T) - T\,F_T(T)], & (2)\\ 0 &= F_{TT}(T)\,S_{[ab]}^{\;\;\;\mu}\,\partial_\mu T, & (3)\end{aligned}$$

with $\overset{\circ}{G}_{ab}$ the Einstein tensor, $\Theta_{(ab)}$ the energy-momentum, g_{ab} the gauge metric and κ the coupling constant. The torsion tensor $T^a{}_{\mu\nu}$, the torsion scalar T and the super-potential $S_a{}^{\mu\nu}$ are defined as [5]:

$$\begin{aligned} T^a{}_{\mu\nu} &= \partial_\mu h^a{}_\nu - \partial_\nu h^a{}_\mu + \omega^a{}_{b\mu}h^b{}_\nu - \omega^a{}_{b\nu}h^b{}_\mu, & (4)\\ S_a{}^{\mu\nu} &= \frac{1}{2}\left(T_a{}^{\mu\nu} + T^{\nu\mu}_{\;\;\;a} - T^{\mu\nu}_{\;\;\;a}\right) - h_a{}^\nu\,T^{\lambda\mu}_{\;\;\;\lambda} + h_a{}^\mu\,T^{\lambda\nu}_{\;\;\;\lambda}, & (5)\\ T &= \frac{1}{2}\,T^a{}_{\mu\nu}S_a{}^{\mu\nu}. & (6)\end{aligned}$$

Equation (4) can be expressed in terms of the three irreducible parts of torsion tensor as:

$$T_{abc} = \frac{2}{3}(t_{abc} - t_{acb}) - \frac{1}{3}(g_{ab}V_c - g_{ac}V_b) + \epsilon_{abcd}A^d \tag{7}$$

where,

$$V_a = T^b_{\;\;ba}, \quad A^a = \frac{1}{6}\epsilon^{abcd}T_{bcd}, \quad t_{abc} = \frac{1}{2}(T_{abc} + T_{bac}) - \frac{1}{6}(g_{ca}V_b + g_{cb}V_a) + \frac{1}{3}V_c. \tag{8}$$

We usually solve in teleparallel $F(T)$ gravity the Equations (2)–(3). Therefore in refs. [55–57], we showed that Equation (3) is trivially satisfied despite a non-zero spin-connection, because the teleparallel geometry is purely symmetric. Only the Equations (2) is non-trivial and will be explicitly solved in detail.

2.2. Teleparallel Robertson-Walker Spacetime Geometry

Any frame-based geometry in teleparallel gravity on a frame bundle is defined by a coframe/spin-connection pair and a field $\mathbf{X}$. The geometry must satisfy the fundamental Lie Derivative-based equations [5,6,55–57]:

$$\mathcal{L}_{\mathbf{X}}\,\mathbf{h}_a = \lambda_a{}^b\,\mathbf{h}_b \text{ and } \mathcal{L}_{\mathbf{X}}\,\omega^a_{\;\;bc} = 0, \tag{9}$$

where $\omega^a_{\;\;bc}$ is the spin-connection in terms of the differential coframe $\mathbf{h}_a$ and $\lambda_a{}^b$ is the linear isotropy group component. In addition for a pure teleparallel $F(T)$-type gravity, we must also satisfy the null

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Riemann curvature condition $R^a{}_{bc} = 0$. For TRW spacetime geometries on an orthonormal frame, the coframe/spin-connection pair $h^a{}_\mu$ and ω_{abc} solutions are [55–57] :

$$h^a{}_\mu = Diag\left[1, a(t)\left(1-k\,r^2\right)^{-1/2}, a(t)\,r,\, a(t)\,r\,\sin\theta\right], \tag{10}$$

$$\omega_{122} = \omega_{133} = \omega_{144} = W_1(t), \quad \omega_{234} = -\omega_{243} = \omega_{342} = W_2(t),$$
$$\omega_{233} = \omega_{244} = -\frac{\sqrt{1-kr^2}}{a(t)r}, \quad \omega_{344} = \frac{\cot(\theta)}{a(t)r}, \tag{11}$$

where W_1 and W_2 are depending on k-parameter and defined by:

1. $k = 0$: $W_1 = W_2 = 0$,
2. $k = +1$: $W_1 = 0$ and $W_2(t) = \pm\frac{\sqrt{k}}{a(t)}$,
3. $k = -1$: $W_1(t) = \pm\frac{\sqrt{-k}}{a(t)}$ and $W_2 = 0$.

For any W_1 and W_2, we will obtain the same symmetric FEs set to solve for each subcases depending on k-parameter. The previous coframe/ spin-connection pair was found by solving the Equations (9) and $R^a{}_{b\mu\nu} = 0$ condition as defined in ref. [5]. These solutions were also used in $F(T)$ TRW spacetime recent works [55–57]. The TRW spacetime structure is typically explanable by a G_6 Lie algebra group. The FEs to be solved in the current paper are defined for each k-parameter cases and will lead to additional new teleparallel $F(T)$ solution classes. The FEs defined by Equations (2)–(3) are still purely symmetric and valid on proper frames as showed in refs. [55–57]. The Equations (3) are trivially satisfied and we will solve the Equations (2) for each k-parameter case.

2.3. Conservation Laws and Field Equations of Cosmological Perfect Fluids

The canonical energy-momentum and its GR CLs are obtained from $\mathcal{L}_{Source}$ term of Equation (1) as [3,7]:

$$\Theta_a{}^\mu = \frac{1}{h}\frac{\delta\mathcal{L}_{Source}}{\delta h^a{}_\mu}, \quad \Rightarrow \quad \overset{\circ}{\nabla}_\nu\left(\Theta^{\mu\nu}\right) = 0, \tag{12}$$

where $\overset{\circ}{\nabla}_\nu$ the covariant derivative and $\Theta^{\mu\nu}$ the conserved energy-momentum tensor. The antisymmetric and symmetric parts of Θ_{ab} are [47–52,55]:

$$\Theta_{[ab]} = 0, \qquad \Theta_{(ab)} = T_{ab}, \tag{13}$$

where T_{ab} is the symmetric part of $\Theta^{\mu\nu}$. The Equation (12) also imposes the symmetry of $\Theta^{\mu\nu}$ and then Equations (13) condition. Equation (13) is valid only when the matter field interacts with the metric $g_{\mu\nu}$ defined from the coframe $h^a{}_\mu$ and the gauge g_{ab}, and is not directly coupled to the $F(T)$ gravity. This consideration is only valid for the null hypermomentum case (i.e. $\mathfrak{T}^{\mu\nu} = 0$) as discussed in refs. [48–52,54,55]. This last condition on hypermomentum is defined from Equations (2)–(3) as [54]:

$$\mathfrak{T}_{ab} = \kappa\Theta_{ab} - F_T(T)\overset{\circ}{G}_{ab} - F_{TT}(T)\,S_{ab}{}^\mu\,\partial_\mu T - \frac{g_{ab}}{2}\left[F(T) - T\,F_T(T)\right] = 0. \tag{14}$$

There are more general teleparallel $\mathfrak{T}^{\mu\nu}$ definitions and $\mathfrak{T}^{\mu\nu} \neq 0$ CLs, but this does not really concern the teleparallel $F(T)$-gravity situation [54,112–114].

For a TRW spacetime geometry defined by Equations (10)–(11), the Equation (12) for a $P = P(\rho)$ fluid is [55–57]:

$$\dot{\rho} + 3\,H\left(P(\rho) + \rho\right) = 0, \tag{15}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. The general FEs system for TRW cosmological spacetimes are [55–57]:

1. **k = 0 flat or non-curved**:

$$\kappa\rho = -\frac{F}{2} + 6H^2 F_T, \tag{16}$$
$$\kappa(\rho + 3P(\rho)) = F - 6\left(\dot{H} + 2H^2\right) F_T - 6H F_{TT} \dot{T}, \tag{17}$$
$$T = 6H^2. \tag{18}$$

The Equation (18) yields to $H = \sqrt{\frac{T}{6}}$, and then Equations (16)–(17) become:

$$\kappa\rho = -\frac{F}{2} + T F_T, \tag{19}$$
$$\kappa(\rho + 3P(\rho)) = F - \left(6\dot{H} + 2T\right) F_T - 12\dot{H}\, T F_{TT}, \tag{20}$$

The pure vacuum solution ($\rho = 0$ and $P = 0$) to Equations (19)–(20) is $F(T) = F_0 \sqrt{T}$. However for any $P = P(\rho)$, we can set $a(t) = a_0 t^n$ as cosmological scale and $F(T) = \sqrt{T}\, G(T)$ as solution ansatz, and we find the unified FE by merging Equations (19)–(20):

$$\frac{3}{2}\left(n - 1 + \frac{n\, P(\rho)}{\rho}\right) G_T = T G_{TT}. \tag{21}$$

Equation (21) is the general $k = 0$ unified FE to solve for any EoS and CL. This is an easy-to-solve and the solution will be some easy-to-compute integral equation.

2. **k = −1 negative curved**:

$$\kappa\rho = -\frac{F}{2} + 6H\left(H + \frac{\delta\sqrt{-k}}{a}\right) F_T, \tag{22}$$
$$\kappa(\rho + 3P(\rho)) = F - 6\left(\dot{H} + H^2 + \left(H + \frac{\delta\sqrt{-k}}{a}\right)^2\right) F_T - 6\left(H + \frac{\delta\sqrt{-k}}{a}\right) F_{TT} \dot{T}, \tag{23}$$
$$T = 6\left(H + \frac{\delta\sqrt{-k}}{a}\right)^2. \tag{24}$$

From Equation (24) and using $a(t) = a_0 t^n$ ansatz, we find a characteristic equation yielding to $t(T)$ solutions:

$$0 = \frac{\delta\sqrt{-k}}{a_0} t^{-n} + n t^{-1} - \delta_1 \sqrt{\frac{T}{6}}. \tag{25}$$

The possible solutions of Equation (25) are:

(a) $\mathbf{n} = \frac{1}{2}$ (slow expansion):

$$t^{-1}(T) = \left[-\frac{\delta\sqrt{-k}}{a_0} \pm \sqrt{-\frac{k}{a_0^2} + \delta_1 \sqrt{\frac{2T}{3}}}\right]^2. \tag{26}$$

(b) $\mathbf{n} = \mathbf{1}$ (linear expansion):

$$t^{-1}(T) = \frac{\delta_1}{\left(\frac{\delta\sqrt{-k}}{a_0} + 1\right)} \sqrt{\frac{T}{6}}. \tag{27}$$

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(c) $\mathbf{n} = \mathbf{2}$ (fast expansion):

$$t^{-1}(T) = -\frac{\delta a_0}{\sqrt{-k}} \pm \sqrt{-\frac{a_0^2}{k} + \delta_1 \delta a_0 \sqrt{-\frac{T}{6k}}}. \tag{28}$$

(d) $\mathbf{n} \to \infty$ (very fast expansion limit):

$$t^{-1}(T) \approx \frac{\delta_1}{n}\sqrt{\frac{T}{6}} \to 0. \tag{29}$$

Then Equations (22)–(23) become by substituting Equation (25) and the ansatz:

$$\kappa\rho = -\frac{F}{2} + \sqrt{6}\, n\delta_1\, t^{-1}(T)\sqrt{T}F_T, \tag{30}$$

$$\kappa\rho\left(1 + \frac{3P(\rho)}{\rho}\right) = F - \left(6n(n-1)t^{-2}(T) + T\right)F_T + 12nt^{-1}(T)\left((1-n)\, t^{-1}(T) + \delta_1\sqrt{\frac{T}{6}}\right)T\, F_{TT}, \tag{31}$$

The unified FE from Equations (30)–(31) is:

$$\left[-\frac{F}{2} + \sqrt{6}\, n\delta_1\, t^{-1}(T)\sqrt{T}F_T\right]\left(1 + \frac{3P(\rho)}{\rho}\right) = F - \left(6n(n-1)t^{-2}(T) + T\right)F_T + 12nt^{-1}(T)\left((1-n)\, t^{-1}(T) + \delta_1\sqrt{\frac{T}{6}}\right)T\, F_{TT}, \tag{32}$$

3. $\mathbf{k} = +\mathbf{1}$ **positive curved**:

$$\kappa\rho = -\frac{F}{2} + 6H^2\, F_T, \tag{33}$$

$$\kappa(\rho + 3P(\rho)) = F - 6\left(\dot{H} + 2H^2 - \frac{k}{a^2}\right)F_T - 6H\, F_{TT}\dot{T}, \tag{34}$$

$$T = 6\left[H^2 - \frac{k}{a^2}\right]. \tag{35}$$

From Equation (35) and using $a(t) = a_0\, t^n$ ansatz, we find the characteristic equation for $t^{-1}(T)$:

$$\frac{k}{a_0^2}t^{-2n} - n^2\, t^{-2} + \frac{T}{6} = 0. \tag{36}$$

The possible solutions of Equation (36) are:

(a) $\mathbf{n} = \frac{\mathbf{1}}{\mathbf{2}}$ (slow expansion):

$$t^{-1}(T) = \frac{2k}{a_0^2} \pm \sqrt{\frac{4k^2}{a_0^4} + \frac{2T}{3}}. \tag{37}$$

(b) $\mathbf{n} = \mathbf{1}$ (linear expansion):

$$t^{-2}(T) = \frac{T}{6\left(1 - \frac{k}{a_0^2}\right)}. \tag{38}$$

 doi:10.20944/preprints202508.2052.v1

(c) **n = 2** (fast expansion):

$$t^{-2}(T) = \frac{2a_0^2}{k} \pm \sqrt{\frac{4a_0^4}{k^2} - \frac{a_0^2\,T}{6k}}. \tag{39}$$

(d) **n** $\to \infty$ (very fast expansion limit):

$$t^{-2}(T) \approx \frac{T}{6n^2} \to 0. \tag{40}$$

Then Equations (33)–(34) become by substituting Equation (36) and the ansatz:

$$\kappa\rho = -\frac{F}{2} + 6n^2\,t^{-2}(T)\,F_T, \tag{41}$$

$$\kappa\rho\left(1+\frac{3P(\rho)}{\rho}\right) = F - \left(6n(n-1)\,t^{-2}(T)+T\right)F_T + 12n^2\,t^{-2}(T)\left(T-6n(n-1)t^{-2}(T)\right)F_{TT}, \tag{42}$$

The unified FE from Equations (41)–(42) is:

$$\left[-\frac{F}{2} + 6n^2\,t^{-2}(T)\,F_T\right]\left(1+\frac{3P(\rho)}{\rho}\right) = F - \left(6n(n-1)\,t^{-2}(T)+T\right)F_T + 12n^2\,t^{-2}(T)\left(T-6n(n-1)t^{-2}(T)\right)F_{TT}, \tag{43}$$

2.4. *Energy Conditions and Thermodynamic Laws in Teleparallel Gravity*

Regardless of the definition of EoS or any pressure-density relationship, there are energy conditions to satisfy for any physical system based on a PF [115]:

- Weak Energy Condition (**WEC**): $\rho \geq 0$, $P_r + \rho \geq 0$ and $P_t + \rho \geq 0$.
- Strong Energy Condition (**SEC**): $P_r + 2P_t + \rho \geq 0$, $P_r + \rho \geq 0$ and $P_t + \rho \geq 0$.
- Null Energy Condition (**NEC**): $P_r + \rho \geq 0$ and $P_t + \rho \geq 0$.
- Dominant Energy Condition (**DEC**): $\rho \geq |P_r|$ and $\rho \geq |P_t|$.

From this consideration and by setting $P_r = P_t$ for a uniform pressure cosmological fluid, we can summarize the energy conditions (ECs):

$$\rho + P \geq 0, \quad \rho + 3P \geq 0, \quad \rho \geq |P|, \quad \rho \geq 0. \tag{44}$$

We will apply in Sections 3 and 4 the Equations (44) for any CL solutions. By this way, we will verify the physical consistency of all CL solutions found in the current paper.

3. Pure Chaplygin Gas Teleparallel Field Equation Solutions

3.1. *Conservation Law Solutions and Energy Conditions*

For a pure Chaplygin gas of EoS $P = \frac{P_0}{\rho}$ (see refs. [98–102,111]) and by using a power-law ansatz $a(t) = a_0\,t^n$, the Equation (15) becomes:

$$\dot{\rho} + \frac{3n}{t}\left(\frac{P_0}{\rho} + \rho\right) = 0,$$
$$\Rightarrow \rho(t) = \left[\left(P_0 + \rho_0^2\right)t^{-6n} - P_0\right]^{1/2}. \tag{45}$$

 doi:10.20944/preprints202508.2052.v1

We will apply the ECs as stated by Equations (44) on Equation (45) and find that:

$$\left(P_0+\rho_0^2\right)t^{-6n}\geq 0,\quad \left(P_0+\rho_0^2\right)t^{-6n}\geq -2P_0,\quad \left(P_0+\rho_0^2\right)t^{-6n}\geq P_0+|P_0|,$$
$$\left(P_0+\rho_0^2\right)t^{-6n}\geq P_0. \tag{46}$$

The dominating condition from Equations (46) is:

$$\left(P_0+\rho_0^2\right)t^{-6n}\geq 2|P_0|. \tag{47}$$

3.2. $k=0$ Cosmological Solutions

Using Equation (45) for CL and EoS and the torsion scalar defined by Equation (18), the unified FEs described by Equation (21) and the solution are:

$$\frac{3}{2}\left[n-1+\frac{nP_0}{\left[\left(P_0+\rho_0^2\right)T^{3n}-P_0\right]}\right]G_T = T\,G_{TT},$$
$$\Rightarrow G_T(T)=G_T(0)\,T^{\frac{3}{2}(n-1)}\sqrt{\beta T^{3n}-P_0}, \tag{48}$$

where $\beta=\frac{P_0+\rho_0^2}{(6n^2)^{3n}}$. By integration and substitution into $F(T)$ ansatz, we find as final solution:

$$F(T)=G(0)\sqrt{T}-2G_T(0)\sqrt{-P_0}\,{}_2F_1\left(-\frac{1}{2},-\frac{1}{6n};1-\frac{1}{6n};\frac{\beta}{P_0}T^{3n}\right). \tag{49}$$

As in refs [55,56], the $k=0$ flat cosmological case yields to an easy to compute analytical teleparallel $F(T)$ solution for any value of n.

3.3. $k=-1$ Cosmological Solutions

The Equation (32) with $P(\rho)=\frac{P_0}{\rho}$ is by substituting Equation (45) solution:

$$\left[-\frac{F}{2}+\sqrt{6}\,n\delta_1\,t^{-1}(T)\sqrt{T}F_T\right]\left(\frac{\left(P_0+\rho_0^2\right)t^{-6n}(T)+2P_0}{\left(P_0+\rho_0^2\right)t^{-6n}(T)-P_0}\right)$$
$$=F-\left(6n(n-1)t^{-2}(T)+T\right)F_T+12nt^{-1}(T)\left((1-n)\,t^{-1}(T)+\delta_1\sqrt{\frac{T}{6}}\right)T\,F_{TT}. \tag{50}$$

The possible solution to Equation (50) are by using a power-law ansatz $F(T)=F_0\,T^r$:

1. $\mathbf{n}=\frac{1}{2}$: By using the approximation $\frac{a_0^2\delta_1}{k}\sqrt{\frac{2T}{3}}\ll 1$ and setting the $+$ root, Equation (50) simplifies as:

$$0\approx C_1T^2F+\left(2C_2C_1T^{7/2}-2C_2\sqrt{T}+C_2^2T-1\right)F_T+2C_2\left(C_2T-\sqrt{T}\right)TF_{TT}, \tag{51}$$

where $C_1=\left(1+\frac{\rho_0^2}{P_0}\right)\frac{a_0^6}{216k^3}$ and $C_2=\frac{\delta_1 a_0^2}{2\sqrt{6k}}$. The solution of Equation (51) is:

$$F(T)\approx F_+\,T^{r_+}+F_-\,T^{r_-} \tag{52}$$

$$r_\pm=\frac{1}{4C_2(C_2-1)}\left[-\left(2C_1C_2-10C_2+7C_2^2-1\right)\pm\left[\left(2C_1C_2-10C_2+7C_2^2-1\right)^2\right.\right.$$
$$\left.\left.-8C_2(C_2-1)\left(C_1(1-C_2)-\frac{25}{2}C_2+6C_2^2-3\right)\right]^{1/2}\right], \tag{53}$$

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where $C_2 \neq \{0, 1\}$.

2. $\mathbf{n} = \mathbf{1}$: For $C_3 \ll 1$, Equation (50) can be approximated as:

$$0 \approx C_3\, T^2 F - \frac{2}{C_4}\left(1 + C_3\, T^3 - \frac{C_4}{2}\right) F_T - \frac{2}{C_4}\, T F_{TT}, \tag{54}$$

where $C_3 = \frac{\left(1+\frac{\rho_0^2}{P_0}\right)}{216\left(\frac{\delta\sqrt{-k}}{a_0}+1\right)^6}$ and $C_4 = \frac{\delta\sqrt{-k}}{a_0} + 1$. The solution of Equation (54) is described by Equation (52) with the $r_{\pm}$ roots:

$$r_{\pm} = -\frac{1}{2}\left(6 + C_3 - \frac{C_4}{2}\right) \pm \left[\frac{1}{4}\left(6 + C_3 - \frac{C_4}{2}\right)^2 - \left(9 - \frac{3C_4}{2} - \frac{C_3 C_4}{2}\right)\right]^{1/2}. \tag{55}$$

3. $\mathbf{n} = \mathbf{2}$: By using the approximation $\frac{\delta\,\delta_1\sqrt{-k}}{a_0\sqrt{6}}\sqrt{T} \ll 1$ and setting the $+$ root, Equation (50) simplifies as:

$$0 \approx C_5\, T^5\, F + \left(\frac{7}{2} + 2C_5\, T^6\right) F_T + 3T F_{TT}, \tag{56}$$

where $C_5 \approx 5,233 \times 10^{-9}\left(1 + \frac{\rho_0^2}{P_0}\right) \ll 1$. The solution of Equation (54) is still the Equation (52) with $r_{\pm}$ roots:

$$r_{\pm} = \frac{1}{12}\left[-(1 + 4C_5) \pm \left[1 + 16C_5^2 - 520C_5\right]^{1/2}\right] \approx \frac{1}{12}[-(1 + 4C_5) \pm (1 - 260C_5)]. \tag{57}$$

Under the $C_5 \to 0$ limit: $r_+ \approx -22C_5 \to 0$ and $r_- \approx -\frac{1}{6} + \frac{64}{3}C_5 \to -\frac{1}{6}$. In this case, Equation (52) leads to $F(T) \to -\Lambda_0 + F_-\, T^{-1/6}$ with the cosmological constant Λ_0. This last solution is similar to those of ref. [62].

4. $\mathbf{n} \to \infty$: Under this limit, Equation (50) simplifies as a simple homogeneous equation for $P_0 \neq = -\rho_0^2$:

$$T\, F_T \approx \frac{F}{2} \quad \Rightarrow \quad F(T) \approx F_0\sqrt{T}, \tag{58}$$

as we can find for any pure flat background cosmological simple solutions [49,55,56].

3.4. $k = +1$ Cosmological Solutions

The Equation (43) with $P(\rho) = \frac{P_0}{\rho}$ is by substituting Equation (45) solution:

$$\left[-\frac{F}{2} + 6n^2\, t^{-2}(T)\, F_T\right]\left(\frac{(P_0 + \rho_0^2)\, t^{-6n}(T) + 2P_0}{(P_0 + \rho_0^2)\, t^{-6n}(T) - P_0}\right)$$
$$= F - \left(6n(n-1)\, t^{-2}(T) + T\right) F_T + 12n^2\, t^{-2}(T)\left(T - 6n(n-1)t^{-2}(T)\right) F_{TT}. \tag{59}$$

The possible solution to Equation (59) are by using the $F(T) = F_0\, T^r$ ansatz:

1. $\mathbf{n} = \frac{\mathbf{1}}{\mathbf{2}}$: By using the approximation $\frac{a_0^4}{6k^2}\, T \ll 1$ and setting the $-$ root, Equation (59) simplifies as:

$$0 \approx C_7\, T^2 F + \left(3C_6\, T - 2C_6 C_7\, T^4 - 1\right) F_T + 2C_6(1 + C_6\, T)\, T^2 F_{TT}, \tag{60}$$

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where $C_6 = \frac{a_0^4}{24k^2}$ and $C_7 = \left(1+\frac{\rho_0^2}{P_0}\right)\frac{a_0^6}{216k^3} \ll 1$. The solution of Equation (60) is Equation (52) with $r_\pm$ roots:

$$r_\pm = \frac{1}{4C_6(1+C_6)}\Bigg[-\left(13C_6 - 2C_6C_7 + 6C_6^2 - 1\right) \pm \Bigg[\left(13C_6 - 2C_6C_7 + 6C_6^2 - 1\right)^2 - 8C_6(1+C_6)\left(4C_6^2 + 21C_6 - 4\right)\Bigg]^{1/2}\Bigg], \tag{61}$$

where $C_6 \neq \{-1, 0\}$.

2. $\mathbf{n = 1}$: Equation (59) becomes:

$$0 \approx C_9\, T^2 F + \left(1 - \frac{2}{C_8}\left(1 + C_9\, T^3\right)\right) F_T - \frac{2T\, F_{TT}}{C_8}, \tag{62}$$

where $C_8 = 1 - \frac{k}{a_0^2}$ and $C_9 = \frac{\left(1+\frac{\rho_0^2}{P_0}\right)}{216\left(1-\frac{k}{a_0^2}\right)^3} \ll 1$. The solution of Equation (62) is Equation (52) with $r_\pm$ roots:

$$r_\pm = -\frac{1}{2}\left(C_9 + \frac{C_8}{2} + 6\right) \pm \left[\frac{1}{4}\left(C_9 + \frac{C_8}{2} + 6\right)^2 - \left(9 + \frac{3C_8}{2} - \frac{C_8C_9}{2}\right)\right]^{1/2}. \tag{63}$$

3. $\mathbf{n = 2}$: By using the approximation $\frac{k}{24a_0^2}\, T \ll 1$ and setting the $-$ root, Equation (59) simplifies under the $C_{10} \ll 1$ approximation as:

$$0 \approx C_{10}\, T^5 F - \left(\frac{1}{2} + 2C_{10}\, T^6\right) F_T - T F_{TT}, \tag{64}$$

where $C_{10} = \left(1+\frac{\rho_0^2}{P_0}\right)\left(\frac{1}{24}\right)^6 \ll 1$. The solution of Equation (64) is Equation (52) with $r_\pm$ roots:

$$r_\pm = -\left(C_{10} + \frac{23}{4}\right) \pm \left[\left(C_{10} + \frac{23}{4}\right)^2 - 33 + C_{10}\right]^{1/2} \approx -C_{10} - \frac{23}{4} \pm \left(\frac{1}{4} + 25C_{10}\right). \tag{65}$$

Under the $C_{10} \to 0$ limit: $r_+ \to -\frac{11}{2}$ and $r_- \to -6$. Then Equation (52) becomes $F(T) \to F_+\, T^{-\frac{11}{2}} + F_+\, T^{-6}$.

4. $\mathbf{n \to \infty}$: Once again for $P_0 \neq -\rho_0^2$, we find under this limit the same differential equation and solution as Equation (58), i.e. $F(T) \approx F_0\, \sqrt{T}$.

4. General Polytropic Gas Teleparallel Field Equation Solutions

4.1. Conservation Law Solutions and Energy Conditions

A polytropic gas is defined as $\rho = \rho(t)$ and $P = -P_0\, \rho^{1+\frac{1}{p}}$ where $\frac{1}{3} < P_0 \leq 1$ and $0 < p < \infty$ [103–110]. Under the $p \to \infty$ limit, we find that $P = -P_0\, \rho$, a linear DE PF. However the $p \to 0$ limit will lead to an infinitely huge pressure universe looking like the very early universe. The Equation (15) for this general type of gas is by using the $a(t) = a_0\, t^n$ ansatz:

$$\dot{\rho} + \frac{3n}{t}\rho\left(1 - P_0\, \rho^{\frac{1}{p}}\right) = 0,$$
$$\Rightarrow \rho(t) = \left[P_0 - \left(P_0 - \rho_0^{-1/p}\right) t^{3n/p}\right]^{-p}. \tag{66}$$

 doi:10.20944/preprints202508.2052.v1

We will apply the ECs defined by Equations (44) for Equation (66) CL solution as:

$$\left(P_0-\rho_0^{-1/p}\right)t^{3n/p}\leq 0,\quad \left(P_0-\rho_0^{-1/p}\right)t^{3n/p}\leq -2P_0,\quad \left(P_0-\rho_0^{-1/p}\right)t^{3n/p}\leq P_0-|P_0|,$$
$$\left(P_0-\rho_0^{-1/p}\right)t^{3n/p}\leq P_0. \tag{67}$$

The dominating energy conditions is exactly $t^{3n/p}\leq \frac{-2|P_0|}{\left(P_0-\rho_0^{-1/p}\right)}$.

4.2. $k=0$ Cosmological Solutions

The unified FEs using Equation (18), $P=-P_0\,\rho^{1+\frac{1}{p}}$ and Equation (66) is [56]:

$$\frac{3}{2}\left(n-1+\frac{n\,P_0}{\left[\beta_p T^{-3n/2p}-P_0\right]}\right)G_T=T\,G_{TT},$$
$$\Rightarrow\;G_T(T)=G_T(0)\,T^{\frac{3}{2}(n-1)}\left[P_0\,T^{3n/2p}-\beta_p\right]^{-2p/3}, \tag{68}$$

where $\beta_p=\left(P_0-\rho_0^{-1/p}\right)(6n^2)^{3n/2p}$. The chaplygin gas case corresponds to the $p=-\frac{1}{2}$, i.e. $\beta_{1/2}=\beta$ in section 3. The general teleparallel $F(T)$ solution is exactly:

$$F(T)=G(0)\sqrt{T}+G_T(0)\sqrt{T}\left[\int dT'\,T'^{3(n-1)/2}\left(P_0\,T^{3n/2p}-\beta_p\right)^{-2p/3}\right],$$
$$=G(0)\sqrt{T}+G_T(0)\sqrt{T}\,I_p(T), \tag{69}$$

where $I_p(T)$ is a new integral-based special function. Some values of $I_p(T)$ are displayed in Table 1. The $p\geq 0$ solutions yield to hypergeometric functions and $p<0$ solution yield to power-law superposition solutions. We confirm in part the refs [55,56] where $k=0$ flat cosmological case still yields to analytical teleparallel $F(T)$ solution for each value of p and n.

Table 1. Values of $I_p(T)$ function for the Equation (69) teleparallel flat polytropic solutions.

p	$I_p(T)$
0 limit	$\frac{2}{(3n-1)}\,T^{(3n-1)/2}$
$\frac{3}{2}$	$-\frac{2\,T^{(3n-1)/2}}{\beta_{3/2}\,(3n-1)}\,{}_2F_1\left(1,\frac{3n-1}{2n};\frac{5n-1}{2n};\frac{P_0}{\beta}\,T^n\right)$
3	$\frac{2\,T^{(3n-1)/2}}{\beta_3^2\,(3n-1)}\,{}_2F_1\left(2,\frac{3n-1}{n};\frac{4n-1}{n};\frac{P_0}{\beta_3}\,T^{n/2}\right)$
$\frac{9}{2}$	$-\frac{2\,T^{(3n-1)/2}}{\beta_{9/2}^3\,(3n-1)}\,{}_2F_1\left(3,\frac{3(3n-1)}{2n};\frac{11n-3}{2n};\frac{P_0}{\beta_{9/2}}\,T^{n/3}\right)$

4.3. $k=-1$ Cosmological Solutions

The Equation (32) for $P=-P_0\,\rho^{1+\frac{1}{p}}$ is by substituting Equation (66) solution:

$$\left[-\frac{F}{2}+\sqrt{6}\,n\delta_1\,t^{-1}(T)\sqrt{T}F_T\right]\left(\frac{\left(P_0-\rho_0^{-1/p}\right)t^{3n/p}(T)+2P_0}{\left(P_0-\rho_0^{-1/p}\right)t^{3n/p}(T)-P_0}\right)$$
$$=F-\left(6n(n-1)t^{-2}(T)+T\right)F_T+12nt^{-1}(T)\left((1-n)\,t^{-1}(T)+\delta_1\sqrt{\frac{T}{6}}\right)T\,F_{TT}. \tag{70}$$

The possible solution to Equation (70) are by using the $F(T)=F_0\,T^r$ ansatz:

1. $\mathbf{n}=\frac{1}{2}$: By using the approximation $\frac{a_0^2\delta_1}{k}\sqrt{\frac{2T}{3}}\ll 1$ and the $+$ root, Equation (70) becomes:

$$0\approx\frac{3F}{2}+\left(C_2^2T+C_2T^{1/2}-1\right)TF_T+2C_2\left(C_2\sqrt{T}-1\right)T^{5/2}\,F_{TT}, \tag{71}$$

where $C_{11}=\left(1-\frac{1}{P_0\,\rho_0^{1/p}}\right)\frac{(-6k)^{3/2p}}{(\delta\delta_1a_0)^{3/p}}\gg 1$ and then $\frac{C_{11}\,T^{-3/p}-2}{C_{11}\,T^{-3/p}+1}\to 1$ under the same limit. Using the same power-law ansatz, we find the Equation (52) as solution with the roots:

$$r_{\pm}=-\frac{1}{4C_2}\pm\left[\frac{1}{16C_2^2}-\frac{1+2C_2}{4C_2(C_2-1)}\right]^{1/2}, \tag{72}$$

where $C_2\neq\{0,1\}$.

2. $\mathbf{n}=\mathbf{1}$: We find:

$$0\approx\frac{3F}{2}-\left(1+\frac{1}{C_4}\right)T\,F_T+\frac{2}{C_4}\,T^2\,F_{TT}, \tag{73}$$

where $C_{12}=\left(1-\frac{1}{P_0\,\rho_0^{1/p}}\right)\left(\sqrt{6}\delta_1\left(\frac{\delta\sqrt{-k}}{a_0}+1\right)\right)^{3/p}\gg 1$ and then $\frac{2+C_{12}T^{-3/2p}}{1-C_{12}T^{-3/2p}}\to -1$. We find the Equation (52) as solution with the roots $r_+=\frac{C_4}{2}$ and $r_-=\frac{3}{2}$.

3. $\mathbf{n}=\mathbf{2}$: By using $\frac{\delta\,\delta_1\sqrt{-k}}{a_0\sqrt{6}}\sqrt{T}\ll 1$ and the $+$ root, Equation (70) simplifies as:

$$0\approx\frac{3F}{2}-\frac{1}{2}TF_T-3T^2\,F_{TT}, \tag{74}$$

where $C_{13}=\left(1-\frac{1}{P_0\,\rho_0^{1/p}}\right)(24)^{3/p}\gg 1$ and then $\frac{2+C_{13}T^{-3/p}}{1-C_{13}T^{-3/p}}\to -1$. We find the Equation (52) as solution with the roots $r_{\pm}=\frac{5\pm\sqrt{97}}{12}$.

4. $\mathbf{n}\to\infty$ limit: For $P_0\neq\rho_0^{-1/p}$, we obtain the Equation (58) with $F(T)\approx F_0\,\sqrt{T}$ as solution.

4.4. $k=+1$ Cosmological Solutions

The Equation (43) for $P=-P_0\,\rho^{1+\frac{1}{p}}$ is by substituting Equation (66) solution:

$$\left[-\frac{F}{2}+6n^2\,t^{-2}(T)\,F_T\right]\left(\frac{\left(P_0-\rho_0^{-1/p}\right)t^{3n/p}(T)+2P_0}{\left(P_0-\rho_0^{-1/p}\right)t^{3n/p}(T)-P_0}\right)$$
$$=F-\left(6n(n-1)\,t^{-2}(T)+T\right)F_T+12n^2\,t^{-2}(T)\left(T-6n(n-1)t^{-2}(T)\right)F_{TT}. \tag{75}$$

The possible solution to Equation (75) are by using the $F(T)=F_0\,T^r$ ansatz:

1. $\mathbf{n}=\frac{1}{2}$: By using the approximation $\frac{a_0^4}{6k^2}\,T\ll 1$ and the $-$ root, Equation (75) simplifies as:

$$0\approx\frac{3F}{2}-TF_T+2C_6(1+C_6\,T)\,T^3F_{TT}, \tag{76}$$

where $C_{14}=\left(1-\frac{1}{P_0\,\rho_0^{1/p}}\right)\left(-\frac{6k}{a_0^2}\right)^{3/2p}\gg 1$. We find the Equation (52) as solution with the roots:

$$r_{\pm}=\frac{(1-2C_6+2C_6^2)}{4C_6(1+C_6)}\pm\left[\frac{(1-2C_6+2C_6^2)^2}{16C_6^2(1+C_6)^2}+\frac{1}{2}\right]^{1/2}, \tag{77}$$

where $C_6\neq\{-1,0\}$.

 doi:10.20944/preprints202508.2052.v1

2. $\mathbf{n=1}$: With the approximation $\frac{a_0^4}{6k^2}\,T \ll 1$, we find that:

$$0 \approx \frac{3F}{2} - \left(1+\frac{1}{C_8}\right) T\,F_T + \frac{2T^2\,F_{TT}}{C_8}, \tag{78}$$

where $C_{15} = \left(1-\frac{1}{P_0\,\rho_0^{1/p}}\right)\left(6\left(1-\frac{k}{a_0^2}\right)\right)^{3/2p} \gg 1$. We find the Equation (52) as solution with the roots $r_+ = \frac{C_8}{2}$ and $r_- = \frac{3}{2}$

3. $\mathbf{n=2}$: By using $\frac{k}{24a_0^2}\,T \ll 1$, $C_{13} \gg 1$ and $-$ root, Equation (75) simplifies as:

$$0 \approx \frac{3F}{2} - \frac{5}{2}TF_T + T^2F_{TT}. \tag{79}$$

We find the Equation (52) as solution with the roots $r_+ = 3$ and $r_- = \frac{1}{2}$.

4. $\mathbf{n} \to \infty$ limit: Here again for $P_0 \neq \rho_0^{-1/p}$, we obtain the Equation (58) with $F(T) \approx F_0\,\sqrt{T}$ as solution.

5. Comparison Between Chaplygin and Polytropic Gases in Teleparallel $F(T)$ Gravity

We compare the teleparallel $F(T)$ solutions for flat cosmological ($k=0$) Chaplygin and polytropic gases cases. We have plotted Equation (49) (Chaplygin gas), $p \to 0$ and 3 subcases of Equation (69) on the Figure 1. The top left subfigure compares for $n=\frac{1}{2}$ case the Chaplygin and polytropic p-type gases solutions. We removed the cosmological geometry (homogeneous) $G(0)\,\sqrt{T}$ term (i.e. the $n \to \infty$ solution) and kept only the source contribution for making the distinction between polytropic and Chaplygin gases for $n=\frac{1}{2}$, 1 and 2. The Chaplygin gas is a specific case of the general polytropic gas where we can consider the parameter $p=-\frac{1}{2}$ in the polytropic EoS (even if the EoS is defined for positive values of p). The Chaplygin gas case leads to some characteristic curves as we can also see on the top-right graphs of Figure 1. This feature makes the comparison more relevant and expresses the real nature of Chaplygin and polytropic gases.

We found for spatially curved $k=\pm 1$ cases some approximated teleparallel $F(T)$ solutions, all described by double power-law from defined by the Equation (52) form. We only found the power $r_\pm$ in each cases after approximating the Equations (50) and (59) for Chaplygin gas, and Equations (70) and (75) for general polytropic gas. A first feature is that the polytropic p-type teleparallel $F(T)$ solutions do not contain any p-dependent term as seen in Sections 4.3 and 4.4 for $k=\pm 1$ solutions. The approximated polytropic teleparallel solutions are greatly simplified by some parameter approximations.

In general, there is no cosmological constant Λ_0 in the teleparallel solutions in almost all the cases. The only exception to this rule is Equations (56)–(57) under the $C_5 \to 0$ limit of $n=2$ subcase, where $r_+ = 0$ leading to $F_+ = -\Lambda_0$, and then a simple polynomial teleparallel $F(T)$ solution.

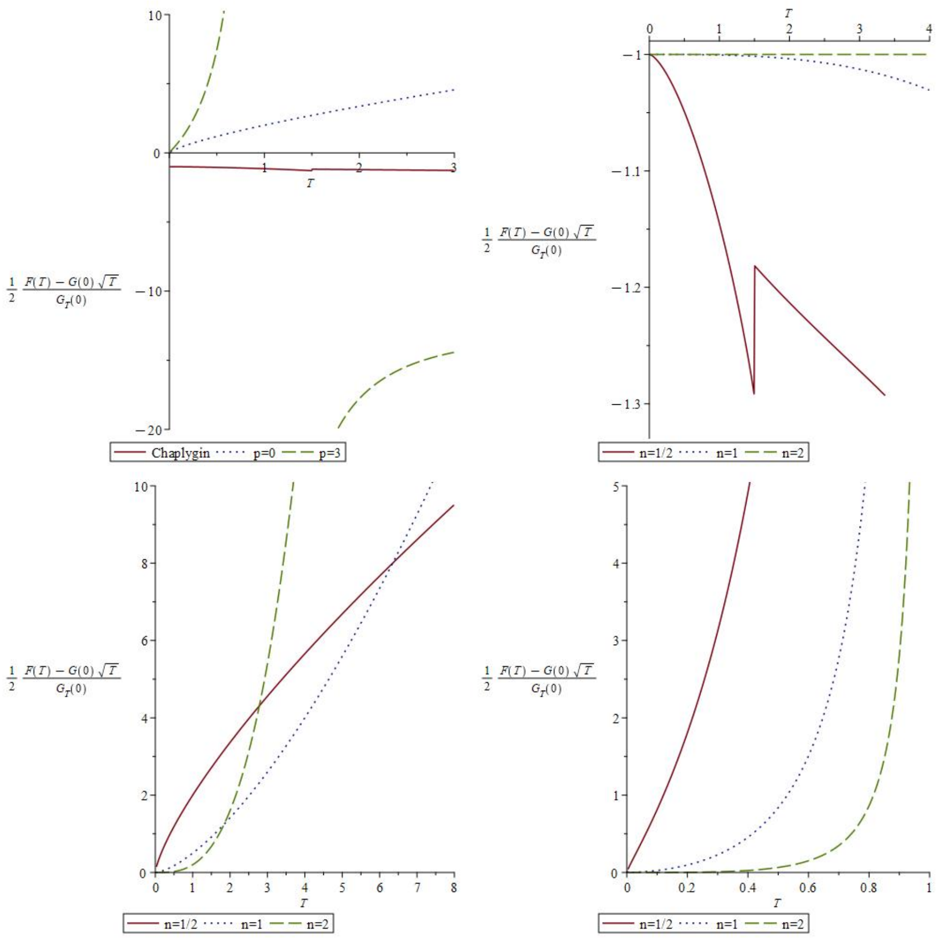

Figure 1. Plot of flat cosmological $k = 0$ teleparallel $F(T)$ solutions for Chaplygin and Polytropic cosmological gas sources (top left: $n = \frac{1}{2}$ and any p, top right: Chaplygin, bottom left: $p \to 0$ limit, bottom right: $p = 3$).

6. Concluding Remarks

We will first conclude the current paper by stating that the flat cosmological $k = 0$ teleparallel $F(T)$ solutions are described by special functions, especially by hypergeometric functions plotted on the Figure 1 for Chaplygin ($p = -\frac{1}{2}$) and polytropic $p \to 0$ limit and 3 gases. For the spatially curved cosmological $k = \pm 1$ teleparallel $F(T)$ solutions are practically all approximated by a double power-law function described by the Equation (52). The $k = \pm 1$ general polytropic solutions found in Sections 4.3 and 4.4 are not p-dependent because of some dominating terms approximations. There is an important feature in the new solutions: no cosmological constant Λ_0 term in almost all the cases. The only exception is the Equations (56)-(57) under the $C_5 \to 0$ limit leading to a simple polynomial with Λ_0 term. Under all previous considerations, we can claim that non-flat $k = \pm 1$ teleparallel $F(T)$ solutions have some common points with those found for cosmological teleparallel $F(T)$ and $F(T, B)$ solutions found in refs [55,56,62], because we use the same coframe/spin-connection pair and ansatz.

Therefore, there are some perspective of future works going further than the current works. We can proceed for example with the same type of polytropic fluid sources for KS and static SS teleparallel spacetimes for new additional classes of solutions by using the same process and coframe ansatz approaches as in refs. [48,49,51,52]. We can also use the same TRW geometry, replace polytropic sources by an electromagnetic source, proceed by the same approach, and finally find the corresponding classes of teleparallel $F(T)$ solutions. A last insight is that we will be able to study the Anti-deSitter (AdS) spacetimes in teleparallel gravity by using polytropic sources. However, we also need to investigate the polytropic principles in teleparallel gravity framework for possible symmetries and additional KVs. More physically, we will be able to elaborate a pure teleparallel quartessence models explaining the dark energy and the dark matter under an unified model. All these mentioned suggestions are feasible in a near future and will allow to study the AdS wormholes and BHs solutions in teleparallel gravity.

Funding: This research received no external funding.

Data Availability Statement: All data are included in the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Lucas, T.G.; Obukhov, Y.; Pereira, J.G. Regularizing role of teleparallelism. *Physical Review D* **2009**, *80*, 064043. [arXiv:0909.2418 [gr-qc]].
2. Krššák, M., van den Hoogen, R. J., Pereira, J. G., Boehmer, C. G. & Coley, A. A. , Teleparallel Theories of Gravity: Illuminating a Fully Invariant Approach, *Classical and Quantum Gravity* **2019**, *36*, 183001 , [arXiv:1810.12932 [gr-qc]].
3. Bahamonde, S., Dialektopoulos, K.F., Escamilla-Rivera, C., Farrugia, G., Gakis, V., Hendry, M., Hohmann, M., Said, J.L., Mifsud, J. & Di Valentino, E., Teleparallel Gravity: From Theory to Cosmology, *Report of Progress in Physics* **2023**, *86*, 026901, [arXiv:2106.13793 [gr-qc]].
4. Krssak, M.; Pereira, J.G. Spin Connection and Renormalization of Teleparallel Action. *The European Physical Journal C* **2015**, *75*, 519. [arXiv:1504.07683 [gr-qc]].
5. Coley, A. A., van den Hoogen, R. J. & McNutt, D.D., Symmetry and Equivalence in Teleparallel Gravity, *Journal of Mathematical Physics* **2020**, *61*, 072503, [arXiv:1911.03893 [gr-qc]].
6. McNutt, D.D., Coley, A.A. & van den Hoogen, R.J., A frame based approach to computing symmetries with non-trivial isotropy groups, *Journal of Mathematical Physics* **2023**, *64*, 032503, [arXiv:2302.11493 [gr-qc]].
7. Aldrovandi, R. & Pereira, J.G. *Teleparallel Gravity, An Introduction*, Springer, 2013.
8. Olver, P. *Equivalence, Invariants and Symmetry*; Cambridge University Press: Cambridge, UK, 1995.
9. Krššák, M. & Saridakis, E. N., The covariant formulation of $f(T)$ gravity, *Classical and Quantum Gravity* **2016**, *33*, 115009, [arXiv:1510.08432 [gr-qc]].
10. Ferraro, R. & Fiorini, F. Modified teleparallel gravity: Inflation without an inflation. *Physical Review D* **2007**, *75*, 084031. [arXiv:gr-qc/0610067].
11. Ferraro, R. & Fiorini, F. On Born-Infeld Gravity in Weitzenbock spacetime. *Physical Review D* **2008**, *78*, 124019. [arXiv:0812.1981 [gr-qc]].
12. Linder, E. Einstein's Other Gravity and the Acceleration of the Universe. *Physical Review D* **2010**, *81*, 127301; Erratum in *Physical Review D* **2010**, *82*, 109902. [arXiv:1005.3039 [gr-qc]].
13. Hayashi, K. & Shirafuji, T. New general relativity. *Physical Review D* **1979**, *19*, 3524.
14. Jimenez, J.B. & Dialektopoulos, K.F. Non-Linear Obstructions for Consistent New General Relativity. *Journal of Cosmological and Atroparticle Physics* **2020**, *2020*, 018. [arXiv:1907.10038 [gr-qc]].
15. Bahamonde, S., Blixt, D., Dialektopoulos, K.F. & Hell A., Revisiting Stability in New General Relativity, Physical Review D **2024** 111, 064080, [arXiv:2404.02972 [gr-qc]].
16. Heisenberg, L. Review on $f(Q)$ Gravit, Physics Reports **2023**, 1-78. [arXiv:2309.15958 [gr-qc]].
17. Heisenberg, L.; Hohmann, M. & Kuhn, S., Cosmological teleparallel perturbations, Journal of Cosmology and Astroparticle Physics **2024**, 03, 063, [arXiv:2311.05495 [gr-qc]].
18. Flathmann, K. & Hohmann, M. Parametrized post-Newtonian limit of generalized scalar-nonmetricity theories of gravity. *Physical Review D* **2022**, *105*, 044002, [arXiv:2111.02806 [gr-qc]].
19. Hohmann, M. General covariant symmetric teleparallel cosmology. *Physical Review D* **2021**, *104*, 124077. [arXiv:2109.01525 [gr-qc]]
20. Jimenez, J.B.; Heisenberg, L.; & Koivisto, T.S. The Geometrical Trinity of Gravity. *Universe* **2019**, *5*, 173. [arXiv:1903.06830 [gr-qc]].
21. Nakayama, Y. Geometrical trinity of unimodular gravity. *Classical and Quantum Gravity* **2023**, *40*, 125005. [arXiv:2209.09462 [gr-qc]].
22. Xu, Y.; Li, G.; Harko, T. & Liang, S.-D., $f(Q,T)$ gravity. *The European Physical Journal C* **2019**, *79*, 708. [arXiv:1908.04760 [gr-qc]].
23. Maurya, D.C. & Myrzakulov, R., Exact Cosmology in Myrzakulov Gravity, *The European Physical Journal C* **2024**, 84, 625, [arXiv:2402.02123 [gr-qc]].
24. Harko, T.; Lobo, F.S.N.; Nojiri, S. & Odintsov, S.D., $f(R,T)$ gravity. *Physical Review D* **2011**, *84*, 024020.[arXiv:1104.2669 [gr-qc]].

25. Momeni, D. & Myrzakulov, R., Myrzakulov Gravity in Vielbein Formalism: A Study in Weitzenböck Spacetime, *Nuclear Physics B* **2025**, 1015, 116903, [arXiv:2412.04524 [gr-qc]].
26. Maurya, D.C.; Yesmakhanova, K.; Myrzakulov, R. & Nugmanova, G., Myrzakulov $F(T, Q)$ gravity: Cosmological implications and constraints. *Physica Scripta* **2024**, *99*, 10. [arXiv:2404.09698 [gr-qc]].
27. Maurya, D.C.; Yesmakhanova, K.; Myrzakulov, R. & Nugmanova, G., FLRW Cosmology in Metric-Affine $F(R, Q)$ Gravity. *Chin. Phys. C* **2024**, *48*, 125101, [arXiv:2403.11604 [gr-qc]].
28. Maurya D.C. & Myrzakulov, R., Transit cosmological models in Myrzakulov F(R,T) gravity theory. *The European Physical Journal C* **2024**, *84*, 534, [arXiv:2401.00686 [gr-qc]].
29. Mandal, S.; Myrzakulov, N.; Sahoo, P.K. & Myrzakulov, R., Cosmological bouncing scenarios in symmetric teleparallel gravity. *The European Physical Journal Plus* **2021**, *136*, 760, [arXiv:2107.02936 [gr-qc]].
30. Golovnev, A. & Guzman, M.-J., Approaches to spherically symmetric solutions in $f(T)$-gravity. *Universe* **2021**, *7*, 121, [arXiv:2103.16970 [gr-qc]].
31. Golovnev, A., Issues of Lorentz-invariance in $f(T)$-gravity and calculations for spherically symmetric solutions. *Classical and Quantum Gravity* **2021**, *38*, 197001, [arXiv:2105.08586 [gr-qc]].
32. DeBenedictis, A.; Ilijić, S. & Sossich, M., On spherically symmetric vacuum solutions and horizons in covariant $f(T)$ gravity theory. *Physical Review D* **2022**, *105*, 084020, [arXiv:2202.08958 [gr-qc]].
33. Bahamonde, S. & Camci, U., Exact Spherically Symmetric Solutions in Modified Teleparallel gravity. *Symmetry* **2019**, *11*, 1462, [arXiv:1911.03965 [gr-qc]].
34. Awad, A.; Golovnev, A.; Guzman, M.-J. & El Hanafy, W., Revisiting diagonal tetrads: New Black Hole solutions in $f(T)$-gravity. *The European Physical Journal C* **2022**, *82*, 972, [arXiv:2207.00059 [gr-qc]].
35. Bahamonde, S.; Golovnev, A.; Guzmán, M.-J.; Said, J.L. & Pfeifer, C., Black Holes in $f(T, B)$ Gravity: Exact and Perturbed Solutions. *Journal of Cosmological and Atroparticle Physics* **2022**, *1*, 037, [arXiv:2110.04087 [gr-qc]].
36. Bahamonde, S.; Faraji, S.; Hackmann, E. & Pfeifer, C., Thick accretion disk configurations in the Born-Infeld teleparallel gravity. *Physical Review D* **2022**, *106*, 084046, [arXiv:2209.00020 [gr-qc]].
37. Nashed, G.G.L. Quadratic and cubic spherically symmetric black holes in the modified teleparallel equivalent of general relativity: Energy and thermodynamics. *Classical and Quantum Gravity* **2021**, *38*, 125004, [arXiv:2105.05688 [gr-qc]].
38. Pfeifer, C. & Schuster, S., Static spherically symmetric black holes in weak $f(T)$-gravity. *Universe* **2021**, *7*, 153, [arXiv:2104.00116 [gr-qc]].
39. El Hanafy, W. & Nashed, G.G.L., Exact Teleparallel Gravity of Binary Black Holes. *Astrophys. Space Sci.* **2016**, *361*, 68, [arXiv:1507.07377 [gr-qc]].
40. Aftergood, J. & DeBenedictis, A., Matter Conditions for Regular Black Holes in $f(T)$ Gravity. *Physical Review D* **2014**, *90*, 124006, [arXiv:1409.4084 [gr-qc]].
41. Bahamonde, S.; Doneva, D.D.; Ducobu, L.; Pfeifer, C. & Yazadjiev, S.S., Spontaneous Scalarization of Black Holes in Gauss-Bonnet Teleparallel Gravity. *Physical Review D* **2023**, *107*, 104013, [arXiv:2212.07653 [gr-qc]].
42. Bahamonde, S.; Ducobu, L. & Pfeifer, C., Scalarized Black Holes in Teleparallel Gravity. *Journal of Cosmological and Atroparticle Physics* **2022**, *2022*, 018, [arXiv:2201.11445 [gr-qc]].
43. Iorio, L.; Radicella, N. & Ruggiero, M.L., Constraining $f(T)$ gravity in the Solar System. *Journal of Cosmological and Atroparticle Physics* **2015**, *2015*, 021, [arXiv:1505.06996 [gr-qc]].
44. Pradhan, S.; Bhar, P.; Mandal, S.; Sahoo, P.K. & Bamba, K., The Stability of Anisotropic Compact Stars Influenced by Dark Matter under Teleparallel Gravity: An Extended Gravitational Deformation Approach. *The European Physical Journal C* **2025**, *85*, 127, [arXiv:2408.03967 [gr-qc]].
45. Mohanty, D.; Ghosh, S. & Sahoo, P.K., Charged gravastar model in noncommutative geometry under $f(T)$ gravity. *Physics of the Dark Universe* **2025**, *46*, 101692, [arXiv:2410.05679 [gr-qc]].
46. Calza, M. & Sebastiani, L., A class of static spherically symmetric solutions in $f(T)$-gravity. *The European Physical Journal C* **2024**, *84*, 476, [arXiv:2309.04536 [gr-qc]].
47. Coley, A.A., Landry, A., van den Hoogen, R.J. & McNutt, D.D., Spherically symmetric teleparallel geometries, *The European Physical Journal C* **2024**, *84*, 334, [arXiv:2402.07238 [gr-qc]].
48. Landry, A., Static spherically symmetric perfect fluid solutions in teleparallel $F(T)$ gravity, *Axioms* **2024**, *13* (5), 333, [arXiv:2405.09257 [gr-qc]].
49. Landry, A., Kantowski-Sachs spherically symmetric solutions in teleparallel $F(T)$ gravity, *Symmetry* **2024** *16* (8), 953, [arXiv:2406.18659 [gr-qc]].
50. van den Hoogen, R.J. & Forance, H., Teleparallel Geometry with Spherical Symmetry: The diagonal and proper frames, *Journal of Cosmology and Astrophysics* **2024**, 11, 033, [arXiv:2408.13342 [gr-qc]].

51. Landry, A., Scalar field Kantowski-Sachs spacetime solutions in teleparallel $F(T)$ gravity, *Universe* **2025**, 11(1), 26, [arXiv:2501.11160 [gr-qc]].
52. Landry, A., Scalar Field Static Spherically Symmetric Solutions in Teleparallel $F(T)$ Gravity, *Mathematics* 2025, 13(6), 1003, [arXiv:2503.14465 [gr-qc]].
53. Coley, A.A., Landry, A., van den Hoogen, R.J. & McNutt, D.D., Generalized Teleparallel de Sitter geometries, *The European Physical Journal C* **2023**, *83*, 977, [arXiv:2307.12930 [gr-qc]].
54. Golovnev, A. & Guzman, M.-J., Bianchi identities in $f(T)$-gravity: Paving the way to confrontation with astrophysics, *Physics Letter B* **2020**, *810*, 135806, [arXiv:2006.08507 [gr-qc]].
55. Landry, A., Scalar field source Teleparallel Robertson-Walker $F(T)$-gravity solutions, *Mathematics* 2025, 13(3), 374, [arXiv:2501.13895 [gr-qc]].
56. Coley, A.A., Landry, A. & Gholami, F., Teleparallel Robertson-Walker Geometries and Applications, *Universe* **2023**, *9*, 454, [arXiv:2310.14378 [gr-qc]].
57. Coley, A.A., van den Hoogen, R.J. & McNutt, D.D., Symmetric Teleparallel Geometries, *Classical and Quantum Gravity* **2022**, *39*, 22LT01, [arXiv:2205.10719 [gr-qc]].
58. Aldrovandi, R., Cuzinatto, R.R. & Medeiros, L.G., Analytic solutions for the Λ-FRW Model, *Foundations of Physics* **2006**, *36*, 1736-1752, [arXiv:gr-qc/0508073 [gr-qc]].
59. Casalino, A., Sanna, B., Sebastiani, L. & Zerbini, S., Bounce Models within Teleparallel modified gravity, *Physical Review D* **2021**, *103*, 023514, [arXiv:2010.07609 [gr-qc]].
60. Capozziello, S., Luongo, O., Pincak, R. & Ravanpak, A., Cosmic acceleration in non-flat $f(T)$ cosmology, *General Relativity and Gravitation* **2018**, *50*, 53, [arXiv:1804.03649 [gr-qc]].
61. Bahamonde, S., Dialektopoulos, K.F., Hohmann, M., Said, J.L., Pfeifer, C. & Saridakis, E.N., Perturbations in Non-Flat Cosmology for $f(T)$ gravity, *European Physical Journal C* **2023**, *83*, 193, [arXiv:2203.00619 [gr-qc]].
62. Gholami, F. & Landry, A., Cosmological solutions in teleparallel $F(T,B)$ gravity, *Symmetry* **2025**, *17* (1), 060, [arXiv:2411.18455 [gr-qc]].
63. Hohmann, M., Järv, L., Krššák, M., & Pfeifer, C., Modified teleparallel theories of gravity in symmetric spacetimes, *Physical Review D* **2019** *100*, 084002, [arXiv:1901.05472 [gr-qc]].
64. Cai, Y.-F., Capozziello, S., De Laurentis, M. & Saridakis, E.N., $f(T)$ teleparallel gravity and cosmology, *Report of Progress in Physics* **2016**, *79*, 106901, [arXiv:1511.07586 [gr-qc]].
65. Dixit, A. &; Pradhan, A., Bulk Viscous Flat FLRW Model with Observational Constraints in $f(T,B)$ Gravity. *Universe* **2022**, *8*, 650.
66. Chokyi, K.K. & Chattopadhyay, S., Cosmological Models within $f(T,B)$ Gravity in a Holographic Framework. *Particles* **2024**, *7*, 856.
67. Hawking, S.W. & Ellis, G.F.R., *The Large Scale Structure of Space-Time*, Cambridge University Press, 2010.
68. Bohmer, C.G. & d'Alfonso del Sordo, A., Cosmological fluids with boundary term couplings, *General Relativity and Gravitation* **2024**, *56*, 75, [arXiv:2404.05301 [gr-qc]].
69. Bahamonde, S., Bohmer, C.G., Carloni, S., Copeland, E.J., Fang, W. & Tamanini, N., Dynamical systems applied to cosmology: dark energy and modified gravity, *Physical Reports* **2018**, *775-777*, 1-122, [arXiv:1712.03107 [gr-qc]].
70. Zlatev, I., Wang, L. & Steinhardt, P., Quintessence, Cosmic Coincidence, and the Cosmological Constant, *Physical Review Letters*, **1999**, *82* (5), 896, [arXiv:astro-ph/9807002 [astro-ph]].
71. Steinhardt, P., Wang, L. & Zlatev, I., Cosmological tracking solutions, *Physical Review D*, **1999**, *59* (12), 123504, [arXiv:astro-ph/9812313 [astro-ph]].
72. Caldwell, R.R., Dave, R. & Steinhardt, P., Cosmological Imprint of an Energy Component with General Equation of State, *Physical Review Letters*, **1998**, *80*, 1582, [arXiv:astro-ph/9708069 [astro-ph]].
73. Carroll, S.M., Quintessence and the Rest of the World, *Physical Review Letters*, **1998**, *81*, 3067, [arXiv:astro-ph/9806099 [astro-ph]].
74. Doran, M., Lilley, M., Schwindt, J. & Wetterich, C., Quintessence and the Separation of CMB Peaks, *Astrophysical Journal*, **2001**, *559*, 501, [arXiv:astro-ph/0012139v2 [astro-ph]]
75. Zeng, X.-X., Chen, D.-Y., Li, L.-F., Holographic thermalization and gravitational collapse in the spacetime dominated by quintessence dark energy, *Physical Review D*, **2015**, *91*, 046005, [arXiv:1408.6632 [hep-th]].
76. Chakraborty, S., Mishra, S. & Chakraborty, S., Dynamical system analysis of quintessence dark energy model, *International Journal of Geometric Methods in Modern Physics*, **2025**, *22*, 2450250, [arXiv:2406.10692 [gr-qc]]
77. Shlivko, D., & Steinhardt, P.J., Assessing observational constraints on dark energy, *Physics Letters B*, **2024**, *855*, 138826, [arXiv:2405.03933 [astro-ph]]

78. Ratra, B. & Peebles, P.J.E., Cosmological consequences of a rolling homogeneous scalar field. *Physical Review D* **1988**, *37*, 3406.
79. Wolf, W.J. & Ferreira, P.G., Underdetermination of dark energy. *Physical Review D* **2023**, *108*, 103519, [arXiv:2310.07482 [gr-qc]].
80. Wolf, W.J.; García-García, C.; Bartlett, D.J. & Ferreira, P.G., Scant evidence for thawing quintessence. *Physical Review D* **2024**, *110*, 083528, [arXiv:2408.17318 [gr-qc]].
81. Wolf, W.J.; Ferreira, P.G. & García-García, C., Matching current observational constraints with nonminimally coupled dark energy, *Physical Review D* **2025**, 111, L041303, [arXiv:2409.17019 [gr-qc]].
82. Wetterich, C., Cosmology and the Fate of Dilatation Symmetry, *Nuclear Physics B* **1988**, *302*, 668, [arXiv:1711.03844 [hep-th]].
83. Chiba, T., Okabe, T., Yamaguchi, M., Kinetically Driven Quintessence, *Physical Review D* **2000**, *62*, 023511, [arXiv:astro-ph/9912463 [astro-ph]].
84. Carroll, S.M., Hoffman, M., Trodden, M., Can the dark energy equation-of-state parameter w be less than -1?, *Physical Review D* **2003**, *68*, 023509, [arXiv:astro-ph/0301273 [astro-ph]].
85. Caldwell, R.R., A phantom menace? Cosmological consequences of a dark energy component with super-negative equation of state, *Physics Letters B* **2002**, *545*, 23, [arXiv:astro-ph/9908168 [astro-ph]].
86. Farnes, J.S., A Unifying Theory of Dark Energy and Dark Matter: Negative Masses and Matter Creation within a Modified ΛCDM Framework, *Astronomy & Astrophysics* **2018**, *620*, A92, [arXiv:1712.07962 [physics.gen-ph]].
87. Baum, L. & Frampton, P.H., Turnaround in Cyclic Cosmology, *Physical Review Letters* **2007**, *98*, 071301, [arXiv:hep-th/0610213 [hep-th]].
88. Hu, W., Crossing the Phantom Divide: Dark Energy Internal Degrees of Freedom, *Physical Review D* **2005**, *71*, 047301, [arXiv:astro-ph/0410680v2 [astro-ph]].
89. Karimzadeh, S. & Shojaee, R., Phantom-Like Behavior in Modified Teleparallel Gravity, *Advances in High Energy Physics*, **2019**, 4026856, [arXiv:1902.04406 [physics.gen-ph]].
90. Pati, L., Kadam, S.A., Tripathy, S.K. & Mishra, B., Rip cosmological models in extended symmetric teleparallel gravity, *Physics of the Dark Universe*, **2022**, *35*, 100925, [arXiv:2112.00271 [gr-qc]].
91. Kucukakca, Y., Akbarieh, A.R. & Ashrafi, S., Exact solutions in teleparallel dark energy model, *Chinese Journal of Physics*, **2023**, *82*, 47.
92. Cai, Y.-F., Saridakis, E.N., Setare, M.R. & Xia, J.-Q., Quintom Cosmology: Theoretical implications and observations, *Physics Report*, **2010**, *493*, 1, [arXiv:0909.2776 [hep-th]].
93. Guo, Z.-K., Piao, Y.-S., Zhang, X. & Zhang, Y.-Z., Cosmological evolution of a quintom model of dark energy, *Physics Letters B*, **2005**, *608*, 177, [arXiv:astro-ph/0410654 [astro-ph]].
94. Feng, B., Li, M., Piao, Y.-S., Zhang, X., Oscillating quintom and the recurrent universe, *Physics Letters B*, **2006**, *634*, 101, [arXiv:astro-ph/0407432 [astro-ph]].
95. Mishra, S. & Chakraborty, S., Dynamical system analysis of quintom dark energy model, *European Physical Journal C*, **2018**, *78*, 917, [arXiv:1811.08279 [gr-qc]].
96. Tot, J., Coley, A.A., Yildrim, B. & Leon, G., The dynamics of scalar-field Quintom cosmological models, *Physics of the Dark Universe*, **2023**, *39*, 101155, [arXiv:2204.06538 [gr-qc]].
97. Bahamonde, S., Marciu, M. & Rudra, P., Generalised teleparallel quintom dark energy non-minimally coupled with the scalar torsion and a boundary term, *Journal of Cosmology and Astroparticle Physics*, **2018**, *04*, 056, [arXiv:1802.09155 [gr-qc]].
98. Gorini, V., Kamenshchik, A.Y., Moschella, U. & Pasquier, V., Chaplygin gas as a model for dark energy, *The Tenth Marcel Grossmann Meeting* **2006**, 840, [arXiv:gr-qc/0403062 [gr-qc]].
99. Bento, M.C., Bertolami, O. & Sen, A.A., Generalized Chaplygin Gas Model: Dark Energy-Dark Matter Unification and CMBR Constraints, *General Relativity and Gravitation* **2003**, 35, 2063, [arXiv:gr-qc/0305086 [gr-qc]].
100. Bilić, N., Tuppe, G.B. & Viollier, R.D., Unification of dark matter and dark energy: the inhomogeneous Chaplygin gas, *Physics Letters B* **2002**, 535, 17, [arXiv:astro-ph/0111325 [astro-ph]].
101. Makler, M., de Oliveira, S.Q. & Waga, I., Observational constraints on Chaplygin quartessence: Background results, *Physical Review D* **2003**, 68, 123521, [arXiv:astro-ph/0306507 [astro-ph]].
102. Zhu, Z.-H., Generalized Chaplygin gas as a unified scenario of dark matter/energy: Observational constraints, *Astronomy and Astrophysics* **2004**, 423, 421, [arXiv:astro-ph/0411039 [astro-ph]].
103. Karami, K., Ghaffari, S. & Fehri, J., Interacting polytropic gas model of phantom dark energy in non-flat universe, *The European Physical Journal C* **2009**, 64, 85, [arXiv:0911.4915 [gr-qc]]..

 doi:10.20944/preprints202508.2052.v1

104. Karami, K., Safari, Z. & Asadzadeh, S., Cosmological constraints on polytropic gas model, *International Journal Theoretical Physics* **2014**, 53, 1248, [arXiv:1209.6374 [astro-ph]].
105. Karami, K. & Abdolmaleki, A., Reconstructing interacting new agegraphic polytropic gas model in non-flat FRW universe, *Astrophysical and Space Science* **2010**, 330, 133, [arXiv:1010.4294 [hep-th]].
106. Karami, K. & Khaledian, M.S., Polytropic and Chaplygin $f(R)$-gravity models, *International Journal of Modern Physics D* **2012**, 21, 1250083, [arXiv:1010.2639 [physics]].
107. Banerjee, S. & Paul, A., Effect of Accretion on the evolution of Primordial Black Holes in the context of Modified Gravity Theories, preprint (2024), arXiv:2406.04605.
108. Aboueisha, M.S., Nouh, M.I., Abdel-Salam, E.A-B., Kamel, T.M., Beheary, M.M. & Gadallah, K.A.K., Analysis of the Fractional Relativistic Polytropic Gas Sphere, *Scientific Reports* **2023**, 13, 14304, [arXiv:2405.19467 [gr-qc]].
109. Cardenas, V.H. & Cruz, M. , Emulating dark energy models with known equation of state via the created cold dark matter scenario, *Physics of the Dark Universe* **2024**, 44, 101452, [arXiv:2401.16905 [gr-qc]].
110. Jia, Y., He, T.-Y., Wang, W.-Q. , Han, Z.-W. & Yang, R.-J., Accretion of matter by a Charged dilaton black hole, The European Physical Journal C **2024**, 84, 501, [arXiv:2401.15654 [gr-qc]].
111. Arun, K., Gudennavar, S.B. & Sivaram, C., Dark matter, dark energy, and alternate models: A review, *Advances in Space Research* **2017**, **60**(1), 166, [arXiv:1704.06155 [physics]].
112. Iosifidis, D., Cosmological Hyperfluids, Torsion and Non-metricity, *European Physical Journal C* **2020**, *80*, 1042, [arXiv:2003.07384 [gr-qc]].
113. Heisenberg, L., Hohmann, M. & Kuhn, S., Homogeneous and isotropic cosmology in general teleparallel gravity, *European Physical Journal C* **2023**, *83*, 315, [arXiv:2212.14324 [gr-qc]].
114. Heisenberg, L. & Hohmann, M., Gauge-invariant cosmological perturbations in general teleparallel gravity, *European Physical Journal C* **2024**, *84*, 462, [arXiv:2311.05597 [gr-qc]].
115. Kontou, E.-A. & Sanders, K., Energy conditions in general relativity and quantum field theory, *Classical and Quantum Gravity* **2020** 37, 193001, [arXiv:2003.01815 [gr-qc]].