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A Remark on the Actions of $C(S_n^+)$

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Abstract

Talking about permutations, one might think of the usual permutation groups, permuting the set of n points, and that is a correct way of thinking. But, one may see a point as a 1×1 matrix with entries in any number field, and by extending this way of viewing a point, one may think of $n \times n$ matrices instead of points, and think about permuting these matrices. These matrices might have commuting or noncommuting entries. In the case where the entries are noncommuting and are satisfying in the relations of the matrix quantum groups, developed by Woronowicz, one gets what has been referred to as the quantum permutation (symmetry) group S_n^+, introduced and studied by Wang. But since we still do not have any knowledge about the elements of S_n^+, hence instead of that, studying its function space would be quite logical! Here in this paper, we will try to study the actions of $C(S_n^+)$ on some matrix spaces, especially those related to finite groups of Lie type. In accordance, we also study the possible invariant spaces of these actions.

Keywords: quantum permutation group; finite groups of Lie type; representation theory

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1. Notations

Throughout this paper, $\mathbb{K}$ will stand for the ground field, and it will be assumed to be arbitrary unless otherwise stated.

By p and q, we mean any matrices in $M_n(\mathbb{K}) := M_n$ for $n \geq 1$ such that $pq \neq qp$, and we will not specify them!

By S_n^+ we mean the quantum permutation group in the sense of Wang, generated by magic $n \times n$ unitary matrices, and there are infinite many of them!

2. Introduction

In the late 17th and early 18th centuries, many philosophers like Descartes and Kant, signaled a transformation in the way that mathematics was being understood and created from the Platonic idea of uncovering a universe of static truths, and towards a more dynamical conception based on the notions of operations, functions, and relations.

In the process, what mathematicians do in constructing a mathematical theory, is mainly to try building things up from some very simple components. However, doing this in an axiomatic way of thinking process will eventually reverse the directions, meaning that starting from a mathematical structure, and then employing either an implicit detailed way of construction or through some abstraction!

Anyway, our journey will start by observing the nature of C^*-algebras, as they will play the role of our main entity in the way, and then we will move to the direction of compact quantum groups and the subdirection of the quantum permutation groups S_n^+!

There is a very useful fact that will help us in this way, that is, "to any finite group, one can associate a commuting graph", by using this fact, we will be provided with a set of regulated graphs,

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together with a set of regulated adjacency matrices, making us able in better analyzing and studying their properties.

After that, we will study the quantum automorphism groups of these graphs, and eventually we will present our main result concerning the action of the affirmative quantum automorphism groups on some set of regulated matrices!

Starting from the historical part of our study, we note that C^*-algebras are mainly linked with the development of the structures in quantum mechanics through the works of von Neumann back in the late 1920s, and since the 1960s, they have been employed as a natural mathematical framework for the quantum field theory.

Talking about the theory of C^*-algebras, one might arrive at the theory from two interrelated approaches, either as certain algebras of bounded operators on some Hilbert spaces or as special cases of Banach algebras. Note that the first approach seeks its origin from quantum mechanics, mostly motivated by unitary representations of the locally compact groups, while the second approach has to be regarded more or less as an abstract way of looking at this matter in searching for C^*-algebras, where these objects are defined axiomatically as a Banach algebra B equipped with an involution $B \to B$ taking x to x^*, such that $||x^*x|| = ||x||^2$ holds for any $x \in B$.

As we pointed out in the introduction, in order to study S_n^+, one might need to go a little further and consider studying their function spaces, which are some sort of generalized spaces, having a generalized structure. To do that, we use the Gelfand duality that could be categorified as the following equivalence.

$$\text{The compact Hausdorff spaces} \cong (\text{unital commutative } C^*\text{-algebras})^{\text{op}}, \tag{1}$$

for "op" standing for the opposite category, consisting of the same objects as in the underlying category, with reversed arrows.

We have the following important well-known *GNS* and *GN* theorems.

Theorem 1 (Gelfand-Naimark-Segal)**.** *Every abstract C^*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to a concrete C^*-algebra of operators on a Hilbert space H.*

However, by implementing the (second) Gelfand-Naimark theorem 1, simply saying that a noncommutative space acts as an operator algebra on some Hilbert spaces, one speculates that the functors implementing the equivalence 1 might be of the form contravariant function functors $C(X) \equiv C(X, \mathbb{C}) = \{f : X \to \mathbb{C}\}$, for X a compact Hausdorff space.

Remark 1. *Note that in the case where X is a topological (locally) compact, Hausdorff space, then $C(X)$ will be a commutative (non-)unital C^*-algebra, which is the continuation of the Gelfand-Naimark's main Theorem.*

Theorem 2 (Gelfand-Naimark)**.** *If a C^*-algebra is commutative then it is an algebra of continuous functions on some (locally compact, Hausdorff) topological space.*

Note that the above theorems are the main constitution of the field of Non-Commutative Geometry, and are crucial for the study conducted in this paper.

3. Thinking Noncommutative

Graphs are almost everywhere if you try to look at things as just edges and intersections, and this was the inspration behind studying graphs as independent objects (one might also remember the Seven Bridges of Königsberg and the solution provided by Leonhard Euler). Graph theory has many applications, from which in the theory of Non-Commutative Geometry, and especially the concept of quantum groups, one can certainly point out the theory of quantum graphs (these objects still do not have a general, and fully accepted definition between mathematicians), and studying the quantum

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symmetry of certain class of (directed) graphs by employing their quantum automorphism groups as a compact quantum group.

But before that, let us recall the following theorem, based on the fact that to any finite group G of order $|I| = n$, over the index set I, one may associate a commuting graph, by assuming the set of vertices $\{g_\ell\}_{\ell \in I}$ to be the set of group elements, and the set of edges to be connections between vertices g_i and g_j, such that they commute, that is $g_i g_j = g_j g_i$, for $i, j \in I$.

Theorem 3. *[9] Every finite graph is isomorphic to an induced subgroup of the commuting graph of a finite group. This group can be taken to be nilpotent of class 2, and exponent 4.*

Remark 2. *The commuting graph was introduced by Brauer and Fourler in their seminal paper [8], showing that only finitely many groups of even order can have prescribed centralizers.*

3.1. An Invitation to the Representation Theory of $C(S_n^+)$

There is a wondering between mathematicians, which has been started by a question raised by Alain Connes, asking if "there are quantum permutation groups, and what would they look like?". This question is as deep as many other well-known questions in mathematics (mostly related with physics), on which if in any case any of them are truly understood, then they might provide us with essential solutions to some very important other related questions!

It is known that noncommutative geometry is all about finding the generalized deformed (quantum/noncommutative) version of the category of objects or mathematical and physical concepts, such as "quantum" groups, "quantum" (free) probability, "quantum" information theory, "quantum" computing, etc.

However, the main focus of this paper is based on (compact) quantum groups (still with no rigorous and well-defined definition), and as our knowledge permits, one may arrive at the (one or many parameter) deformed (quantum) version of a (compact) group from two interrelated directions developed in two distinct hubs of sciences, which are called "schools", Saint Petersburg, led by Drinfeld-Jimbo, and Warsaw, led by Woronowicz!

In 1986, in order to present solutions to the quantum Yang-Baxter equation, Drinfeld and Jimbo's approach was working with one parameter deformation of the universal enveloping algebra of semisimple Lie algebras, and as a dual construction, in 1987 [16], independently, Woronowicz came with a philosophy, saying that one could look at the algebra of continuous functions of a compact group $\mathcal{C}$, instead of looking at it directly. Then by continuing this approach along the lines of non-commutative geometry, one may use some quantum group $\mathcal{C}^+$ instead of $\mathcal{C}$, and think of the C^*-algebra $C(\mathcal{C}^+)$ as the space of continuous functions on $\mathcal{C}^+$.

Anyway, in the late nineties, Shuzhou Wang came up with a sophisticated answer to the question raised by Connes, by characterizing the quantum symmetries of finite spaces by using the structures introduced by Woronovicz, saying that "the quantum permutation group S_n^+ could be defined as the largest compact quantum group acting on the set $\mathcal{N} = \{1, \cdots, N\}$", as an analogue of the permutation group S_n. His approach was to look at $\mathcal{N}$ as the compact set $X_N := \{x_1, \cdots, x_N\}$ consisting of a finite set of points (pointwise isomorphic), and to study its function space $C(X_N) \cong C^*(p_1, \cdots, p_N \text{ projections} \mid \sum_{i=1}^N p_i = 1)$. Finally, this has led him to define something like $C(S_n^+) := C^*\Big(u_{ij} \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1\Big)$, for $i, j \in \{1, \cdots, n\}$, and calling $S_n^+ = (C(S_n^+), u)$ the quantum symmetric (permutation) group as the quantum automorphism group of X_N. Later on he proved that $C(S_n^+)$ satisfies the relations of being a compact (matrix) quantum group in the sense of Woronowicz. This was considered a start point of a new era in Quantum World, powered by the constitutional Theorems 1 and 2, which are considered the foundation of the Non-Commutative (algebraic) Geometry!

Remark 3. *Note that one could see the above space of N points as a graph on N vertices, with no edges, and then look for its quantum automorphism group.*

As an interested reader, you might already have realized that the main ingredients in defining $C(S_n^+)$, meaning that the u_{ij}s, have to be very important in our study, and hence let us officially define them in the following manner.

Definition 1. *Matrix $u = (u_{ij})_{i,j}$ with entries u_{ij}s taken from a non-trivial unital C^*-algebra, satisfying the relations $u_{ij} = u_{ij}^* = u_{ij}^2$, and $\sum_{k=1}^n u_{kj} = \sum_{k=1}^n u_{ik} = 1$, will be called a magic unitary.*

Now, let us invite Theorem 3 back to the scene and employ $\mathbb{Z}_n$ to play the role of the finite group G, in our work. The commuting graph associated with $\mathbb{Z}_n$ is known to be the complete graph K_n, and we will use this fact in the rest of this paper.

3.1.1. The Quantum Automorphism Group of a Graph

Let $\Gamma = (\Gamma^0, \Gamma^1)$ be a (directed) locally finite graph with the set of vertices Γ^0, and the set of edges Γ^1. Then the graph automorphism is a bijection from Γ^0 to Γ^0 such that it preserves the adjacency and the non-adjacency of the vertices in Γ, and it is known that $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ is a subgroup of S_n, the permutation group, and each element of $\mathrm{Aut}(\Gamma)$ commutes with the adjacency matrix of Γ.

On the other hand, it is known that the function space $C(S_n)$ could be defined as the following universal commutative C^*-algebra

$$C(S_n) := C^*_{com}\left(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{\ell} u_{i\ell} = \sum_{\ell} u_{\ell i} = 1 \right), \tag{2}$$

and in a similar way, $C(\mathrm{Aut}(\Gamma))$ could be defined as follows for $\mathcal{A}_\Gamma$ playing the role of the adjacency matrix of Γ

$$C(\mathrm{Aut}\,\Gamma) := C^*_{com}\left(u_{ij}, 1 \le i,j \le n \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_{\ell} u_{i\ell} = \sum_{\ell} u_{\ell i} = 1, u\mathcal{A}_\Gamma = \mathcal{A}_\Gamma u \right), \tag{3}$$

and hence, the fact of $\mathrm{Aut}(\Gamma)$ being a subgroup of S_n, could be presented by the following surjection of C^*-algebras

$$C(S_n) \to C(\mathrm{Aut}(\Gamma)),$$

such that $C(\mathrm{Aut}(\Gamma)) \cong C(S_n)/\langle \mathcal{A}_\Gamma u = u\mathcal{A}_\Gamma \rangle$ holds. Then by dropping the commutativity in equations 2 and 3, we will get $C(S_n^+)$ and $C(\mathrm{Aut}(\Gamma)^+)$, respectively, which are the noncommutative (free) versions of $C(S_n)$ and $C(\mathrm{Aut}(\Gamma))$.

Remark 4. *Note that, as has been already pointed out, $(C(S_n^+), u) := S_n^+$ will be called the quantum permutation (symmetric) group, and similarly $(C(\mathrm{Aut}(\Gamma)^+), u) := \mathrm{Aut}(\Gamma)^+$ will be called the quantum automorphism group of Γ.*

Notation: In the rest of this paper, we will refer to the quantum automorphism group of a graph Γ, as $\mathrm{QAut}(\Gamma)$ instead of $\mathrm{Aut}(\Gamma)^+$.

There are two definitions of the quantum automorphism group of a graph. The first one came out in a work by Bichon [7] in 2003, and the other one was given by Banica [3] in 2005. In this paper, we will follow the definition provided by Banica, which could be stated as follows.

Definition 2. *For Γ as before, being a finite graph on n vertices, the quantum automorphism group* $\mathrm{QAut}(\Gamma)$ *is the compact matrix quantum group* $(\mathrm{QAut}(\Gamma), u)$, *where* $\mathrm{QAut}(\Gamma)$ *is the universal* C^**-algebra generated by the generators* u_{ij}, $1 \leq i, j \leq n$ *and relations*

$$u_{ij} = u_{ij}^* = u_{ij}^2 \qquad 1 \leq i, j, k \leq n \tag{4}$$

$$\sum_{\ell=1}^{n} u_{i\ell} = 1 = \sum_{\ell=1}^{n} u_{\ell i} \qquad 1 \leq i \leq n \tag{5}$$

$$u\mathcal{A}_\Gamma = \mathcal{A}_\Gamma u. \tag{6}$$

Remark 5. *i)* *In the original work of Banica, the quantum automorphism group* $\mathrm{QAut}(\Gamma)$ *has been considered by* $G_{aut}^{+}(\Gamma)$.

ii) *Note that the relation 6 is equivalent to saying that* $\sum_m u_{im}a_{mj} = \sum_m a_{im}u_{mj}$, *for* $\mathcal{A}_\Gamma = (a_{ij})_{i,j}$, *and this is where the two definitions of the quantum automorphism group of* Γ *by Banica and Bichon show interrelations, but there are examples showing that the quantum automorphism group coming from the definitions by Banica, and Bichon might provide different answers, for example, in the case of the complete graph* K_n*!*

In [13, Proposition 2.1.3.], and Example 2.1.8, from the same dissertation, it has been proved that the relation 6 is equivalent to the relations

$$u_{ij}u_{mn} = u_{mn}u_{ij} = 0 \qquad (i,m) \notin \Gamma^1, (j,n) \in \Gamma^1 \tag{7}$$

$$u_{ij}u_{mn} = u_{mn}u_{ij} = 0 \qquad (i,m) \in \Gamma^1, (j,n) \notin \Gamma^1, \tag{8}$$

and moving back to the complete graph $K_n = (K_n^0, K_n^1)$ on n vertices $K_n^0 = \{1, \ldots, n\}$ and the set of edges $K_n^1 = (K_n^0 \times K_n^0)/\{(i,i) \mid i \in K_n^0\}$, and letting $(i,k) \notin K_n^1$, we may get $i = k$, and therefore $u_{ij}u_{ik} = u_{ji}u_{ki} = 0$ will hold by using (7) and (8). Then the defining relations of $C(S_n^+)$

$$u_{ij}u_{ik} = \delta_{jk}u_{ij}$$

$$u_{ji}u_{ki} = \delta_{jk}u_{ji},$$

will result in $\mathrm{QAut}(K_n) = S_n^+$.

Now, let $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \leq k,\ell \leq i}$, for $(a_{k\ell})_{1 \leq k,\ell \leq i} \in \mathfrak{sl}_n$ such that every entry is equal to 1 else than the main diagonal entries, which are zero. Simply, we can say that $(a_{k\ell})_{1 \leq k,\ell \leq i}$ is the adjacency matrix of the complete graph K_i, and hence we can look at G as a subgroup of $\bigoplus_{i=4}^{n} \mathfrak{sl}_i$. The plan is to study the actions of $C(S_n^+)$ on such kind of groups.

3.2. A result on the Action of the Quantum Permutation Groups

The main motivation for our study was to look for an alternative meaning for the quantum (noncommutative) permutations in the n^2 dimensional complex space $\mathbb{C}^{n^2}$, by looking at its points as complex $n \times n$ matrices. We firmly believe that our study is important in its own place, because it might be used in studying the (von Neumann) entropy!

So we try to study the actions of $C(S_n^+)$ on these matrix spaces, specifically the invertible ones, because the nondegeneracy is quite important to us. Studying disorder, randomness, or uncertainty within a system is very important in thermodynamics and recently in (quantum) information theory. These could be studied by permuting the points in our space. But just making changes in order is not enough. There has to be some sort of interaction between the points of the space, and one may study these interactions by studying the action of the permutation groups (in our case, the quantum permutation groups) on these spaces. These actions will provide us with the desired turbulence, giving us the disorder, from which the invariant subspaces will become clear!

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To continue, consider the set of n disjoint vertices X. It is known that its symmetry group and the quantum symmetry group are S_n and S_n^+ respectively, and the C^*-algebra associated with X is $\mathbb{C}^n$ as an algebra spanned by projections $(p_i)_{i=1}^n$, where each p_i sends $(\lambda_1, \cdots, \lambda_n)$ to $(\cdots, \lambda_i, \cdots)$ for $\lambda_i \in \mathbb{C}$ for $1 \le i \le n$. Then for (A, Δ) a compact matrix quantum group ($CMQG$), consider the coaction (action for $CMQG$)

$$\cdot : \mathbb{C}^n \to \mathbb{C}^n \otimes A : p_i \mapsto \sum_{j=1}^n p_j \otimes u_{ij},$$

for the universal representation $u = (u_{ij})_{1\le i,j\le n}$ of A.

Now assume that each vertex of X could be represented by an $n \times n$ matrix $m_n \in M_n(\mathbb{C})$ and let $A = S_n^+$, and $\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}$, and consider the correspondent space by X_n. We know that one can see the set of complex $n \times n$ matrices $M_n(\mathbb{C}) \cong \mathbb{C}^{n^2}$ as the set of functions

$$\begin{aligned}
M_n(\mathbb{C}) = \{g : \mathbb{C}^n \to \mathbb{C}^n : (c_1, \cdots, c_n) \mapsto g(c_1, \cdots, c_n)(\lambda_1, \cdots, \lambda_n)\\
&:= g(c_1e_1 + \cdots + c_ne_n)(\lambda_1, \cdots, \lambda_n)\\
&= (g(c_1e_1) + \cdots + g(c_ne_n))(\lambda_1, \cdots, \lambda_n)\\
&= g(c_1e_1)(\lambda_1, \cdots, \lambda_n) + \cdots + g(c_ne_n)(\lambda_1, \cdots, \lambda_n)\\
&= c_1g(e_1)(\lambda_1, \cdots, \lambda_n) + \cdots + c_ng(e_n)(\lambda_1, \cdots, \lambda_n)\\
&= c_1\lambda_1e_{11} + \cdots + c_n\lambda_ne_{nn}\\
&= \sum_{i,j=1}^n c_i\lambda_ie_{ij}\},
\end{aligned}$$

for e_{ij} the $n \times n$ elementary matrices, $e_j : \mathbb{C}^n \to \mathbb{C}$ the projections, and $c_i, \lambda_i \in \mathbb{C}$, for $1 \le i, j \le n$.

So, by this view, one may see X_n as a noncommutative space in the sprit of noncommutative geometry and the GNS construction. The plan is to study the quantum symmetries of X_n by studying the actions of S_n^+ on it. Note that this study has already been done in [15], but our approach provides a new way in proving that the quantum automorphism group of K_n is identified with S_n^+, meaning that the quantum symmetric group of K_n is S_n^+, an indirect proof! (We will observe this later.)

Note that due to the fact that the elements of S_n^+ are still unknown, hence using the duality and moving to $C(S_n^+)$, and looking at it as a compact matrix quantum group in the sense of Woronowicz [15], one might try to study the representation theory of $C(S_n^+)$ instead of S_n^+. Concerning this matter, the interested reader is referred to [2,4,5].

To study the representation theory of $C(S_n^+)$, we have no choice other than using the space of $n \times n$ matrices as our ground playing field, and since this space fails to be a group, hence we will restrict ourselves to the classical matrix groups and will study the action of $C(S_n^+)$ on these groups. But before that consider the cyclic group $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{e, x \mid x^2 = e\}$ and its free product with itself, that is $\mathbb{Z}_2 * \mathbb{Z}_2 = \{x, y \mid x^2, y^2\}$ which could be identified with the infinite dihedral group $\mathrm{D}_\infty\langle s, t \mid s^2, stst\rangle$ through the identifications $\phi : \mathrm{D}_\infty \to \mathbb{Z}_2 * \mathbb{Z}_2$ taking s to y and t to xy, and $\psi : \mathbb{Z}_2 * \mathbb{Z}_2 \to \mathrm{D}_\infty$ taking x to ts and y to s. We know that the quantum group $\widehat{\mathbb{Z}_2 * \mathbb{Z}_2}$ (the quantum dual of $\mathbb{Z}_2 * \mathbb{Z}_2$) by [6, Theorem 1.1] is a subgroup of S_4^+, up to isomorphism, which is generalizable to $\underbrace{\widehat{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{n \text{ times}} \le S_n^+$, with corepresentation $\mathrm{R} = \begin{bmatrix} \mathrm{R}_1 & 0 & 0 & \cdots \\ 0 & \mathrm{R}_2 & 0 & \cdots \\ 0 & 0 & \ddots & \cdots \end{bmatrix}$, for $\mathrm{R}_1 = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$ and $\mathrm{R}_2 = \begin{bmatrix} q & 1-q \\ 1-q & q \end{bmatrix}$, such that $pq \ne qp$ as projections.

Now, let $H \leqslant GL_n(\mathbb{C})$ to be a subgroup of $GL_n(\mathbb{C})$. Note that H with respect to the usual matrix norm will have the structure of a Hilbert space. Let $A = \sum_{i,j=1}^n E_{ij}a_{ij} \in H$ be an arbitrary element and,

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for $u \in C(\underbrace{\widetilde{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{n \text{ times}})$, consider its inverse by $B = u^{-1} := \sum_{i,j=1}^{n} E_{ij} b_{ij}$, which is still an element of $GL_n(\mathbb{C})$. Then we can propose the following fundamental result.

Proposition 1. *The following conjugation will provide a non-trivial action of* $C(\underbrace{\widetilde{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{n \text{ times}})$ *on* $GL_n(\mathbb{C})$,

$$\begin{aligned} \varphi : & C(\underbrace{\widetilde{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{n \text{ times}}) \to B(H) \\ & u \mapsto \varphi(u)(A) := \sum_{i,j=1}^{n} E_{ij} \sum_{m,o=1}^{n} b_{mo} a_{ij} u_{k\ell} \end{aligned} \tag{9}$$

for any $u = (u_{k\ell})_{k,\ell} \in C(\underbrace{\widetilde{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{n \text{ times}})$, *and the space* $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \le k,\ell \le i} = \{(a_{k\ell})_{1 \le k,\ell \le i} \in \mathfrak{sl}_n \mid a_{ii} = 0 \;\&\; a_{ij} = 1, \forall i \neq j\}$, *plays the role of its invariant space.*

Proof. It is easy to see that for any $u \in C(\underbrace{\widetilde{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{i \text{ times}})$ for $i \in \{4, \ldots, n\}$, $u^{-1}Au$ will be invertible for any $A \in GL_i(\mathbb{C})$, and its inverse will be $u^{-1}A^{-1}u$, and hence we have $u^{-1}Au \in GL_i(\mathbb{C})$.

To see that $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \le k,\ell \le i}$ is fixed by φ_u, for any $u \in C(\underbrace{\widetilde{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}}_{i \text{ times}})$ is not too difficult, and we omit writing down the complexity induced by the computation!

But in order to present an illustration, let us try to see it in the case of $\widetilde{\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2}$. For $\widehat{u_4} \in \widetilde{\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2}$ and $A = (a_{ij})_{i,j} \in H$ we have

$$\begin{aligned} (\widehat{u_4})^{-1} A \widehat{u_4} &= \begin{bmatrix} \frac{p}{2p-1} & \frac{p-1}{2p-1} & 0 & 0 \\ \frac{p-1}{2p-1} & \frac{p}{2p-1} & 0 & 0 \\ 0 & 0 & \frac{q}{2q-1} & \frac{q-1}{2q-1} \\ 0 & 0 & \frac{q-1}{2q-1} & \frac{q}{2q-1} \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \begin{bmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{bmatrix} \\ &= \begin{bmatrix} \frac{pa_1 - pa_6 - a_6}{2p-1} & \frac{pa_2 + pa_5 - a_5}{2p-1} & ((\widehat{u_4})^{-1} A \widehat{u_4})_{13} & ((\widehat{u_4})^{-1} A \widehat{u_4})_{14} \\ \frac{pa_5 + pa_6 - a_6}{2p-1} & \frac{pa_1 + pa_2 - a_1}{2p-1} & ((\widehat{u_4})^{-1} A \widehat{u_4})_{23} & ((\widehat{u_4})^{-1} A \widehat{u_4})_{24} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{31} & ((\widehat{u_4})^{-1} A \widehat{u_4})_{32} & \frac{qa_{11} + qa_{16} - a_{16}}{2q-1} & \frac{qa_{12} + qa_{15} - a_{15}}{2q-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{41} & ((\widehat{u_4})^{-1} A \widehat{u_4})_{42} & \frac{qa_{15} + qa_{12} - a_{12}}{2q-1} & \end{bmatrix} \end{aligned}$$

for

$$\begin{cases} ((\widehat{u_4})^{-1} A \widehat{u_4})_{13} = \frac{pqa_3 + pqa_7 - qa_7 + pa_4 + pa_8 - a_8 - pqa_4 - pqa_8 + qa_8}{2p-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{14} = \frac{pa_3 + pa_7 - a_7 - pqa_3 - pqa_7 + qa_7 + pqa_4 + pqa_8 + qa_8}{2p-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{23} = \frac{pqa_3 + pqa_7 - qa_3 + pa_4 + pa_8 - a_4 - pqa_4 - pqa_8 + qa_4}{2p-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{24} = \frac{pa_8 + pa_7 - a_3 - pqa_3 - pqa_7 + qa_3 + pqa_4 + pqa_8 - qa_4}{2p-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{31} = \frac{qpa_9 + qpa_{13} - pa_{13} + qa_{10} + qa_{14} - a_{14} - pqa_9 - pqa_{13} + pa_{13}}{2q-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{32} = \frac{qa_{11} + qa_{15} - a_{15} - pqa_{11} - pqa_{15} + pa_{15} + pqa_{12} + pqa_{16} - pa_{16}}{2q-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{41} = \frac{qa_{11} + qa_{15} - a_{15} - pqa_{11} - pqa_{15} + pa_{15} + pqa_{12} + pqa_{16} - pa_{10}}{2q-1} \\ ((\widehat{u_4})^{-1} A \widehat{u_4})_{42} = \frac{qa_9 + qa_{13} - a_9 - pqa_9 - pqa_{13} + pa_9 + pqa_{10} + pqa_{14} - pa_{10}}{2q-1}. \end{cases}$$

And it is not too difficult to see that only the space $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \le k,\ell \le i} = \{(a_{k\ell})_{1 \le k,\ell \le i} \in \mathfrak{sl}_n \mid a_{ii} = 0 \;\&\; a_{ij} = 1 \forall i \neq j\}$ will remain unchanged by the above action.

□

Remark 6. *(i)* *Let $A \in M_n$, and let B be an $n \times n$ matrix which has been obtained from A by a finite set of elementary row and column operations, such that we have $AX = X$ for some set of matrices $X \in M_n$. Then it is not too difficult to verify that $BX = X$ also satisfies.*

(ii) *Let $\mathsf{u} = (\mathsf{u}_{ij})_{1 \leq i,j \leq n}$ be the main generator of $C(S_n^+)$. Then one may rewrite u by using a finite number of elementary row and column operators as an element $\widehat{u_n} \in C(\underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{n \text{ times}})$.*

Using 6(ii), we obtain the generalized version of Proposition 1 as follows for H as before, and we will present matrices with their usual presentation using elementary matrices, and based on [15], we will use $A_s(n)$ instead of $C(S_n^+)$ and $A_o(n)$ instead of $C(O_n^+)$.

Theorem 4. *The following conjugation will provide a non-trivial action of $A_s(n)$ on $GL_n(\mathbb{C})$,*

$$\begin{aligned} \varphi :& A_s(n) \to B(H) \\ & u \mapsto \varphi(u)(A) := \sum_{i,j=1}^{n} E_{ij} \sum_{m,o=1}^{n} b_{mo} a_{ij} u_{k\ell} \end{aligned} \tag{10}$$

for any $u = (u_{k\ell})_{k,\ell} \in A_s(n)$, and the space $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \leq k,\ell \leq i} = \{(a_{k\ell})_{1 \leq k,\ell \leq i} \in \mathfrak{sl}_n \mid a_{ii} = 0 \;\&\; a_{ij} = 1, \forall i \neq j\}$, plays the role of its invariant space.

Proof. The proof will follow exactly the approaches used in Proposition 1, by using Remark 6(ii). □

Remark 7. *Note that the approach provided in Theorem 4 also presents a new way of proving the truthfulness of $\mathsf{G}_{\mathrm{QAut}}(K_n) = S_n^+$, meaning that the quantum automorphism group of K_n is S_n^+, an indirect proof!*

Note that the original proof of this statement could be found in [15].

Following the above approach by moving forward and fixing a triangular decomposition $\mathfrak{sl}_n = \mathfrak{sl}_n^+ \oplus \mathfrak{h} \oplus \mathfrak{sl}_n^-$, for $\mathfrak{sl}_n^+$ and $\mathfrak{sl}_n^-$ being the upper and lower triangular matrices with trace zero, one might think of $\mathscr{G}_+$ and $\mathscr{G}_-$ as the upper and the lower triangular matrices satisfying the conditions of G, from Theorem 4, for their entries.

Note that one might obtain an almost analogous result by repeating the proposed direction in Theorem 4 for $\psi : A_s(i) \to B(H)$, still being a conjugation for $H \leqslant SL_n(\mathbb{C})$, but this time with $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \leq k,\ell \leq i}$ playing its invariant space such that it could be the upper or the lower triangular matrices $\mathscr{G}_+$, and $\mathscr{G}_-$ respectively, with the same conditions as in G from the Proposition 1.

Corollary 1. *The following conjugation will provide a non-trivial action of $A_s(n)$ on $SL_n(\mathbb{C})$,*

$$\begin{aligned} \varphi :& A_s(n) \to B(H) \\ & u \mapsto \varphi(u)(A) := \sum_{i,j=1}^{n} E_{ij} \sum_{o,p=1}^{n} b_{op} a_{ij} u_{k\ell} \end{aligned} \tag{11}$$

for any $u = (u_{k\ell})_{k,\ell} \in A_s(n)$ such that the invariant space $G = \bigoplus_{i=4}^{n} (a_{k\ell})_{1 \leq k,\ell \leq i}$, might be the upper or the lower triangular matrices $\mathscr{G}_+$, and $\mathscr{G}_-$, associated with $\mathfrak{sl}_n^+$ and $\mathfrak{sl}_n^-$, respectively.

Proof. The proof will exactly follow the directions illustrated in the proof of the Proposition 1. □

This study is important because of the study of the almost similar structures implied by Lie's theorem on the finite dimensional simple $U_q(\mathfrak{sl}_n^+)$ modules in the context of the Corollary 1.

As the elements of $C(S_n^+)$ are orthogonal, hence based on the above results we have the following very interesting result.

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Lemma 1. *For the special orthogonal group $SO_n(\mathbb{R})$ (or the special unitary group $SU_n(\mathbb{C})$), the conjugation will provide a nontrivial action of $A_s(n)$ on $SO_n(\mathbb{R})$ (or $SU_n(\mathbb{C})$) as stated in Proposition 1, with $A_s(n)$ itself playing its invariant space.*

Proof. In order to avoid the complexity posed by the matrix multiplication in the case of $A_s(n)$, the interested reader should note that the proof will exactly follow the directions proposed in the proof of proposition 1, and the fact that $A_s(n)$ will remain unchanged by the proposed action could easily be verified by the way of construction, that is, how the action has been defined! □

Cl1. The conjugation studied in Proposition 1 does not provide a valid action of $C(\underbrace{\mathbb{Z}_2 * \cdots * \mathbb{Z}_2}_{n \text{ times}})$ on $SP_n(\mathbb{C})$, and most probably the conditions studied in Theorem 4 do not satisfy as well!

4. Concluding Remarks

In our previous works [10–12], we tried to open a new research direction based on a previously known very powerful theory as a natural extension of the category of Hopf algebras from the nonunital point of view, which is called the multiplier Hopf algebras, introduced and studied by A. Van Daele [14]. In our opinion, the new direction could pretty well play as an extension that could host a part of the category of multiplier Hopf algebras, or even the whole of it in a nontrivial way!

One could say that the multiplier Hopf algebras and the quantum groups belong to an almost same category, and we know that quantum groups have been studied almost in a nonaxiomatic way in mathematics, meaning that the theory is mainly based on examples rather than axioms! However, this fact is not generalizable to the whole class of quantum groups! For example, a class of quantum groups which are called compact quantum groups, introduced by Woronowicz in 1987 [16], could be excluded, as they have a rigorous definition and have been structured quite well!

In 1998, in order to present an answer to a question by Alain Connes "asking if there is any sort of quantization of the usual permutation groups and what they might look like?", Wang came with a smart positive answer, stating that the quantization of the permutation groups has to be some sort of deformed object presented by S_n^+ and generated by magic unitary matrices $u = (u_{ij})_{i,j}$, whose entries satisfy the properties of being orthogonal projections [15]. Accordingly, in his paper, Wang raised another interesting question, asking "if there is any deformation of the finite groups of the Lie type into the category of finite quantum groups?"

As I pointed out earlier in this section, in our paper [11], we developed and studied some examples, which we called multiplier Hopf $*$-graph algebras of type n, and we believe that these structures (still not axiomatized) might provide us with an answer to the question raised by Wang!

About the continuation of the current paper, note that the cases related to $SP_n(\mathbb{C})$, as formulated in claim **??**, and the other finite groups of Lie type still have not been considered, and studying these cases might move us one more step closer to understanding the behaviors of finite groups of Lie type in order to be able to propose one or many parameter deformations of these groups!

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