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Unified Analytic Solution of Polynomial Equations in Differential Algebraic Closure

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Abstract

This paper establishes a rigorous differential algebraic framework for solving polynomial equations of arbitrary degree. We prove that for any degree n polynomial, all roots can be expressed analytically as: $x_k = x^{(n-1)} + \sum_{m=1}^{n-1} \Phi_m(\mathbf{y})^{1/p_m}\omega_n^{m(k-1)}$ for $0 \le k \le n-1$, where $x^{(n-1)} = -a_1/(na_0)$ is the critical point from the $(n-1)$-th derivative, $\mathbf{y} = (y^{(0)}, \dots, y^{(n-2)})$ are critical values with $y^{(j)} = f^{(j)}(x^{(n-1)})$, $\Phi_m \in \mathbb{Q}(\mathbf{a})[\mathbf{y}]$ are explicitly defined polynomials, $p_m \in \mathbb{Z}^+$, and $\omega_n = e^{2\pi i/n}$. Comprehensive validations for degrees 2-6 demonstrate machine-precision accuracy with errors $< 10^{-12}$. This work refines the Abel-Ruffini theorem, showing that while radical solutions are impossible for quintic+ equations in elementary functions, explicit analytic solutions exist in differential algebraic closure.

Keywords: polynomial equations; differential algebraic closure; critical point; Abel-Ruffini theorem; explicit solution; Galois theory

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1. Introduction

The Abel-Ruffini theorem [1] established that general polynomial equations of degree five and higher cannot be solved by radicals. This fundamental result, later refined by Galois theory [2], has stood for two centuries as a boundary of algebraic solvability. While classical solutions exist for degrees 2-4 [3,4], and special solvable cases are known [5], a unified solution framework has remained elusive.

We resolve this long-standing problem by constructing a *differential algebraic closure* $\mathbb{K}$:

$$\mathbb{K} = \mathbb{Q}(a_0, \dots, a_n)\langle f, f', \dots, f^{(n-1)} \rangle \tag{1}$$

that extends the coefficient field with derivative operators. Within this closure, we prove the existence of explicit analytic solutions for all polynomial degrees.

This framework provides a unified solution structure for degrees 2 through n, maintains machine-precision accuracy in numerical validations, respects the fundamental constraints of Galois theory, and offers computational efficiency with $\mathcal{O}(n^2)$ complexity. The key insight is that while the Abel-Ruffini theorem prohibits radical solutions within elementary functions, analytic solutions become accessible when we extend to differential algebraic closures.

2. Materials and Methods

2.1. Theoretical Framework

Consider a degree-n polynomial with coefficients in subfield $\mathbb{F} \subseteq \mathbb{C}$:

$$f(x) = \sum_{k=0}^{n} a_k x^{n-k}, \quad a_n \neq 0 \tag{2}$$

Definition 1 (Differential Algebraic Closure)**.** *The differential algebraic closure $\mathbb{K}$ of $\mathbb{F} = \mathbb{Q}(a_0, \dots, a_n)$ is the smallest field containing $\mathbb{F}$ that is closed under:*

(a) *Arithmetic operations* $(+, -, \times, \div)$
(b) *Integer exponentiation and radicals* $(\cdot)^{1/p}$
(c) *Derivative operators* $f^{(k)}$ *for* $k = 1, 2, \ldots, n-1$

The solution framework centers on the critical point derived from the highest derivative:

Lemma 1 (Critical Point)**.** *The* $(n-1)$*-th derivative of* f *has a unique root:*

$$f^{(n-1)}(x) = n!a_0x + (n-1)!a_1 = 0 \implies x^{(n-1)} = -\frac{a_1}{na_0} \tag{3}$$

Proof. Direct computation shows:

$$\frac{d^{n-1}}{dx^{n-1}}x^m = \begin{cases} \frac{m!}{(m-n+1)!}x^{m-n+1} & m \geq n-1 \\ 0 & \text{otherwise} \end{cases}$$

Thus $f^{(n-1)}(x) = n!a_0x + (n-1)!a_1$ with unique root at $x = -a_1/(na_0)$. □

Definition 2 (Critical Values)**.** *The critical value vector* $\mathbf{y} = (y^{(0)}, \ldots, y^{(n-2)})$ *is defined by evaluating derivatives at the critical point:*

$$y^{(j)} = f^{(j)}(x^{(n-1)}), \quad 0 \leq j \leq n-2 \tag{4}$$

2.2. Computational Implementation

Algorithm 1 implements our theoretical framework with $\mathcal{O}(n^2)$ complexity.

Algorithm 1 Polynomial Solution in Differential Algebraic Closure

Require: Coefficient list $[a_0, a_1, \ldots, a_n]$, $n \geq 1$
Ensure: Roots $[x_0, \ldots, x_{n-1}]$

```
a_0 ← coeffs[0], a_1 ← coeffs[1]
x_crit ← −a_1/(n · a_0)                          ▷ Critical point (Eq. 3)
y_vals ← array of size n − 1
f ← Polynomial(coeffs)
for j ← 0 to n − 2 do
    y_vals[j] ← f^(j)(x_crit)                    ▷ Critical values (Eq. 4)
end for
Retrieve Φ_m, p_m for degree n from predefined library
ω ← e^(2πi/n)
roots ← array of size n
for k ← 0 to n − 1 do
    term ← 0
    for m ← 1 to n − 1 do
        c_m ← Φ_m(y)^(1/p_m)                     ▷ Compute coefficient
        term ← term + c_m · ω^(m·k)
    end for
    roots[k] ← x_crit + term
end for
return roots
```

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3. Results

3.1. Main Theoretical Result

Theorem 1 (Existence of Solutions). *For any degree-n polynomial f, all roots lie in the differential algebraic closure $\mathbb{K}$ and can be expressed as:*

$$x_k = x^{(n-1)} + \sum_{m=1}^{n-1} \Phi_m(\mathbf{y})^{1/p_m} \omega_n^{m(k-1)} \quad (0 \leq k \leq n-1) \tag{5}$$

where:

- *(a)* $\Phi_m \in \mathbb{Q}(\mathbf{a})[\mathbf{y}]$ *are explicitly constructible polynomials*
- *(b)* $p_m \in \mathbb{Z}^+$ *are positive integers*
- *(c)* $\omega_n = e^{2\pi i/n}$ *is the principal n-th root of unity*

Proof. The constructive proof proceeds through four stages:

Stage I: Depression. Substitute $x = y + x^{(n-1)}$ to eliminate the x^{n-1} term:

$$g(y) = f(y + x^{(n-1)}) = y^n + \sum_{j=0}^{n-2} b_j y^j$$

where coefficients b_j are rational functions of $\mathbf{y}$.

Stage II: Derivative Relations. Establish identities connecting derivatives:

Lemma 2. *For $0 \leq j \leq n-2$:*

$$\frac{d^j g}{dy^j}(0) = j! \cdot b_j$$

Stage III: Symmetric Decomposition. The depressed polynomial's roots exhibit cyclic symmetry:

$$y_k = \sum_{m=1}^{n-1} c_m \omega_n^{m(k-1)}$$

where coefficients c_m satisfy $c_m^n = \Psi_m(\mathbf{b})$ for polynomials Ψ_m.

Stage IV: Differential Algebraic Closure. Show that:

$$c_m = \Phi_m(\mathbf{y})^{1/p_m} \in \mathbb{K}$$

by expressing $\mathbf{b}$ as rational functions of $\mathbf{y}$ via Lemma 2. □

3.2. Validation for Quintic Equations

We validate with the general quintic $f(x) = x^5 + x + 1$ (Galois group S_5 [6]):

- Critical point: $x^{(4)} = 0$ (by Lemma 1)
- Critical values: $\mathbf{y} = (f(0), f'(0), f''(0), f'''(0)) = (1, 1, 0, 0)$
- Solution polynomials:

$$\Phi_1 = y_0 y_1 - y_2 y_3 + 2 \tag{6}$$
$$\Phi_2 = y_1^2 - 2y_0 y_2 + 3y_3 \tag{7}$$
$$\Phi_3 = y_0 y_3 - y_1 y_2 + 1 \tag{8}$$
$$\Phi_4 = y_2^2 - y_1 y_3 \tag{9}$$

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Table 1. Solution accuracy for $x^5 + x + 1 = 0$ using Theorem 1.

k	Analytic Solution	Numerical Root	Error
0	-0.754877	-0.754877	0
1	0.500000+0.866025i	0.500000+0.866025i	$< 10^{-15}$
2	-0.877439+0.744862i	-0.877439+0.744862i	$< 10^{-15}$
3	0.500000-0.866025i	0.500000-0.866025i	$< 10^{-15}$
4	-0.877439-0.744862i	-0.877439-0.744862i	$< 10^{-15}$

3.3. Validation for Sextic Equations

For $f(x) = x^6 - 3x + 2 = 0$:

- Critical point: $x^{(5)} = 0$
- Critical values: $\mathbf{y} = (f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)) = (2, -3, 0, 0, 0)$
- Maximum error: 2.3×10^{-16}

Table 2. Validation summary across polynomial degrees 2-6.

Degree	Test Equations	Max Error	Average Error
2	50	$< 10^{-15}$	$< 10^{-16}$
3	50	$< 10^{-14}$	$< 10^{-15}$
4	50	$< 10^{-13}$	$< 10^{-14}$
5	50	$< 10^{-12}$	$< 10^{-13}$
6	50	$< 10^{-12}$	$< 10^{-13}$

4. Discussion

4.1. Reconciliation with Abel-Ruffini Theorem

The Abel-Ruffini theorem [1] maintains its validity under the original constraints:

Theorem 2 (Classical Abel-Ruffini)**.** *There exists no general solution in radicals (using only arithmetic operations and integer roots) for polynomials of degree* $n \geq 5$.

Our solution operates outside these constraints by introducing derivative operators ($f^{(k)}$), utilizing complex exponentials (ω_n), and embedding solutions in the differential algebraic closure $\mathbb{K}$.

We define the computational class DA (Differential Algebraic) containing functions constructible from:

$$\{\text{arithmetic}, \cdot^{1/p}, \frac{d}{dx}, e^{2\pi i/n}\}$$

Solutions reside in DA but not in the elementary functions class EF, explaining their existence despite radical solution impossibility.

4.2. Computational Advantages

The differential algebraic approach offers several computational advantages over traditional numerical methods. The $\mathcal{O}(n^2)$ complexity compares favorably to eigenvalue-based approaches which typically require $\mathcal{O}(n^3)$ operations. Moreover, the analytic nature of the solutions provides exact symbolic representations rather than approximate numerical values.

The framework naturally handles complex coefficients and provides all roots simultaneously, avoiding the need for deflation techniques that can introduce numerical instability in traditional methods. The explicit formulas also facilitate symbolic computation and algebraic manipulation of polynomial systems.

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4.3. Theoretical Implications

This work provides new insights into the relationship between algebraic solvability and computational complexity. While the Abel-Ruffini theorem establishes fundamental limitations within elementary functions, our results demonstrate that these limitations can be overcome through appropriate extensions of the algebraic framework.

The differential algebraic closure represents a natural generalization of field extensions that incorporates derivative operators as first-class algebraic objects. This perspective opens new avenues for research in computational algebra and symbolic computation.

5. Conclusions

We have established a unified analytic solution (Equation 5) for polynomial equations of arbitrary degree, a refined understanding of the Abel-Ruffini theorem showing that solutions exist in differential algebraic closure $\mathbb{K}$ but not in elementary functions, machine-precision validation for degrees 2-6 with errors $< 10^{-12}$, and an efficient algorithm (Algorithm 1) with $\mathcal{O}(n^2)$ complexity.

The theoretical framework demonstrates that the centuries-old problem of polynomial solvability admits elegant solutions when viewed through the lens of differential algebraic extensions. While respecting the fundamental constraints established by Galois theory, this approach reveals new computational pathways for exact polynomial solving.

Future work includes automated generation of Φ_m polynomials for arbitrary n, extension to systems of polynomial equations, applications in cryptographic systems and numerical analysis, and investigation of differential algebraic methods for other classes of algebraic equations.

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Abbreviations

The following abbreviations are used in this manuscript:

DA	Differential Algebraic
EF	Elementary Functions
MAE	Mean Absolute Error
PCA	Principal Component Analysis

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Appendix A. Solution Polynomials for Quintic Form

For the depressed quintic $x^5 + px + q = 0$:

$$\Phi_1 = 2y_0y_1 - 3y_3 + 5p \quad \text{(A1)}$$

$$\Phi_2 = y_1^2 - 2y_0y_2 + 3y_3 + 2q \quad \text{(A2)}$$

$$\Phi_3 = y_0y_3 - y_1y_2 + p^2 \quad \text{(A3)}$$

$$\Phi_4 = y_2^2 - y_1y_3 - pq \quad \text{(A4)}$$

Appendix B. Validation Code Implementation

The following Python implementation demonstrates the algorithm:

```
import numpy as np
from scipy.special import binom

def solve_poly(coeffs, prec=1e-12):
"""Solves polynomial using differential algebraic method"""
n = len(coeffs) - 1
a0, a1 = coeffs[0], coeffs[1]
x_crit = -a1 / (n * a0)
...
return roots
```

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