Review

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Role of Qubits in Quantum Entanglement and Quantum Teleportation

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Abstract

A qubit is an exhibition of quantum entanglement and a key element in the quantum teleportation process. In this paper, we review the role of qubits in quantum entanglement and quantum teleportation.

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1. Introduction

Nearly ninety years ago, Erwin Schrödinger introduced the term verschränking to describe a correlation in quantum mechanics [1,2]. In German, the term verschränking is commonly used by non-physicists to refer to the act of folding one's arms, but it has been translated to entanglement in a way that evokes greater inspiration and its interpretation has evolved over the years. The issue of anticipated locality in entangled quantum systems highlighted by the EPR paradox [3,4] enabled John Stewart Bell to formulate his well-known inequalities, which act as a test and illustration of the peculiar characteristics of the simplest entangled wave function, specifically the singlet state [5,6]. However, it took a considerable amount of time before proposals and practical applications for quantum entanglement emerged. Before 1975, there was a notable absence of experiments that tested the violation of Bell's inequalities to confirm the reality of quantum entanglement. The groundbreaking experiment conducted by French physicist Alain Aspect at the École Supérieure d'Optique in Orsay between 1980 and 1982 was the first in the realm of quantum mechanics to demonstrate such a violation of Bell's inequalities [7,8]. This experiment is referred to as Aspect's experiment. It validated the predictions made by quantum mechanics and consequently established its incompatibility with local theories. Quantum entanglement is a phenomenon unlike anything found in classical physics. It stands out as the most unusual characteristic of quantum mechanics, prompting a multitude of philosophical, physical, and mathematical inquiries since the inception of quantum theory. Entanglement represents a quantum mechanical type of correlation that manifests in various fields, including condensed matter physics, quantum chemistry, and other branches of physics [9]. Not only is quantum entanglement one of the most extraordinary aspects of quantum mechanics, but it also serves as the foundation for numerous applications, such as quantum teleportation, quantum information theory, quantum cryptography, and quantum computing [10]. In this article, we discuss quantum teleportation. The concept of teleportation originates from science fiction, referring to the ability to make an individual or an object vanish while a duplicate appears in a different location. Quantum teleportation was proposed by Bennett and colleagues, and unlike some fictional portrayals, it adheres to established physical laws. Specifically, it cannot occur instantaneously or across a spacelike interval, as it necessitates, among other things, the transmission of a classical message from a sender to a receiver [11]. Today, quantum teleportation is widely recognized as a significant application of quantum entanglement in relation to quantum information processing. The role of qubits in quantum entanglement and quantum teleportation is reviewed in this paper because the system of two qubits is the simplest system to exhibit quantum entanglement, and quantum teleportation entails moving a qubit's quantum state from one place to another using entanglement and classical communication. Section 2 is about qubits, and then

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we present in section 3 the criteria of separability of entanglement in bipartite systems. The protocol of quantum teleportation and the fidelity of teleportation are exhibited in section 4. Concluding remarks are given in 5.

2. Qubits

The Hilbert space with n dimensions $\mathbb{C}^n$ is the quantum analog of a system with n states- The term qubit (quantum bit) is used to describe a quantum system with two states. The two most common states for a qubit (quantum bit) are

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad 1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1}$$

just like a classical bit can have either 0 or 1. Any two-level quantum-mechanical system can be used as a qubit. The simplest two-state quantum mechanical system is represented by qubits, which can exist in a coherent superposition of both states simultaneously. This means that the state vectors in $\mathcal{H} = \mathbb{C}^2$ are of the form

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{2}$$

for some $\alpha, \beta \in \mathbb{C}$, with $|\alpha|^2 + |\beta|^2 = 1$. Without losing anything, the coefficients α and β can be considered real numbers. A small amount of algebra gives

$$|v\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}. \tag{3}$$

- Quantum superposition principle: *If a quantum system can be in the state $|0\rangle$ and can also be in the state $|1\rangle$, then quantum mechanics allows the system to be in any arbitrary state $|v\rangle = \alpha|0\rangle + \beta|1\rangle$. The state $|v\rangle$ is said to be in a superposition of $|0\rangle$ and $|1\rangle$ with probability amplitude α and β.* Two well-known states are provided by

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle, \quad |-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle. \tag{4}$$

- What is the physical realization of qubits?
  1. One possible representation of the qubits would be two states of an electron orbiting an atom.
  2. Two directions of a particle's spin could be used to represent the qubits; for instance, up or down would be used to measure a particle's spin along the z-axis, that is ($z+$ direction) and ($z-$ direction) or $|\uparrow\rangle$ and $|\downarrow\rangle$. For computational purpose it is convenient to use $|0\rangle$ and $|1\rangle$.
  3. A photon with two polarizations could be used to represent the qubits.
- The computational basis $\{|0\rangle,\ |1\rangle\}$ is typically used to represent two exclusive states of a quantum system used as quantum-0 and quantum-1. For instance, if our quantum bit is the energy of an electron in an atom, we could say that our quantum-0 is the ground state (the state with the lowest energy) and our quantum-1 is the excited state (the state with the highest energy). Since the ground state and the excited states are mutually exclusive, the representation could be: ground state $\leftrightarrow |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, excited state $\leftrightarrow |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

A key component of quantum computing is the ability of the qubit to exist in a coherent superposition of several states at once, which is made possible by quantum mechanics. In the sense that quantum entanglement is a property of two or more qubits that permits a set of qubits to express higher correlation than is feasible in classical systems, the qubit is an example of quantum entanglement. In a

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conversation with William Wootters, Benjamin Schumacher came up with the term "qubit," which was created in jest [12].

3. Quantum Entanglement

The fundamental question in quantum entanglement theory is which states are entangled and which are not. This question, known as the separability problem, has been solved for pure states [13] and for 2×2 and 2×3 systems [14], but it remains an open problem from both a theoretical and experimental standpoint. According to [15], a separability condition may be necessary or necessary and sufficient for separability. Every separable state needs to meet a prerequisite for separability. We cannot draw a conclusion if a state satisfies the condition, but if it does not, it must be entangled.

If a quantum system's wave function cannot be expressed as a product of subsystem states, it is entangled. The singlet state of two spin-$\frac{1}{2}$ particles is the most basic example [16].

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle). \tag{5}$$

It can be proven that $|\psi\rangle \neq |\varphi\rangle|\phi\rangle$ for any $|\varphi\rangle$ and $|\phi\rangle$ describing subsystems, $|0\rangle$ standing for "spin up" state and $|1\rangle$ standing for "spin down" state.

The " nonfactorisability" of any bipartite pure state implies that its reduced density matrices are mixed. The above definition is naturally generalized to the entanglement of multiparticle pure state. A bipartite pure quantum state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled when it cannot be written as

$$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B, \tag{6}$$

for some $|\psi\rangle_A \in \mathcal{H}_A$ and $|\psi\rangle_B \in \mathcal{H}_B$. A mixed state or density matrix ρ_{AB} which is semi-definite operator on $\mathcal{H}_A \otimes \mathcal{H}_B$ is called entangled when it cannot be written in the following form

$$\rho = \sum_i p_i |\psi_i\rangle_{AA}\langle\psi_i| \otimes |\psi_i\rangle_{BB}\langle\psi_i| \tag{7}$$

Here the coefficients p_i are probabilities, that means $0 \leq p_i \leq 1$ and $\sum_i P_i = 1$. Note that in general neither $\{|\psi_i\rangle_A\}$ nor $\{|\psi_i\rangle_B\}$ have to be orthogonal.

3.1. Bell-CHSH Inequalities

The original purpose of the Bell inequality was to compare the predictions of a local hidden variables theory with those of quantum mechanics [5]. At first, Bell's inequalities applied to two qubits, i.e. e, two-level systems, and offer a prerequisite for the separability of states with two qubits. Bell's inequalities are also adequate for separability in pure states. Gisin has demonstrated that a Bell inequality is violated by any non-product state of two-particle systems [17]. In 1969, Clauser, Horne, Shimony, and Holt (CHSH) extended this inequality for the case of four vectors. This inequality involves three vectors in real space $\mathbb{R}^3$ that determine which component of a spin to be measured by each party [18]. Additionally, a test to differentiate between entangled and non-entangled states is provided by the Bell-CHSH inequality.

Consider a system of two qubits. Let observables on the first qubit be represented by A and A', and observables on the second qubit by B and B'. According to the Bell-CHSH inequality, the following inequality is true for non-entangled states, which include states with the form $\rho = \rho_1 \otimes \rho_2$ or mixtures of such states

$$|\langle\, A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B' \,\rangle_\rho| \leq 2, \tag{8}$$

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where $\langle A \otimes B \rangle_\rho := \text{Tr}\, \rho(A \otimes B)$ and $\langle A \otimes B \rangle_\psi = \langle \psi | A \otimes B | \psi \rangle$ for the expectation value of $A \otimes B$ in the mixed states ρ or pure state $|\psi\rangle$. As an example, we consider a two qubits state $|\phi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, and the observables

$$A = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z),\ A' = \frac{1}{\sqrt{2}}(\sigma_x - \sigma_z),\ B = \sigma_x,\ B' = \sigma_z, \tag{9}$$

where σ_x and σ_z are Pauli matrices. We have then explicitly

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{10}$$

and

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix};\ A' = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix};\ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ B' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

It is easy to check that

$$\langle \phi |\ A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B'\ | \phi \rangle = 2\sqrt{2}. \tag{12}$$

One of the Bell pairs, a maximally entangled state, is the well-known entangled state $|\phi\rangle$, which violates the Bell-CHSH inequality. An inequality of Cirelson [19] leads to the maximal violation of (8) for entangled states

$$|\langle A \otimes B + A \otimes B' + A' \otimes B - A' \otimes B' \rangle_\rho| \leq 2\sqrt{2}. \tag{13}$$

The singlet state can achieve the equality in equation (13). Although it has long been known that breaking a Bell-CHSH inequality is merely a sufficient condition for entanglement and not a necessary one, and that many entangled states satisfy them, Bell-CHSH inequalities were historically the first tool for recognizing entanglement [20]. A criterion to differentiate separable states from entangled states is provided by the violations of Bell-CHSH inequalities, which were extended to N qubits [21–23].

3.2. Criteria of Separability

The following criteria have been discussed in [24], which provides an overview of the progress made in solving the separability problem in two qubit systems

1. Schmidt decomposition criterion [25–28].
2. The Positive Partial Transpose (PPT) criterion [14,28–30].
3. Entropy of entanglement criterion.
4. The reduction criterion [14,31].
5. Concurrence criterion [32–35].
6. The majoration criterion [36,37].
7. The computable cross norm or realignment (CCNR) criterion [38–40].
8. A matrix realignment criterion [40,41].
9. The correlation matrix criterion [41–44].
10. Enhanced entanglement criterion via SIC POVMs [45,46].
11. Positive maps criterion [47–51].
12. The entanglement witnesses [14,52–55].
13. Local uncertainty relations (LURs) criterion [56–59].
14. The Li-Qiao criterion [14,41,60–63].

4. Quantum Teleportation

The process of quantum teleportation uses long-range EPR correlations in an entangled state to transport all of the information of a single quantum state from one place to another. This entangled state is shared between the two parties, which are known as the sender and the receiver. At first, the sender

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makes some measurements with the information state and her or his shared part of the entangled state. In this process, the information state disappears at the sender's end and instantly appears at the receiver's end. This is obtained when the receiver makes some unitary transformation that depends on some result of the sender's measurement, which is received through some classical channel. The transmission of quantum states can be accomplished without using any entanglement, whether it is only classical communication or by the transmission of classical bits, where the sender and the receiver share entanglement. We are interested in the second situation. In that regard, quantum teleportation is the process of moving a qubit's quantum state from one place to another by means of classical communication and entanglement. It's not about physically moving the qubit, but rather transferring its quantum information. In the context of a qubit's teleportation, the impact of nonmaximality of a quasi-Bell state-based quantum channel is studied [64,65].

4.1. Teleportation Through a Nonmaximally Entangled Channel

Amy has a qubit $|\psi\rangle_C$ that she wishes to send to Bouba at the start of the teleportation protocol. This is

$$|\psi\rangle_C = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}\exp(i\phi)|1\rangle. \tag{14}$$

In the protocol, Amy and Bouba share this nonmaximal entangled channel

$$|\psi\rangle_{AB} = \alpha|00\rangle + \beta|11\rangle, \tag{15}$$

where we assume that α and β are real and that $\alpha \leq \beta$. The subscripts A and B in the entangled state refer to Amy's or Bouba's state, while the subscript C refers to the initial state in Amy's possession. Since the entangled channel shared by Amy and Bouba is nonmaximal, it is not possible to perform this qubit teleportation perfectly, so we would like to compute the fidelity. Amy receives one of the two states, while Bouba receives the other. Amy currently possesses two states: A, the state she wishes to teleport, and one of the entangled pairs, while Bouba possesses one particle, B.

The state of the total system, consisting of the input qubit in the equation (14) and the shared entanglement in the equation (15) becomes

$$\begin{aligned} |\psi\rangle_C \otimes |\psi\rangle_{AB} &= (\cos\frac{\theta}{2}|0\rangle_C + \sin\frac{\theta}{2}\exp(i\phi)|1\rangle_C) \\ &\otimes (\alpha|0\rangle_A|0\rangle_B + \beta|1\rangle_A|1\rangle_B). \end{aligned} \tag{16}$$

The initial state of the total system is then

$$\begin{aligned} |\psi\rangle_{CAB} &= \alpha\cos\frac{\theta}{2}\,|0\rangle_C|0\rangle_A|0\rangle_B + \beta\cos\frac{\theta}{2}\,|0\rangle_C|1\rangle_A|1\rangle_B \\ &+ \alpha\sin\frac{\theta}{2}\,e^{i\phi}\,|1\rangle_C|0\rangle_A|0\rangle_B + \beta\sin\frac{\theta}{2}\,e^{i\phi}\,|1\rangle_C|1\rangle_A|1\rangle_B. \end{aligned} \tag{17}$$

Amy uses the two states that she possesses to make a local measurement on the Bell basis. Amy's two qubit states should ideally be expressed as a superposition of the Bell basis in order to clearly convey the outcome of her measurement. The following general identities are used to make this possible:

$$|0\rangle|0\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle); \tag{18}$$

$$|0\rangle|1\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle); \tag{19}$$

$$|1\rangle|0\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle); \tag{20}$$

$$|1\rangle|1\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle). \tag{21}$$

Applying the general identities to the qubits with A and C subscripts, we have

$$|0\rangle_C|0\rangle_A = \frac{1}{\sqrt{2}}(|\phi^+\rangle_{CA} + |\phi^-\rangle_{CA}); \quad (22)$$

$$|0\rangle_C|1\rangle_A = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{CA} + |\psi^-\rangle_{CA}); \quad (23)$$

$$|1\rangle_C|0\rangle_A = \frac{1}{\sqrt{2}}(|\psi^+\rangle_{CA} - |\psi^-\rangle_{CA}); \quad (24)$$

$$|1\rangle_C|1\rangle_A = \frac{1}{\sqrt{2}}(|\phi^+\rangle_{CA} - |\phi^-\rangle_{CA}). \quad (25)$$

The state of the total system A, B and C together form the following four-term superposition.

$$\begin{aligned} |\psi\rangle_{CAB} &= \frac{1}{\sqrt{2}}\Big[|\phi^+\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right) \\ &+ |\phi^-\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right) \\ &+ |\psi^+\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right) \\ &+ |\psi^-\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right)\Big]. \quad (26) \end{aligned}$$

Since no operations have been carried out, all three states remain in the same total state. Amy's portion of the system is merely a change of basis in the equation above. The entanglement has shifted from particles A and B to particles C and A as a result of this modification. When Amy measures her two qubits in the Bell basis, the teleportation happens: $|\phi^+\rangle_{CA}$, $|\phi^-\rangle_{CA}$, $|\psi^+\rangle_{CA}$, $|\psi^-\rangle_{CA}$. The result of Amy's local measurement is a collection of two classical bits 00, 01, 10, 11 related to one of the four states after the three-particle state has collapsed into one of the states:

$$|\phi^+\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \quad (27)$$

$$|\phi^-\rangle_{AC} \otimes \left(\alpha\cos\frac{\theta}{2}|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \quad (28)$$

$$|\psi^+\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right); \quad (29)$$

$$|\psi^-\rangle_{AC} \otimes \left(\beta\cos\frac{\theta}{2}|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B\right). \quad (30)$$

The entanglement that was previously shared between Amy's and Bouba's states has been broken, and Amy's two states are now intertwined in one of the four Bell states. One of the four superposition states is assumed by Bouba's qubit:

$$\alpha\cos\frac{\theta}{2}|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B; \quad (31)$$

$$\alpha\cos\frac{\theta}{2}|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B; \quad (32)$$

$$\beta\cos\frac{\theta}{2}|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B; \quad (33)$$

$$\beta\cos\frac{\theta}{2}|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B. \quad (34)$$

At this point, Bouba's qubit is in a state that is similar to the one that will be teleported. There are four possible states for Bouba's qubit, which are unitary images of the state that requires teleportation. Bouba guesses which of the four states his qubit is in after receiving the message from Amy. Bouba selects one of the unitary transformations in accordance with this information: $\{\mathbb{I}, \sigma_x, i\sigma_y, \sigma_z\}$, to perform his part of the channel. Here

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{35}$$

$\mathbb{I}$ represents the identity operator, and $\sigma_x, \sigma_y, \sigma_z$ are the Pauli operators. The correspondence between the measurement outcomes and the unitary operations are

$$|\phi^+\rangle_{CA} \Rightarrow \mathbb{I}; \quad |\phi^-\rangle_{CA} \Rightarrow \sigma_z; \quad |\psi^+\rangle_{CA} \Rightarrow \sigma_x; \quad |\psi^-\rangle_{CA} \Rightarrow i\sigma_y\,. \tag{36}$$

The teleportation is accomplished, and we calculate the fidelity of this teleportation to gauge the effectiveness of the teleportation protocol.

4.2. Quantum Teleportation Fidelity

4.2.1. Definition

The concept of quantum fidelity, which is equivalent to taking into account the behavior in time of the overlap of two quantum states—one evolving according to a specific dynamics and the other evolving by a slight perturbation of it—is comparable to the fidelity measure of the quality of the teleported state. Both states began at time zero with the same initial state. The fidelity between two quantum states σ_1 and σ_2, represented as density matrices, is commonly defined as follows [66].

$$F(\sigma_1, \sigma_2) = \left(\text{Trace}[\ (\sqrt{\sigma_1}\sigma_2\sqrt{\sigma_1})^{\frac{1}{2}}\ ]\right)^2. \tag{37}$$

In quantum teleportation, the fidelity is commonly defined as in the equation (37). However there are alternative definitions, for instance in [67] fidelity is defined as

$$F(\sigma_1, \sigma_2) = \text{Trace}[\ (\sqrt{\sigma_1}\sigma_2\sqrt{\sigma_1})^{\frac{1}{2}}\ ]. \tag{38}$$

There are other definitions of fidelity as follows.

- $$F = \langle\psi|\rho|\psi\rangle\,, \tag{39}$$

  that is the overlap of the input information state $|\psi\rangle$ with the normalized output teleported state ρ.
- $$F = \text{Tr}[\rho_{\text{out}}|\phi\rangle\langle\phi|], \tag{40}$$

  where $|\phi\rangle$ is the input information state, and ρ_{out}, the output information state.
- The following definition has been used in [68]

  $$F = |\langle T|I\rangle|^2, \tag{41}$$

  where $|I\rangle$ represents the information state, to be teleported, and $|T\rangle$ represents the teleported copy of the initial information state that the receiver has after application of the unitary transformation.

In this work, we adopt the following definition of fidelity, which is compatible with the teleportation protocol.

$$F = \sum_{i=1}^{4} P_i\,|\,\langle I\,|\,\zeta_i\rangle\,|^2, \tag{42}$$

$|I\rangle$ is the input state and $P_i = \text{Tr}(\langle\Omega|M_i|\Omega\rangle)$ with $|\Omega\rangle = |I\rangle \otimes |\psi_{channel}\rangle$ $M_i = |\psi_i\rangle\langle\psi_i|$ is a measurement operator on a Bell basis. $|\zeta_i\rangle$ is the teleported state corresponding to i^{th} projective measurement on a Bell basis, where F depends on the parameters of the state to be teleported. In some references, $|\langle I \,|\, \zeta_i\rangle|^2$ is considered as fidelity and F in the equation (42) as average fidelity [69].

The quality of the teleportation can also be studied through other measures such as the minimum assured fidelity (MASF), the average teleportation fidelity (F_{ave}), and the optimal fidelity (f). We do not discuss these measures here. Although the many definitions of fidelity, it satisfies the following properties: i) the fidelity is state-dependent, and $(0 \leq \text{F} \leq 1)$, ii) if the output state is the same as the input information, then the fidelity of the teleportation is equal to unity, $(\text{F} = 1)$, iii) if the output state is imprecise copy of the input information, then the fidelity is smaller than 1, $(\text{F} < 1)$, vi) if the output state is completely orthogonal to the input state, then the fidelity is zero, $(\text{F} = 0)$ and the teleportation is not possible, v) the less entangled states reduce the fidelity of teleportation [69–71].

4.2.2. Computation of the Fidelity

The fidelity of this teleportation is calculated in order to assess the protocol's effectiveness

$$F(|\psi\rangle_C) = \sum_{i=1}^{4} P_i \,|\langle\psi_C|\zeta_i\rangle|^2 , \tag{43}$$

where $P_i = \text{Tr}({}_{CAB}\langle\psi|M_i|\psi\rangle_{CAB})$, $M_i = |\phi_i\rangle\langle\phi_i|$ the measurement operator on the quasi-Bell basis $|\phi_i\rangle \in \{|\phi^+\rangle_{CA}, |\phi^-\rangle_{CA}, |\psi^+\rangle_{CA}, |\psi^-\rangle_{CA}\}$, and $|\zeta_i\rangle$ Bouba's normalized and corrected outcome given the measurement result in i.

The probabilities of the measurement are explicitly the following:

$$\begin{aligned}
P_1 &= \text{Tr}({}_{CAB}\langle\psi|\phi_1\rangle\langle\phi_1|\psi\rangle_{CAB}) = \text{Tr}({}_{CAB}\langle\psi|\phi^+_{CA}\rangle\langle\phi^+_{CA}|\psi\rangle_{CAB});\\
P_2 &= \text{Tr}({}_{CAB}\langle\psi|\phi_2\rangle\langle\phi_2|\psi\rangle_{CAB}) = \text{Tr}({}_{CAB}\langle\psi|\phi^-_{CA}\rangle\langle\phi^-_{CA}|\psi\rangle_{CAB});\\
P_3 &= \text{Tr}({}_{CAB}\langle\psi|\phi_3\langle\phi_3|\psi\rangle_{CAB}) = \text{Tr}({}_{CAB}\langle\psi|\psi^+_{CA}\langle\psi^+_{CA}|\psi\rangle_{CAB});\\
P_4 &= \text{Tr}({}_{CAB}\langle\psi|\phi_4\rangle\langle\phi_4|\psi\rangle_{CAB}) = \text{Tr}({}_{CAB}\langle\psi|\psi^-_{CA}\rangle\langle\psi^-_{CA}|\psi\rangle_{CAB}).
\end{aligned}$$

The probability of Amy measuring $|\phi^+\rangle_{CA}$ or $|\phi^-\rangle_{CA}$ is

$$P_1 = P_2 = \frac{1}{2}\left(\alpha^2\cos^2\frac{\theta}{2} + \beta^2\sin^2\frac{\theta}{2}\right); \tag{44}$$

The probability of Amy measuring $|\psi^+\rangle_{CA}$ or $|\psi^-\rangle_{CA}$ is

$$P_3 = P_4 = \frac{1}{2}\left(\alpha^2\sin^2\frac{\theta}{2} + \beta^2\cos^2\frac{\theta}{2}\right). \tag{45}$$

The normalized qubits of Bouba after each measurement are given as follows:

- for the mesurement of $|\phi^+\rangle_{CA}$, Bouba applies the unit operator to $\alpha\cos\frac{\theta}{2}\,|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B$ and normalizes to obtain

$$|\zeta_1\rangle = \frac{1}{\sqrt{\alpha^2\cos^2\frac{\theta}{2} + \beta^2\sin^2\frac{\theta}{2}}}\left(\alpha\cos\frac{\theta}{2}|0\rangle_B + \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \tag{46}$$

- for the mesurement of $|\phi^-\rangle_{CA}$, Bouba applies the operator σ_x on $\alpha\cos\frac{\theta}{2}\,|0\rangle_B - \beta\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B$ and after normalization the outcome is

$$|\zeta_2\rangle = \frac{1}{\sqrt{\alpha^2\cos^2\frac{\theta}{2} + \beta^2\sin^2\frac{\theta}{2}}}\left(-\beta\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B + \alpha\cos\frac{\theta}{2}|1\rangle_B\right); \tag{47}$$

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- for the mesurement of $|\psi^+\rangle_{CA}$, Bouba applies the operator $i\sigma_y$ on $\beta\cos\frac{\theta}{2}\,|1\rangle_B + \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B$; and after normalization the outcome is

$$|\zeta_3\rangle = \frac{1}{\sqrt{\alpha^2\sin^2\frac{\theta}{2} + \beta^2\cos^2\frac{\theta}{2}}}\left(\beta\cos\frac{\theta}{2}|0\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|1\rangle_B\right); \tag{48}$$

- for the mesurement of $|\psi^-\rangle_{CA}$, Bouba applies the operator σ_z on $\beta\cos\frac{\theta}{2}\,|1\rangle_B - \alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B$.; and after normalization the outcome is

$$|\zeta_4\rangle = \frac{1}{\sqrt{\alpha^2\sin^2\frac{\theta}{2} + \beta^2\cos^2\frac{\theta}{2}}}\left(-\alpha\sin\frac{\theta}{2}e^{i\phi}|0\rangle_B - \beta\cos\frac{\theta}{2}|1\rangle_B\right). \tag{49}$$

Using the equations in (44), (45), and the equations in (46), (47), (48), (49) in the equation (43), the fidelity is given by

$$F(|\psi\rangle_C) = \cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2} + \alpha\beta\sin^2\theta. \tag{50}$$

For $\alpha\beta = 1/2$, $F(|\psi\rangle_C) = 1$, in that case, there is perfect teleportation. This situation implies that $\alpha = 1/\sqrt{2}$ and $\beta = 1/\sqrt{2}$ and then the state (15) is maximally entangled. Assuming the state (15) non maximally entangled state excludes this situation.

5. Conclusion

In this paper, we review the role of qubits in quantum entanglement and quantum teleportation. The fidelity of teleportation is also discussed. As concluding remarks, the following challenges are raised:

1. The separability of quantum states is directly linked to unsolved challenges of mathematics concerning linear algebra and geometry, functional analysis and, in particular, the theory of $C^\star$-algebra. The distillability problem, which involves determining when a composite quantum system's state can be converted into an entangled pure state via local operations, is yet another issue connected to difficult unresolved questions in contemporary mathematics.
2. Deterministic perfect teleportation is not possible in the case of entangled non-orthogonal coherent states.
3. The fidelity of teleportation depends on the parameters of the initial state to be teleported. For the protocol, the unitary operators used by the receiver are the Pauli matrices. It may be interesting to find the convenient unitary operators that give a perfect teleportation.

References

1. E. Schrödinger, Naturwissenschaften 23, 807(1935).
2. E. Schrödinger, Discussion of probability relations between separated systems, Proc. Camb. Phil. Soc., 31(555), 1935.
3. Einstein attacks quantum theory. New York Times, May 4th, 1935.
4. A. Einstein, B. Podolsky, and N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev., 47(777), 1935.
5. John S. Bell, On the Einstein-Podolsky-Rosen paradox. Physics, 1:195 - 200, 1964.
6. John S. Bell, On the problem of hidden variables in quantum mechanics, Rev. Mod. Phys., 38(3): 447 - 452, Jul 1966.
7. A. Aspect, P. Grangier and G. Roger, Experimental tests of realistic local theories via Bell's theorem, Phys. Rev. Lett. 47, 460, 1981.
8. A. Aspect, P. Grangier, and G. Roger. Experimental realization of Einstein-Podolsky-Rosen-Bohm gedankenexperiment: A new violation of Bell's inequalities. Phys. Rev. Lett., 49(91), 1982
9. Luigi Amico, Rosario Fazio, Andrea Osterloh, Vlatko Vedral, Entanglement in Many-Body Systems, Rev. Mod. Phys. 80, 517, (2008).

 doi:10.20944/preprints202508.0756.v1

10. M. A. Nielsen, I. L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press (2000).
11. C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, W. K. Wootters, *Teleporting an unknown quantum state vis dual classical and Einstein-Podolsky-Rosen channels Phys. Rev. Lett* 70 (1993) 1895.
12. Schumacher, B. (1995) “Quantum coding". Physical Review A. 51 (4): 2738-2747.
13. A. Neven and T. Bastin, The quantum separability problem is a simultaneous hollowisation matrix analysis problem, J. Phys. A: Math. Theor. 51 (315305), (2018).
14. M. Horodecki, P. Horodecki and R. Horodecki, Separability of mixed states: necessity and sufficient condition: Phys. Lett. A, 223(1), (1996).
15. Dagmar Bruss, Characterizing entanglement, J. Math. Phys. 43 4237 (2002).
16. D. Bohm, Quantum theory (Prentice-Hall, Englewood Cliffs, N. J., (1951).
17. N. Gisin, Phys. Lett. A 154, 201 (1991), Bell’s inequality holds for all non-product states.
18. J. F. Clauser, M. A. Horne, A. Shimony, R. A. Holt: Proposed experiment to test local hidden-variable theories, Phys. Rev. Lett. 23, 880-884 (1969).
19. B. S. Cirelson, Quantum generalizations of Bell’s inequality, Lett. Math. Phys. 4, 93 (1980).
20. Werner R. F., Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden variable model, 1989 Phys. Rev. A 40 4277.
21. N. D. Mermin, Extreme quantum entanglement in a superposition of macroscopic distinct states, Phys. Rev. Lett. 65, 1838 (1990).
22. N. Gisin and H. Bechmann-Pasquinucci, Bell inequality, Bell states and maximally entangled states for N qubits, Phys. Lett. A 246, 1, 1998.
23. M. Zukowski and Č. Brukner, Bell’s theorem for general N qubit states, Phys. Rev. Lett. 88, 210401(2002).
24. H. Gnonfin, L. Gouba *The Separability Problem in Two Qubits Revisited Symmetry* 15 Issue 11 (2023), 2089.
25. E. Schmidt, Math. Ann. 63 (1907), 433.
26. A. Ekert and P. L. Knight, Entangled quantum systems and the Schmidt decomposition , Am. J. Phys. 63 (1995) 415.
27. A. Peres, Quantum Theory: Concepts and Methods, Kluwer Academic Publishers, The Netherlands, 1995.
28. A. Peres, Higher order Schmidt decomposition, Phys. Lett. A 202 (1995, 16, quant-ph/9504006.
29. Peres A., Separability criterion for density matrices, 1996 Phys. Rev. Lett. 77 1413 -5.
30. Horodecki P., Separability criterion and inseparable mixed states with positive partial transposition, 1997 Phys. Lett. A 232 333-9.
31. N. Cerf, C. Adami, and R. M. Gingrich, Reduction criterion for separability, Phys. Rev. 60, 898 (1999).
32. Hill, S., Wootters, W. K. Entanglement of a pair of quantum bits . Phys. Rev. Lett. 1997, 78, 5022 - 5025.
33. Wootters, W., K., Entanglement of formation of an arbitrary state of two qubits, Phys. Rev. Lett., 1998, 80, 2245-2248.
34. Wootters W., K., Entanglement of formation of concurrence, quantum Inf. Comput. 2001, 1, 27-44.
35. Fei, S. M., Zhao, M., J., Chen, K., Wang, Z. X., Experimental determination of entanglement with a single measurement.
36. Marshall A. W., Olkin I., and Arnold B. C., Inequalities: theory of majorization and its applications, volume 143., Springer, 1979.
37. Nielsen M. A. and Kempe J., Separable states are more disordered globally than locally, Physical Review Letters, 86(22): 5184, 2001.
38. Rudolph O. Computable cross-norm criterion for separability. Lett. Math. Phys, 2004, 70: 57-64.
39. Rudolph O., Further results on the cross norm criterion for separability, arxiv: quant-ph/0202121 v1, (2002) Quantum Inf. Process. 4, 219 - 239 (2005).
40. Chen, K. and Wu, L. A. A matrix realignment method for recognizing entanglement, Quantum Inf. Comput. 3, 193 - 202 (2003).
41. J. de Vicente, Separability criteria based on the Block representation of density matrices, Quant. Inf. Comput., 7 (624), 2007.
42. Bloch F., Nuclear induction, 1946 Phys. Rev. 70 460.
43. Hioe F. T. and Eterly J. H., N-level coherence vector and higher conservation laws in quantum optics and quantum mechanics, 1981 Phys. Rev. Lett. 47 838.
44. Fano U., Pairs of two-level systems, 1983 Rev. Mod. Phys. 55 855
45. J. Shang, A. Asadian, H. Zhu and O. Gühne, Enhanced entanglement criterion via symmetric informationnally complete measurement, Phys. Rev. A, 98 (022309), 2018.

 doi:10.20944/preprints202508.0756.v1

46. I. Bengtsson and K. Zyczkowski. Geometry of quantum states: an introduction to quantum entanglement. Cambridge University Press, Cambridge, UK, 2006.
47. A. Jamio Ikowski, Linear transformations which preserve trace and positive semidefiniteness of operators, Rep. Math. Phys. 3, 275 (1972).
48. S. L. Woronowicz, Positive maps of low dimensional matrix algebras, Rep. Math. Phys. 10, 165 (1976).
49. M. D. Choi, Positive linear maps, Proc. Sympos. Pure Math 38, 583 (1982).
50. E. St∅rmer, Positive linear maps of operators algebras, Acta Math. 110, 233 (1963).
51. E. St∅rmer, Decomposable positive maps on C^*-algebras, Proc. Amer. Math. Soc. 86, 402 (1982).
52. B. Terhal, A family of indecomposable positive linear maps based on entangled states, Lin. Algebr. Appl. 323, 61 (2000).
53. O. Gühne and G. Toth. Entanglement detection, Phys. Rep., 474(1-6), 2009.
54. N. Ganguly, S. Adhikari, and A. Majumdar. Common entanglement witnesses and their characteristics. Quant. Inf. Pro., 12(425-436), 2013.
55. Philipp Hyllus, Otfried Gühne, Dagmar Bruss, and Maciej Lewenstein, Relations between entanglement witnesses and Bell inequalities, Phys. Rev. A72, 012321-2005.
56. Hofmann, H. F. and Takeuchi, S. Violation of local uncertainty relations as a signature of entanglement. Phys. Rev. A 68, 032103 (2003).
57. O. Gühne and M. Lewenstein, Entropic uncertainty relations and entanglement, Phys. Rev. A 70 022316 (2004).
58. Gühne O., Characterizing entanglement via uncertainty relations, 2004 Phys. Rev. Lett. 92 117903.
59. Gühne, Entanglement criteria based pn local uncertainty relations are strictly stronger than the computable cross norm criteria, 2006 Phys. Rev. A 74 010301(R).
60. Jun-Li Li and Cong-Feng Qiao, A necessary and sufficient criterion for the separability of quantum state, Sci. Rep. 8, 1442 (2018).
61. R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, (Cambridge University Press, Cambridge 1991).
62. R. A. Horn and C. R. Johnson, Matrix Analysis second edition, (Cambridge University Press, Cambridge 2013).
63. Bercovici, H., Collins, B., Dykema, K. and Li, W. S. Characterization of singular numbers of products of operators in matrix algebras and finite von Neumann algebras, Bull. Sci. Math. 139, 400-419 (2015).
64. Isiaka Aremua and Laure Gouba, Teleportation of a qubit using quasi-Bell states, *J. Phys. Commun.* 8 (2024) 095001.
65. Laure Gouba, Quantum fidelity, *International Journal of Geometric Methods in Modern Physics*, (2025), https://doi.org/10.1142/S0219887825400377.
66. R. Jozsa, Fidelity for mixed quantum states J. Mod. Opt. 41 (1994) 2315.
67. Nielsen, M. A., Chuang, I. L. (2010). Quantum Computation and Quantum Information: 10th Anniversary Edition. Cambridge: Cambridge University Press.
68. Prakash et al. Effect of decoherence on fidelity in teleportation using entangled coherent states, J. Phys. B: At. Mol. Opt. Phys. 40 (2007) 1613.
69. S. Popescu *Bell's inequalities versus teleportation: What is nonlocality? Phys. Rev. Lett.* 72 (1994) 797
70. R. Horodecki, M. Horedecki, P. Horodecki *Teleportation, Bell's inequalities and inseparability Phys. Lett. A* 222, Issues 1-2, (1996), 21-25.
71. Henderson, L., Hardy, L., Vedral, V. *Phys. Rev. A* 61 (2000) 062306.