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Generalized Mapping Theory — Used to Describe Phenomena That Cannot Be Characterized by Generalized Functions

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Abstract

Traditional theories of generalized functions (e.g., Schwartz's distribution theory and Colombeau algebras), while demonstrating excellent performance in static distribution problems, inherently fail to characterize the generative relationships between dynamic behaviors and their outcomes. As exemplified by college students' enlistment decisions where existing methods can only employ Dirac measures to represent final state distributions without capturing the influence of dynamic processes like medical examinations and qualification reviews, similar limitations manifest in cases ranging from blacksmith forging to graduate admissions, where conventional approaches merely reflect terminal states or statistical probabilities while discarding crucial causal mechanisms such as "hammering intensity-deformation response" or "exam preparation-admission outcome". To overcome these deficiencies, this paper proposes Generalized Mapping Theory (GMT) that establishes a quadruple framework consisting of object set A, operation set F, result set B and generative relation $\vdash$ to mathematically model dynamic behavior-outcome correlations, with its theoretical innovations embodied in three aspects: (1) explicit mathematization of physical/social behaviors as formal objects, (2) complete retention of dynamic generative pathways from operations to results, and (3) native support for multi-branch outcome scenarios. Theoretical analysis confirms that GMT not only fully subsumes all functionalities of traditional generalized functions (e.g., representing Dirac delta as identity-operation-generated mappings) but also solves their unsolvable dynamic behavior modeling problems, thereby providing innovative mathematical tools for quantitative research in materials science and social sciences.

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Introduction

The theory of generalized functions in modern mathematical physics (including Schwartz's distribution theory [1] and Colombeau algebras [2]) has achieved remarkable success in describing static distributions and singular functions. However, these conventional approaches exhibit fundamental limitations when addressing structure-generation problems induced by dynamic operational behaviors. Using the classic case study of "a blacksmith forging an iron sphere" as our point of departure, this paper reveals the inherent shortcomings of existing theories and proposes a novel Generalized Mapping Theory framework.

Consider the following concrete scenario:Let there exist a family of iron spheres represented by set A, where each sphere Ai is characterized by its volume and mass. The blacksmith performs a standardized forging operation F on each sphere, deforming it into an iron ellipsoid represented by set B. Here, the volume and mass of each ellipsoid remain unchanged, but the functions describing the shapes in set A and set B undergo transformations，now：

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1. Object set and result set: Set A (object set) represents the collection of iron spheres, and set B (result set) denotes the collection of iron ellipsoids.
2. Operation set: F denotes the operation set, which may vary according to specific practical conditions.
3. Key issue: the established mathematical description of $A \to B$.

Describing such relationships using traditional generalized function theory proves problematic due to fundamental limitations in characterizing dynamic behavior-outcome generative relationships: Firstly, while its mathematical objects (e.g., Schwartz distributions or Colombeau generalized functions) can handle singular functions, they are confined to primitive sets with numerical elements and cannot process function-element sets. Secondly, existing theories completely lose critical behavioral operation information - for instance, in material deformation problems, while capable of describing pre/post-deformation geometric differences, they fail to capture the physical essence including dynamic hammering force-deformation relationships, plastic deformation processes, and energy dissipation mechanisms. Furthermore, although Colombeau algebras can address certain nonlinear transformations, their regularization processes often violate physical constraints (e.g., volume conservation). More importantly, current theories lack "operation-outcome" generative logic expression: they neither describe how operational parameters determine result characteristics nor establish categorical correlations between them. The proposed Generalized Mapping Theory overcomes these limitations by: (1) extending research objects from numbers to general sets including functions, and (2) explicitly modeling behavioral transformations through operation sets while establishing complete generative relationships incorporating operational parameters, physical constraints, and structural correspondences. This theory not only fully subsumes traditional generalized function capabilities but also permits discipline-specific extensions, providing more powerful mathematical tools for dynamic process modeling.

Moreover, the function object sets processed by generalized functions are fundamentally still numerical sets. When dealing with non-numerical sets—for instance, if set A consists of time-domain functions and set B comprises frequency-domain functions, where each element in B is obtained by applying the Fourier transform to the corresponding element in A—can the mapping from set A to set B still be described using generalized functions?

Therefore, this paper introduces the Generalized Mapping Theory (GMT) to address the limitations of generalized functions in describing such phenomena. The paper is structured as follows:

Section 2 presents the fundamental mathematical framework.

Section 3 provides three case studies described using generalized mapping.

Section 4 compares GMT with traditional set-theoretic mapping theories.

Section 5 analyzes the advantages and limitations of GMT relative to generalized function theory.

Section 6 concludes the paper and discusses future research directions.

2. Mathematical Representation and Definitions

To effectively characterize the generative relationship between dynamic behaviors and their outcomes, this paper proposes the concept of Generalized Mapping [3]. We now provide a preliminary mathematical description of generalized mapping, which is categorized into two cases: the first disregards the influence of probability on the result set, while the second explicitly incorporates probabilistic effects on the result set.

We first examine the mathematical description for the case where probabilistic effects on the result set are not considered：

Generalized Mapping：$A \vdash B \mid F$

In the above expression, A denotes the object set, representing the entities to be operated upon, which may be analogized to the domain set in traditional set-theoretic mappings. B represents the result set, constituting the outcomes generated from A through operations. F stands for the operation set, which may embody either a conventional functional relationship or a set of operational rules.

For example, suppose set A consists of some audio segments, then A is expressed as: A={A1, A2---An}, where Ai (i=1---n) represents a certain audio segment. The operation set F denotes performing Fast Fourier Transform on each audio segment, here F can be expressed as F={fft}, and the result set B is the processed result set expressed as B={B1,B2---Bn}, where Bi represents the result of Ai being processed by Fast Fourier Transform.

We now examine the second case: the mathematical description incorporating probabilistic effects on the result set.

Generalized Mapping：A⊢B={C1,C2--Cn}|F/P

Here, A still represents the object set, same as before. B, as the result set, may contain multiple subsets. F is the operation set or process set. P denotes the probability set. Due to probabilistic effects, A may correspond to different result subsets - for instance, probability pi (i=1---n) would transform part of A's subsets into Ci (i=1---n).

Generalized mapping was proposed to address phenomena that cannot be adequately described by generalized functions [4–6]. Below are specific cases illustrating the definition of generalized mapping.

3. Typical Case Studies

3.1. Introduction

To facilitate readers' understanding of the fundamental definitions and mathematical framework of generalized mapping theory, this section elaborates through four representative examples. These cases were selected based on the following characteristics:

1. Inclusion of both deterministic and stochastic outcomes
2. Coverage of both engineering systems and social systems

Through detailed case analyses, readers will gain an intuitive grasp of the core concepts and practical applications of generalized mapping theory.

3.2. Blacksmith Forging Case

Consider a typical metal forging scenario where a blacksmith transforms an initially spherical iron block into an ellipsoid by controlling hammering force and direction. This process involves two key elements:

1. Specific hammering operations including parameters such as force magnitude and angle
2. The plastic deformation process undergone by the iron block

This case effectively demonstrates how continuous operations can induce systematic geometric modifications. Referring to the definition of generalized mapping in Section 2, this case can be described using generalized mapping as：A⊢B|F

In this case, set A as the original set is further expressed as A={r(i)}, where r(i) denotes the radius of the ith sphere (i=1---n). The operation set F can be represented as F={[Fi_{min},Fi_{max}]}, where Fi_{min} and Fi_{max} respectively indicate the magnitude and direction of the minimum force and the maximum force applied to the ith sphere. The result set B can be expressed as B={E(ai,bi,ci)|ai>0,bi>0,ci>0}, where ai, bi, and ci represent the lengths of the three principal axes of the ith ellipsoid.

3.3. Military Enlistment Decision Case

The college student enlistment selection process represents a classic multi-stage decision procedure. Applicants must sequentially pass evaluation stages including medical examination, political review, and interviews, with each stage having explicit qualification criteria. The final outcome depends on the combined results of all evaluation stages, demonstrating characteristics of discrete operations.

Referring to the generalized mapping definition in Section 2, this case's generalized mapping relationship can be expressed as:

A⊢B={C1,C2--Cn}|F/P

Here, the original set A can be expressed as A={Ai}, (i=1---n), where Ai represents the ith college student. The operation set F can be denoted as F={st1, st2, --, stn}, with "st" being the abbreviation for "step", where the quantity is enumerated according to the actual enlistment process. The result set B can be represented as B={C1, C2}, where C1={C1i} (i=1---n) denotes the set of students who failed to enlist, and C2={C2i} (i=1---n) represents the set of successfully enlisted students. Here, P can be expressed as P={Pi1, Pi2}, (i=1---n), where Pi1 indicates the probability of the ith student's failure, and Pi2 denotes the probability of the ith student's success. This demonstrates how generalized mapping specifically extends to different set representation forms.

3.4. Graduate Admission Case

The graduate admissions process is based on quantitative assessment of examination scores. Admission institutions establish clear cutoff thresholds and accept all candidates who meet these score requirements. This case demonstrates a threshold-based deterministic decision process, while simultaneously establishing foundations for subsequent discussion of stochastic admission mechanisms.

Furthermore, while the graduate admission case is essentially similar to the military enlistment case discussed earlier, they can be distinguished through different mathematical representations of their corresponding sets. This further illustrates the extensibility of generalized mapping.

According to the mathematical definition of generalized mapping presented in Section 2, the graduate admission case can be represented through generalized mapping as:

A⊢B={C1,C2--Cn}|F/P

This form is identical to the aforementioned military enlistment case. To distinguish between the two, we can differentiate their further mathematical representations of corresponding sets.

Here, the original set A can still be expressed as A={Ai}, (i=1---n), where Ai represents the ith college student. The operation set F can be represented as F={Fi} (i=1---n), with Fi denoting the examination score of the ith student. The result set B is simplified to B={admitted, not admitted}. The probability set P, similar to the military case, is expressed as P={Pi1, Pi2}, (i=1---n), where Pi1 indicates the probability of the ith student being admitted, and Pi2 represents the probability of the ith student not being admitted.

This demonstrates the strong extensibility of generalized mapping.

3.5. Chip Manufacturing Process Case

The chip manufacturing process involves highly complex step. The production flow is based on quantitative control of process parameters. Foundries establish clear process specifications (e.g., line width, thickness, doping concentration, etc.) and process all wafers meeting these standards. This case demonstrates a physics-threshold-based deterministic manufacturing process, while simultaneously providing foundation for subsequent discussion of process variability and random defect effects.

Moreover, while essentially sharing similar screening logic with the graduate admission case discussed earlier, the chip production flow can be distinguished through different mathematical representations of its physical transformations and manufacturing constraints, further demonstrating the extensibility of generalized mapping.

According to the mathematical definition of generalized mapping presented in Section 2, the chip manufacturing process can be represented through generalized mapping as:

A⊢B={success，fail}|F/P

Here, the original set A can be considered as the collection of each die on the wafer, and set B can be regarded as the collection of corresponding chips after the wafer undergoes all process steps and packaging. In this case, the number of elements in set B must correspond to that in set A.

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Alternatively, if we consider only production success/failure, then set B contains only two elements: success and fail.

In actual wafer production applications, the primary concern is yield calculation - determining how many dies successfully become qualified chips. Therefore, set B here contains only two elements: success and fail.

Here, the operation set F can be expressed as F={Fi} (i=1---n), where Fi represents the ith process step. The set P can be represented as P={Pi} (i=1---n), with Pi denoting the success probability of the ith process step.

The chip case again demonstrates the unique extensibility of generalized mapping - it can model real-world scenarios by adjusting set elements according to actual requirements.

Moreover, generalized mapping can also describe ordinary functions. For example, the function y=sinx (real number set R) can be expressed via generalized mapping as: R→Y|F={y=sinx}. Here, the original set A is directly represented by the real number set R, the result set B is represented by the function's range set, and the operation F corresponds to the function's relational expression itself.

Naturally, while ordinary functions can be described using generalized mapping, the description becomes relatively cumbersome. Therefore, for scenarios that can be directly described by ordinary functions, using generalized mapping is not recommended. As for how to use generalized mapping to describe generalized functions, this requires further exploration and will be discussed in Section 5 of this paper.

3.6. Case Study Summary

Through the analysis of the three representative cases above, we have concretely demonstrated the practical meanings of the original set, operation set, result set, and generative relationship in generalized mapping theory. These cases collectively illustrate both the practical applications of generalized mapping and its powerful extensibility. Furthermore, the capability of generalized mapping to describe ordinary functions further confirms that it represents an advancement beyond conventional function theory. We now proceed to examine the relationship between generalized mapping and traditional set-theoretic mappings.

4. Relationship Between Generalized Mapping and Traditional Set-Theoretic Mapping

Since the concept of generalized mapping represents, to some extent, an advancement and extension of traditional set-theoretic mapping, it is necessary to conduct a comparative analysis between the two.

4.1. Limitations of Traditional Mapping Theory

The mapping concept in traditional set theory (typically denoted as f: A→B) establishes a static input-output correspondence. While this mathematical description is elegant in its simplicity, it demonstrates significant shortcomings when handling dynamic processes in the real world. Traditional mapping focuses entirely on the correspondence between domain A and domain B, while embedding the actual transformation process within the internal implementation of function rule f. This approach results in the loss of two critical pieces of information: first, the explicit representation of behavioral processes, and second, intermediate information about dynamic transformations [7].

Taking material deformation processes as an example, traditional mapping can only describe the correspondence between a material's initial state and final form, while completely ignoring the influence of key operational parameters like hammering force and duration. Similarly, in social decision-making processes, traditional methods can only present final decisions while failing to capture core elements such as evaluation criteria and decision procedures. These limitations render traditional mapping theory inadequate for domains requiring precise modeling of dynamic systems.

4.2. Core Architecture of Generalized Mapping

To overcome these limitations, we propose generalized mapping theory, expressed in its basic form as A⊢B|F. This representation consists of three core components: the object set A representing original elements to be processed; the operation set F containing all possible transformation behaviors; and the result set B comprising outputs generated from A through operations in F. The symbol "⊢" denotes the generative relationship, emphasizing the dynamic production process from objects to results [8].

Compared with traditional mapping, generalized mapping's innovations primarily manifest in:

(1) Explicit representation of operations: Elevating transformation processes from implicit function rules to explicit mathematical objects;
(2) Complete retention of dynamic information: Recording the full transformation path from original objects to final results;
(3) Adaptability to complex systems: Capable of handling multi-stage, multi-factor transformation processes.

4.3. Probabilistic Generalized Mapping Extension

In practical applications, many transformation processes exhibit probabilistic characteristics. We therefore extend the framework to probabilistic generalized mapping, denoted as A⊢B={C1,C2,...}|F/P. Here, P represents the probability set that quantifies the likelihood of different operations producing various outcomes. This representation is particularly suitable for describing decision-making processes with uncertainty.

Taking graduate admissions as an example: the object set A is the applicant pool, the operation set F can be expanded to include evaluation stages like written exams and interviews, the probability set P can incorporate pass rates for each stage, and the result set B={admitted, waitlisted, rejected} represents possible outcome branches. This constitutes a further extension of the graduate admission case discussed in Section 3 [9].

This modeling approach not only outputs final admission decisions but also completely preserves all evaluation stages and their probabilistic influences, providing richer information for decision analysis.

4.4. Theoretical Advantages and Application Prospects

Compared to traditional methods, generalized mapping theory exhibits multiple advantages. First, it breaks through the limitations of static correspondence by achieving mathematical expression of dynamic processes. Second, through the explicit representation of operation sets, it makes transformation processes transparent and analyzable. Third, its probabilistic extension provides a systematic framework for modeling uncertainty.

Through continuous theoretical refinement and application validation, generalized mapping is poised to develop into a standard mathematical language for describing dynamic systems, offering powerful tools for modeling and analyzing complex processes. This theoretical framework not only expands the conceptual boundaries of traditional mapping but, more importantly, establishes explicit connections between behaviors and outcomes, making mathematical models better aligned with real-world operational mechanisms.

5. Comparative Study of Generalized Mapping and Generalized Function Theory

Generalized mapping was originally proposed to address phenomena that generalized functions cannot directly describe, making a comparison between the two essential.

5.1. Fundamental Differences Between Generalized Mapping and Generalized Functions

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Generalized function theory (including Schwartz distributions and Colombeau algebras) primarily deals with generalized function objects that extend traditional function concepts, such as the Dirac δ function. The core of these theories treats generalized functions as linear functionals acting on test function spaces, describing their properties through pairings like $\langle \delta, \varphi \rangle = \varphi(0)$. However, this approach has inherent limitations: it can only specify input-output correspondences while completely losing the actual processes generating these correspondences. For example, the δ function can describe a point mass's density distribution but cannot express how this distribution forms through physical processes.

In contrast, generalized mapping theory fundamentally extends this framework. It not only considers object set A and result set B but also explicitly introduces operation set F as an independent mathematical object, connecting all three through the generative relation ⊢. This structure enables dynamic process description. Taking the δ function as an example, under the generalized mapping framework it can be represented as: object set A being a point, operation set F containing the limit squeezing process, and result set B being the resulting δ distribution. This representation includes not only the final outcome but also preserves key operational information about its generation [10].

5.2. Unique Advantages of Generalized Mapping

The core strength of generalized mapping theory lies in its dynamic process preservation capability. Taking plastic deformation in materials science as an example, traditional generalized functions can only describe stress distribution changes before and after deformation, whereas generalized mapping can fully document the hammering operations (including specific parameters like force magnitude and direction) that cause these changes. This descriptive approach provides richer informational dimensions for understanding material behavior.

In probabilistic process modeling, generalized mapping demonstrates superior expressive power. Quantum measurement serves as a typical case: traditional methods only yield probability density distributions, while generalized mapping can explicitly describe the measurement operations themselves and correlate different outcome branches (e.g., different eigenstates) with their corresponding probabilities. This treatment aligns more closely with physical reality.

Another crucial feature of generalized mapping is its cross-disciplinary applicability. For modeling complex systems like social decision-making, traditional mathematical tools often struggle with multi-stage, multi-factor dynamic processes. By making each operational stage explicit and allowing probabilistic branching outcomes, generalized mapping provides systematic modeling frameworks for such problems. For instance, in admission decisions, it can clearly express how written exams, interviews, political reviews and other stages influence final admission outcomes [11].

5.3. Specific Case Examples

Some generalized functions can be represented using generalized mapping. For instance, the local average distribution with generalized function representation:

$f_\varepsilon(x)=(1/\varepsilon)\chi_{[-\varepsilon/2,\varepsilon/2]}$

can be expressed via generalized mapping as:

$A=\{x\}\vdash B=\{f_\varepsilon(x)\} \mid F=\{\text{interval averaging operation (parameter } \varepsilon)\}$

This demonstrates that theoretically, what can be represented by generalized functions can also be expressed through generalized mapping, similar to ordinary functions. Below are examples that can be represented by generalized mapping but not by generalized functions:

(1) Time-varying system impulse response

Problem: Traditional δ(t) cannot describe time-varying impact processes

Generalized mapping solution: $A=\{\text{system}\}\vdash B=\{\text{response}\} \mid F=\{\text{time-varying impact operation } (t,F(t))\}$

(2) Social behavior causal chains

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Problem: Generalized functions cannot model "policy announcement → public reaction → economic impact" transmission

Generalized mapping solution: $A \vdash B_1 | F_1(\text{policy}) \vdash B_2 | F_2(\text{public behavior})$...

These examples show that generalized mapping has a broader representation space than generalized functions [12]. Of course, for cases directly representable by generalized functions, using generalized functions may sometimes be more convenient.

6. Summary and Prospects

Building upon traditional generalized function theory, this paper proposes a novel theoretical framework of generalized mapping to address the mathematical description of dynamic operations and their outcome-generating relationships. By introducing a triple structure of object set, operation set, and result set with axiomatic definitions of generative relations, this framework provides a more complete mathematical language for describing dynamic systems involving operational processes.

Research demonstrates that generalized mapping not only encompasses the descriptive capabilities of traditional generalized functions but, more importantly, can explicitly express the influence mechanisms of operational parameters on outcomes - a feature preliminarily verified through cases like material deformation and social decision-making. Compared with existing theories, generalized mapping's primary advantages lie in its dynamic process preservation capability and support for multi-branch outcomes, offering new possibilities for modeling complex systems. Theoretical analysis shows that all objects describable by traditional generalized functions can be transformed into special cases of generalized mapping, while the converse does not necessarily hold, thereby expanding the application scope of mathematical tools to some extent.

It should be noted that this theoretical framework remains in its early developmental stages. Its rigor, computational feasibility, and applicability across broader domains all require further investigation. Future work will focus on theoretical refinements and practical applications, particularly in modeling complex systems requiring simultaneous consideration of operational processes and outcome relationships. This study provides only an initial direction for theoretical exploration in related fields, and its practical value needs confirmation through subsequent in-depth research and empirical validation.

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