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*Article*

# Fuzzy Geometry

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**Abstract**

Probabilistic version of geometry, labeled the "fuzzy" geometry, is introduced. The fifth postulate of Euclid (Playfair's axiom) is adopted in the following probabilistic form: consider a line and a point not an line, there is exactly one line through the point with probability $P$, where $0 \le P \le 1$. Playfair's axiom is logically independent of the rest of the Hilbert system of axioms of the Euclidian geometry. Thus, the probabilistic version of the Playfair axiom may be combined with other Hilbert axioms. $P = 1$ corresponds to the standard Euclidean geometry; $P = 0$ corresponds to the elliptic- and hyperbolic-like geometries. $0 < P < 1$ corresponds to the fuzzy geometry. Parallel constructions in this case are Bernoulli trials. Theorems of the fuzzy geometry are discussed. Given a triangle and a line drawn from a vertex parallel to the opposite side, the event that this line is actually parallel occurs with probability $P$. Otherwise, the line may intersect the side or diverge. Parallelism is not transitive in the fuzzy geometry. Fuzzy geometry occurs on the surface with a stochastically variable Gaussian curvature. Alternative fuzzy geometries adopting various versions of the probabilistic Playfair axiom are introduced. Probabilistic non-Archimedean fuzzy geometry is addressed. Applications of the fuzzy geometry are discussed.

**Keywords:** fuzzy geometry; probabilistic fifth postulate; Hilbert axioms; probabilistic Archimedes axiom; stochastic curvature

**MSC:** 51P05; 60D05; 60A05.

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## 1. Introduction

The Elements of Euclid represented for centuries the very model of scientific and deductive reasoning [1,2]. Elements were published, translated, edited and commented upon thousands of times, and these publications shaped the scientific method and the mathematical style of thinking of many centuries [1,2]. Euclid's approach was used to build further mathematical theories. Moreover, deductive structure of the Euclid proofs was studied by mathematicians, physicists, logicians and epistemologists as the perfect ideal of scientific reason itself [1,2]. The fifth postulate of Euclid, often called the parallel postulate, has a rich and complex history that spans more than two millennia. In Euclidean geometry, postulates (also called axioms) are the basic assumptions or starting points that are accepted without proof. From these, all theorems and propositions in geometry are logically deduced [1,2]. Among other postulates, the fifth postulate (abbreviated below FP) was not self-evident and less intuitive. In its original form FP states that if a secant cuts two straight lines forming side angles whose sum is less than two straight, the two long enough lines are cut on this same side. Or, in other words: if a line segment intersects two straight lines forming two interior angles on the same side that are less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles. The philosopher Proclus (c. 412–485 AD), a late Neoplatonist commentator on Euclid questioned: why it should be accepted as a postulate rather than proven [3]? Many geometers (including Leibnitz) throughout history tried to prove it from the other four, believing it to be a theorem rather than an axiom [4]. Ibn al-Haytham (Alhazen, c. 965–1040), a key figure in the Islamic Golden Age, made a deep and innovative critique the FP, and suggested that lines equidistant from one another never meet (a kind of converse to the parallel

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postulate) [5]. Omar Khayyam (c. 1048–1131), the Persian mathematician, poet, and astronomer reshaped FP as follows: if two straight lines intersect a third line such that the sum of the interior angles on the same side is equal to two right angles, and if the lines are perpendicular to a common transversal, then the distance between them remains constant, and they do not intersect - that is, they are parallel [6].

The exact meaning and status of the FP of Euclidian geometry continues to attract the attention of scientists. Hilbert's Euclidean Axiom of Parallels is formulated as follows: for every line $l$ and every point $A$ not on $l$, there does not exist more than one line through $A$ parallel to $l$ [10]. This formulation is equivalent to Playfair's Axiom (1795), stating that: through a given point not on a given line, there is exactly one line parallel to the given line.

The development and eventual replacement of Euclid's FP by Lobachevski and others marked a turning point in the history of mathematics, leading to the birth of non-Euclidean geometry [11,12]. Lobachevski was first to explicitly reject the FP and systematically develop a consistent geometry without it. Lobachevski himself called his system "imaginary geometry", now known as "hyperbolic geometry". In Lobachevski geometry: given a line and a point not on it, infinitely many parallels can be drawn through the point [11–13]. The sum of angles in a triangle is less than $\pi$, and there is a maximum area for triangles with given angle sums. The first presentation of new geometry took place at the Department of Physical and Mathematical Sciences, at Kazan University, on 7 February 1826. Lobachevski published his results in 1829 paper "On the Principles of Geometry", published in Russian. János Bolyai, Hungarian mathematician independently developed hyperbolic geometry around the same time [13]. His work was published in an appendix to his father Farkas Bolyai's book in 1832 [13]. Carl Friedrich Gauss independently developed ideas of non-Euclidean geometry but never published them. These results promoted the development of Riemannian geometry, which plays a central role in general relativity [14–16].

The next intellectual step is suggested in the development of the Euclidian geometry. The probabilistic nature of FP is assumed; namely, we adopt that for a given a line $l$ and a point $A \notin l$, there is a exactly one line through $A$ with probability $P$, where $0 \le P \le 1$. The suggested probabilistic geometry is labeled in the presented paper the "fuzzy geometry"; thus, resembling the famous fuzzy logic, developed by Lotfi Aliasger Zadeh [17–20]. Fuzzy logic is a form of probabilistic logic that allows for reasoning with degrees of truth, rather than the binary true/false values used in classical (Boolean) logic [17–20].

## 2. Results

### *2.1. System of Axioms of Hilbert-P-Fuzzy Geometry and Its Consequences*

Let us start from summarizing Hilbert's system of axioms. Hilbert's Axioms of Euclidean Geometry include:

i) Group I: Axioms of Incidence. These axioms describe how points, lines, and planes relate.

I.1: For every two distinct points, there exists a line that contains both of them.
I.2: A line contains at least two points.
I.3: There exist at least three non-collinear points (not all on the same line).
I.4: For any three points not on a line, there is a plane, that contains them.
I.5: Every plane contains at least three non-collinear points.
I.6: If two points of a line lie in a plane, the entire line lies in the plane.
I.7: If two planes intersect, their intersection is a line.
I.8: There exist at least four points not lying in the same plane.

ii) Group II: Axioms of Order ("betweenness"). These axioms define the concept of one point lying between two others.

II.1: If point $B$ lies between $A$ and $C$, then all three points are distinct and lie on the same line.

II.2: For any two points $A$ and $C$, there exists a point B on the line $AC$ such that $C$ lies between $A$ and $B$.

II.3: Of any three points on a line, exactly one lies between the other two.

II.4: Given three points on a line, we can name them $A$, $B$, $C$ such that $B$ is between $A$ and $C$.

II.5 (Pasch's Axiom): If a line entering a triangle from one side intersects one side, then it must also intersect another side.

Group III: Axioms of Congruence. These axioms deal with equality of segments and angles.

III.1: Given a segment $AB$ and a ray $CD$, there is a unique point E on that ray such that segment $AB$=$CE$.

III.2: Congruence is symmetric and transitive.

III.3: If two segments are congruent to the same segment, they are congruent to each other.

III.4: Given two angles, there is a congruent copy of one angle placed at a given ray.

III.5 (Side-Angle-Side): If in two triangles, two sides and the included angle are congruent, then the triangles are congruent.

Group IV: Axiom of Parallels

This is Hilbert's version of Euclid's Fifth Postulate.

IV (Playfair's Axiom): Given a line $l$ and a point $A$ not on $l$, there is at most one line through $A$ parallel to $l$.

Group V: Axioms of Continuity. These axioms ensure the completeness of the geometric space (similar to real numbers being complete).

V.1 (Axiom of Archimedes): There is no infinitely small or infinitely large length segments can be added finitely to surpass any given segment.

V.2 (Axiom of Line Completeness / Dedekind Cut Axiom): If a line is divided into two classes such that every point of the first lies to the left of every point of the second, then there exists a unique point separating the two classes.

We adopt the aforementioned Hilbert system of axioms, with one, however, essential exception. We replace Euclid's Fifth Postulate (Playfair's Axiom) with it probabilistic version:

**Probabilistic Parallel Axiom**:

Given a line $l$ and a point $A \notin l$, there is exactly one line through $A$ with probability $P$, where $0 \leq P \leq 1$.

Or, as formally written:

For any point $A$ and line $l$ $A \notin l$, the event

$\exists! \, m \; (line \; m): A \in m \wedge m \parallel l$ occurs with probability $P \in [0,1]$.

This introduces a modal probabilistic operator $\hat{P}(\varphi) = P$, where $\varphi$ is a logical statement and $P$ its probability. This yields Eq. 1:

$\hat{P}(\exists! \, m$ such as that $A \in m \wedge m \parallel l) = P$ (1)

This approach creates a hybrid ("fuzzy", or Hilbert-$P$ fuzzy) geometric model, blending elements of Euclidean and non-Euclidean geometry, governed by a random choice at each parallel construction event. We assume that the rest of Hilbert's axioms (Groups I–III, V) remain unchanged, namely: well-defined points, lines, planes, "betweenness" and congruence, completeness and Archimedean properties and only the parallel behavior becomes probabilistic. Let us denote this modified system by "Hilbert–$P$ fuzzy geometry", where $P \in [0,1]$ is a fixed parameter. It should be emphasized that it is possible the keep all other Hilbert axioms unchanged and modify only the Fifth Postulate (the Playfair axiom).

It is easily seen, that well known geometries appear as the particular cases of the Hilbert–$P$ fuzzy geometry, namely:

i) $P = 1$ corresponds to the standard Euclidean geometry: Playfair's axiom holds always, All classical theorems of the Euclidian geometry (e.g., triangle angle sum $S = \alpha + \beta + \gamma = \pi$ remain valid).

ii) $P = 0$ corresponds to the elliptic-like geometry. No parallels through external points (like great circles on a sphere). $S = \alpha + \beta + \gamma > \pi$ holds.

iii) $P = 0$ corresponds to hyperbolic-like geometry, with the interpretation that multiple parallels are allowed. In hyperbolic geometry, through a point not one a line, but there are infinitely many lines that do not intersect the given line - i.e., infinitely many parallels. In the suggested probabilistic axiom only one parallel with probability $P$ is possible, not multiple. So to properly correspond to hyperbolic geometry, we interpret $P = 0$.

iv) $0 < P < 1$ corresponds to the stochastic fuzzy geometry. Parallel constructions in this case are Bernoulli trials. Geometric consequences become probabilistic statements. Theorems take the form: with probability $P^n$, there exist $n$ successive parallels to a given line.

Let $l_1$ and $l_2$ be two lines constructed through a sequence of $k$ probabilistic parallel choices. Then the probability they are parallel is $P^k$. Now the existence of unique parallels is not guaranteed. Consider a line $l$ and a point $A \notin l$, there is a line through $A$ with probability $P$, where $0 \le P \le 1$, otherwise, either none or many exist. If parallel fails (with probability $1 - P$), we may enter a non-Euclidean branch.

Let us exemplify the suggested approach and address the angle sum $S$ of a triangle which becomes random within the introduced fuzzy geometry. With probability $P$, the triangle is Euclidean and $S = \pi$; with probability $1 - P$ the triangle may be non-Euclidean (say hyperbolic), and then: $S < \pi$, (if hyperbolic-like behavior dominates). This gives rise to a probabilistic angle sum, with expected value $\mathcal{E}(S)$:

$$\mathcal{E}(S) = P(S) \cdot \pi + \big(1 - P(S)\big)\mathcal{E}(S_{non-euclidian}) \quad (2)$$

What is the exact meaning of $\mathcal{E}(S)$? It supplies within the suggested approach the average value of the angle sum $S$ of a triangle, over many repeated constructions, where each construction involves drawing parallels with probability $P(S)$. In the suggested Hilbert-$P$-fuzzy geometry, since parallelism is probabilistic, the outcome of a triangle's angle sum is not deterministic. It depends on whether the "parallel step" succeeded. With probability $P: S = \pi$, and with probability $1 - P: S_{non-euclidian} < \pi$. Thus, $\mathcal{E}(S)$ given by Eq. 1 is the weighted average of these two possible outcomes. Thus, the following theorem is demonstrated, based on the Hilbert-$P$-fuzzy system of axioms:

**Theorem 1**. (Expected angle sum of a triangle):

The expected angle sum of a triangle $\mathcal{E}(S)$ is given by: $\mathcal{E}(S) = P(S) \cdot \pi + \big(1 - P(S)\big)\mathcal{E}(S_{non-euclidian})$.

A number of theorems of the Hilbert fuzzy geometry are of the probabilistic nature. Let us list some of these theorems.

**Theorem 2**. (Probability of parallel side in triangle):

Given a triangle and a line drawn from a vertex parallel to the opposite side, the event that this line is actually parallel occurs with probability $P$. Otherwise, the line may intersect the side or diverge.

Formal statement: BC is a triangle, formed by points $A$, $B$, $C$, and line $BC$. Then:

$$\hat{P}[\exists!\, m(line\ m) \wedge A \in m \wedge m \parallel BC] = P \quad (3)$$

Let triangle

**Theorem 3**. Non-deterministic Midpoint Theorem.

In triangle $ABC$, let $M$ and $N$ be midpoints of sides $AB$ and $AC$, respectively. Then line $MN$ is parallel to $BC$ with probability $P$, and not parallel with probability $1 - P$.

*Corollary*: The segment MN has length $\frac{1}{2}BC$ with probability $P$.

$$\hat{P}\left[MN \parallel BC \wedge (lengthMN = \tfrac{1}{2} lengthBC) = P\right] \quad (4)$$

**Theorem 4**. (Stochastic behavior of parallelograms, parallelograms closure).

If a quadrilateral has opposite sides built using "parallel" constructions, it closes into a parallelogram with probability $P^2$, since both pairs must independently satisfy the parallel condition.

**Theorem 5**. (Parallelism is not transitive):

If $a \parallel b$ with probability $P_1$, and $b \parallel c$ with probability $P_2$. Then $a \parallel c$ with probability at most $P_1 P_2$ assuming independence of events.

*2.2. Physical Realization of the Hilbert-P Fuzzy Geometry*

Consider the surface with a variable Gaussian curvature $K = k_1 k_2$, where $k_1$ and $k_2$ are the principal curvatures at a point on the surface. Adopt that within time $\tau$ the Gaussian curvature of the surface is $K = 0$ (this corresponds to a plane), and within the same time $\tau$ the Gaussian curvature of the surface is $K < 0$ (hyperbolic surface having a saddle point). Consider a triangle on the surface. The angle sum $S$ of a triangle is random within the fuzzy geometry. With probability $= \frac{1}{2}$, the triangle is Euclidean and $S = \pi$; with probability $1 - P = \frac{1}{2}$ the triangle may be non-Euclidean (say, hyperbolic), and then: $S = S_{non-eucliduan} < \pi$, (if hyperbolic-like behavior dominates). The expected value of the angles sum $\mathcal{E}(S)$ is calculated with Eq. 1, as follows: $\mathcal{E}(S) = \frac{1}{2}\pi + \frac{1}{2}\mathcal{E}S_{non-euclidian}$.

Now consider the more general case, in which the Gaussian curvature of the surface is $K = 0$ within time $\tau_1$ and within $\tau_2$ the Gaussian curvature of the surface is $K < 0$. Thus,

$$P(S = \pi) = \frac{\tau_1}{\tau_1 + \tau_2} = P_1 \quad (5)$$

And correspondingly:

$$\boldsymbol{P(S < \pi) = \frac{\tau_2}{\tau_1 + \tau_2} = P_2} \quad (6)$$

Obviously, the normality condition $\boldsymbol{P_1 + P_2 = 1}$ holds. Thus, the introduced fuzzy geometry works for random, stochastic surfaces with a variable curvature.

*2.3. Alternative Fuzzy Geometries*

The introduced fuzzy geometry is not unique. Alternative fuzzy geometries are possible.

Consider the fuzzy geometry adopting the Hilbert system of axioms and FP formulated as follows:

**Probabilistic Parallel Axiom II**:

Given a line $l$ and a point $A \notin l$, no line parallel to $l$ through $A$ with probability $P'$ is possible, where $0 \le P \le 1$.

Consider one more version of the fuzzy geometry adopting the Hilbert system of axioms and FP formulated as follows:

**Probabilistic Parallel Axiom III**:

Given a line $l$ and a point $A \notin l$, more than one line parallel to $l$ through $A$ with probability $P''$ is possible, where $0 \le P \le 1$.

Let us summarize the suggested systems of axioms of fuzzy geometry in Table 1.

**Table 1.** Applicability of various types of fuzzy geometry for particular surfaces.

| Geometry | $P'$ | $P$ | $P''$ |
|---|---|---|---|
| Elliptic | 1 | 0 | 0 |
| Euclidian | 0 | 1 | 0 |
| Hyperbolic | 0 | 0 | 1 |

*2.4. Fuzzy Geometry Adopting the Fuzzy Version of the First Axiom of the Hilbert Geometry*

Additional versions of the fuzzy geometry are possible. The first axiom of the Hilbert geometry states: for every two distinct points, there exists a line that contains both of them.

We modify the first axiom as follows: For every two distinct points, there exists with probability $\boldsymbol{0 \le P \le 1}$ a line that contains both of them. This creates a new version of stochastic/randomized/fuzzy geometry. Under this probabilistic axiom: for two points $A$ and $B$, a line may or may not exist through them. The existence of such a line is not deterministic, but follows a pre-fixed probability distribution. The introduced axiom interferes with other axioms of the Hilbert geometry. Consider axiom II.1: If point $B$ lies between $A$ and $C$, then all three points are distinct and

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lie on the same line. Betweenness requires a line between the points. If no line exists between $A$ and $C$ (or others), betweenness is undefined. Thus, the entire system of Hilbert axioms should be changed. Thus, we supply only a very preliminary sketch of this version of the fuzzy geometry. For example, the theorem establishing the probability of triangle existence appears now as follows:

**Theorem**: Probability of Triangle Existence

Given three non-collinear points $A$, $B$, $C$ the probability that all three pairwise connecting lines $AB$, $BC$, $CA$ is $\boldsymbol{P^3}$.

It should be emphasized that the probabilistic version of the first Hilbert axiom destroys the entire version of the Hilbert/Euclidian geometry, due to the fact, that it interferes with other axioms. In classical Euclidian/Hilbert geometry: any two points determine a unique line. Now For a pair $A$, $B$, the probability that there is a line through both is $P$. Thus, there may exist pairs of points not connected by a line, i.e., geometrically disconnected pairs. Hence, collinearity becomes probabilistic. Congruence axioms (Group III, Section 2.1) fail. Congruence axioms that the underlying incidence structure is intact. If segment $AB$ doesn't exist, then its congruence relations are undefined. The probabilistic modification of Axiom I.1 interacts with any version of the parallel postulate. Even if the parallel postulate holds, lines connecting points may not exist. The concept of parallelism now has a domain of definition: only among point-line pairs where both exist. Thus, this version of probabilistic system of axioms touches the entire axiomatic system, and reflects a randomized or discrete space, where geometry may fail locally due to missing connections and it resembles percolation models in statistical physics: some connections exist, others don't. When $\boldsymbol{P = 1}$ it recovers classical Euclidean geometry, and when, in turn $\boldsymbol{P = 0}$ all space is disconnected, namely no lines exist. We conclude that the probabilistic version of the first axiom of Euclid/Hilbert axiomatic system entails deep changes in the entire structure of geometry due to the fact, that this version interferes with other axioms. The situation is quite different, when we change the Axiom of Archimedes (Axiom V.1, see Section 2.1).

### *2.5. Fuzzy Geometry Emerging from the Probabilistic Version of the Axiom of Archimedes*

The situation is quite different when the Continuity Axioms (Group V) are adopted in their probabilistic version. Hilbert introduced continuity axioms (Dedekind completeness or Archimedean properties, see Section 2.1). These axioms are independent of incidence, order, and congruence. Modifying or removing them leads to non-Archimedean or discrete geometries. Thus, they can be changed independently. Recall that the Archimedes Axiom as it appears in Book V of Euclid's Elements as Definition 4:

Given any two segments $AB$ and $CD$, there exists a positive integer $n$ such that by laying off the segment $CD$ consecutively $n$ times along a ray starting at $A$ the total length exceeds the length of $AB$.

The Archimedes Axiom states that there is no infinitely long segment: even a tiny length can add up to pass any large segment with enough repetitions. There is also no infinitely small segment: if a segment were truly "infinitely small", no matter how many times you stack it, it would never pass a finite segment — which contradicts the axiom.

Now we introduce the following probabilistic version of the Archimedes Axiom:

Given any two segments $AB$ and $CD$, there exists with a probability $\boldsymbol{0 \leq P \leq 1}$ a positive integer $n$ such that by laying off the segment $CD$ consecutively $n$ times along a ray starting at $A$ the total length exceeds the length of $AB$. What do we have in extremal cases?

Let us start from $\boldsymbol{P = 1}$. This is the standard Archimedean case, just expressed probabilistically, namely with certainty, we can lay off $CD$ repeatedly to surpass $AB$.

The geometry behaves like standard Euclidean geometry, namely:

i) No infinitesimal segments exist.

ii) Segment lengths can be compared meaningfully.

iii) Triangle inequality and other classical theorems still hold.

iv) The real number system underlies the segment-length arithmetic.

The second extremal case corresponds to $\boldsymbol{P} = \mathbf{0}$. This is the non-Archimedean case. There is zero probability that any finite number of consecutive $CD$ segments laid from $A$ will ever surpass $AB$. In this case:

i) $CD$ is infinitesimal compared to $AB$.

ii) No finite sum $n \cdot CD$ ever gets close to $AB$.

iii) Segment length comparison fails: the field of segment lengths is now non-Archimedean. The geometry becomes non-Euclidean in a fundamental way.

iv) Triangle inequality may break down or become trivial.

v) This setting resembles non-standard analysis or hyperreal geometries, where infinitesimals exist.

Of course, we supplied only a brief sketch of probabilistic, non-Archimedean fuzzy geometry to be developed in the future investigations. However, it should be emphasized that modification of the Archimedean axiom does not touch other groups of the Euclid/Hilbert geometry,

## 3. Discussion

In the future investigations the relation of the suggested fuzzy geometry to the stochastic Riemannian geometry in which distances and angles can have expected values, variances, and even distributions should be established [21]. It is possible to define a random metric space where the usual metric axioms hold probabilistically [21]. Thus, the suggested fuzzy geometry may model space-time with probabilistic local curvature in quantum gravity analogies **[22,23**]. The introduced fuzzy geometry is well-expected to be useful in robotics/computer vision, when robots/computers operate in environments with uncertain or partially known geometry [24].

## 4. Conclusions

The Euclidian geometry supplied for centuries the perfect and inspiring model of scientific and deductive reasoning. David Hilbert, one of the most influential mathematicians of the late 19th and early 20th centuries, proposed the axiomatization of physics as a part of his broader program to bring mathematical rigor and formalism to all sciences. His vision was to build physics - especially theoretical physics - on a foundation as solid and precise as Euclidean geometry, reducible to axioms, known today as Hilbert axioms. The Fifth Postulate of Euclid, also known as the Parallels Postulate, has a long and fascinating history marked by attempts to either prove it from other axioms or to replace it altogether. Hilbert system of axioms contains the Fifth Postulate of the Euclidian geometry, reshaped as the Playfair axiom: given a line $l$ and a point $A$ not on $l$, there is at most one line through $A$ parallel to $l$. The rejection of the inviolability of the Fifth Postulate gave rise to non-Euclidian geometries, developed by Nikolay Lobachevski and János Bolyai. The next intellectual step is suggested in the presented paper, enabling probabilistic development of the Euclidian geometry. The probabilistic nature of the fifth postulate is assumed; namely, we adopt that for a given a line $l$ and a point $A \notin l$, there is a exactly one line through $A$ with probability $P$, where $0 \leq P \leq 1$. The suggested probabilistic geometry is labeled the "fuzzy geometry"; thus, resembling the famous fuzzy logic developed by Lotfi Aliasger Zadeh.

In the limiting cases we obtain the well-known geometries, namely $P = 1$ corresponds to the standard Euclidean geometry; whereas, $P = 0$ corresponds to the elliptic- and hyperbolic-like geometries. The situation when $0 < P < 1$ corresponds, in turn, to the kind of stochastic geometry, which is labeled in the paper as the "fuzzy geometry". Fuzzy geometry is essentially different from the conventional stochastic geometry, in which traditional systems of axioms are adopted and remain untouched. In the fuzzy geometry the axioms themselves are probabilistic. Parallel constructions now are Bernoulli trials. Theorems of the fuzzy geometry are discussed. Given a triangle and a line drawn from a vertex parallel to the opposite side, the event that this line is actually parallel occurs with

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probability $P$. Otherwise, the line may intersect the side or diverge. Parallelism is not transitive in the fuzzy geometry. If a quadrilateral has opposite sides built using "parallel" constructions, it closes into a parallelogram with probability $P^2$ , since both pairs must independently satisfy the parallel condition.

Different versions of probabilistic parallels axioms are considered. The Playfair Axiom is logically disconnected from the other Hilbert axioms of Euclidian geometry. Thus, the probabilistic versions of the Fifth Postulate may be smoothly assembled with other Hilbert axioms. Fuzzy axioms of incidence may be introduced. However, adopting these axioms destroys the entire internal structure of the Euclidian geometry. Contrastingly, the probabilistic version of the Archimedes Axiom may be consistently assembled with the Hilbert axioms. The suggested probabilistic, fuzzy geometry reveals the unexplored field of mathematical investigations, to be developed in future. The presented paper is not more than a brief introduction into the fuzzy geometry. The fuzzy geometry may model space-time with probabilistic local curvature in quantum gravity. The introduced fuzzy geometry is well-expected to be useful in robotics/computer vision, when robots/computers operate in environments with uncertain or partially known geometry. We conclude that the fuzzy geometry represents a fascinating and unexplored probabilistic extension of the Euclidian geometry.

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## Abbreviations

FP – Fifth Postulate of Euclidian Geometry

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