Article

Not peer-reviewed version

On the Local Splitting into Incoming and Outgoing Waves and the Integral Representation of Regular Scalar Waves

Didier Felbacq * and Emmanuel Rousseau

Posted Date: 28 July 2025

doi: 10.20944/preprints202507.2310.v1

Keywords: integral representation; scalar waves

Article

On the Local Splitting into Incoming and Outgoing Waves and the Integral Representation of Regular Scalar Waves

Didier Felbacq * and **Emmanuel Rousseau**

University of Montpellier
* Correspondence: didier.felbacq@umontpellier.fr

Abstract

The problem of the integral representation over a bounded surface of a regular field satisfying the Helmholtz equation in all space is investigated. This problem is equivalent to the local splitting into an incoming field and an outgoing field. This splitting is not possible in general.

Keywords: scattering theory; scalar waves; integral representation

---

1. Introduction and Setting of the Problem

1.1. Introduction

Let us consider the following harmonic scattering problem, where a time-dependence of $e^{-i\omega t}$ is implied, with $\omega = kc$, $k = 2\pi/\lambda$ is the wavenumber and λ is the wavelength in vacuum. In the following, the proofs of the results are given for a space dimension of 3 but can be easily adapted for any other dimension > 1.

Let Ω be a bounded domain of $\mathbb{R}^3$ and let $\kappa, \rho \in L^\infty(\Omega; \mathbb{R})$ be such that $\kappa - 1$ and $\rho - 1$ have compact support in Ω. Let u^{reg} be a regular field satisfying $\Delta u^{\text{reg}} + k^2 u^{\text{reg}} = 0$ in $\mathbb{R}^n$. This field is the "incident field". The total field u satisfies $\text{div}(\rho \nabla u) + k^2 \kappa u = 0$. The scattered field is defined as $u^s = u - u^{\text{reg}}$ and it satisfies a radiation condition at infinity.

We have proven in [1] that, given any smooth bounded surface Γ enclosing Ω, there is a density $\sigma \in H^{1/2}(\Gamma)$ such that: $u^s(\mathbf{x}) = \int_\Gamma \sigma(\mathbf{x}') g^+(\mathbf{x} - \mathbf{x}') d\mathbf{x}'$ (see also [2]). In this work, we address the question of the possibility of representing the regular field by an integral over Γ. Assuming that Γ is the boundary of a domain Ω_0, it is well-known that there exists $\sigma^{\text{reg}} \in H^{1/2}(\Gamma)$ such that: $u^{\text{reg}}(\mathbf{x}) = \int_\Gamma \sigma^{\text{reg}}(\mathbf{x}') g^+(\mathbf{x} - \mathbf{x}') d\mathbf{x}'$, for $\mathbf{x} \in \Omega$. A representation of the field outside Ω is however not obvious at all. The main result of the present work is that such a representation is not possible in general.

1.2. Setting for the Scattering Problem

Let us specify a few notations. The unit sphere of $\mathbb{R}^3$ is denoted S^2. For $\mathbf{x} \in \mathbb{R}^3 \setminus \{0\}$, we denote $x = |\mathbf{x}|$ the norm of $\mathbf{x}$, and $\hat{x} = \mathbf{x}/x$. The fundamental solution g^+ of the Helmholtz equation: $\Delta g^+ + k^2 g^+ = \delta_0$ with outgoing wave condition is: $g^+(\mathbf{x}) = -\frac{1}{4\pi x} e^{ikx}$ The Green function with the incoming wave condition is denoted $g^-(\mathbf{x})$. Explicitly: $g^-(\mathbf{x}) = -\frac{1}{4\pi x} e^{-ikx}$. The following expansion of the Green functions over the spherical harmonics holds:

$$g^+(\mathbf{x} - \mathbf{x}') = -\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x} - \mathbf{x}'|} = -ik \sum_{n,m} j_n(kx_<)\, h_n^{(1)}(kx_>)\, \overline{Y_n^m}(\hat{x}')\, Y_n^m(\hat{x}), \tag{1}$$

where $x_< = \min(x, x')$ and $x_> = \max(x, x')$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 July 2025 doi:10.20944/preprints202507.2310.v1

Let Ω be a bounded domain of $\mathbb{R}^3$ with boundary $\partial\Omega = \Gamma$. For $u \in H^1(\Omega)$ (the Sobolev space of function of $L^2(\Omega)$ with gradient in $L^2(\Omega)$, see ([8], chap. 2) for more results on Sobolev spaces), the interior traces ([8], chap. 2) of u and its normal derivative on Γ are denoted by:

$$\gamma^-(u) = u|_\Gamma,\ \gamma^-(\partial_n u) = \partial_n u|_\Gamma. \tag{2}$$

For fields belonging to $H^1_{\text{loc}}(\Omega \setminus \mathbb{R}^3)$, we denote the exterior traces by:

$$\gamma^+(u) = u|_\Gamma,\ \gamma^+(\partial_n u) = \partial_n u|_\Gamma. \tag{3}$$

Given a field $u \in H^1_{\text{loc}}(\mathbb{R}^3)$, we denote $[u]_\Gamma$ the jump of u across Γ, i.e.:

$$[u]_\Gamma = \gamma^+(u) - \gamma^-(u) \text{ and } [\partial_n u]_\Gamma = \gamma^+(\partial_n u) - \gamma^-(\partial_n u). \tag{4}$$

2. A Simple Case

For the sake of clarity, let us present the simple situation of a 2D problem with a circle as the curve Γ [5,6]. The regular field u^{reg} satisfies a Helmholtz equation $\Delta u^{\text{reg}} + k^2 u^{\text{reg}} = 0$ in $\mathbb{R}^2$. The field can be expanded in Fourier series in the form:

$$u^{\text{reg}}(r,\theta) = \sum_n i_n J_n(kr) e^{in\theta}, \tag{5}$$

where J_n is a Bessel function [7]. As far as local splitting into an outgoing field and an incoming field is involved, one could think of using the decomposition into Hankel functions: $J_n = \frac{1}{2}\left(H_n^{(1)} + H_n^{(2)}\right)$ and write, for $r > 0$:

$$u^{\text{reg}}(r,\theta) = \sum_n \left(\frac{i_n}{2} H_n^{(1)}(kr) + \frac{i_n}{2} H_n^{(2)}(kr)\right) e^{in\theta}. \tag{6}$$

However this series cannot be split into two converging series, in general, due to the asymptotic behavior: $H_n(kr) \simeq \left(\frac{n}{kr}\right)^n$ as $n \to \infty$. Indeed, (i_n) is not a rapidly decreasing sequence in general. For example, for a plane wave, it has constant modulus, since it holds that:

$$e^{ikr\sin(\theta)} = \sum_n J_n(kr) e^{in\theta}.$$

As a matter of fact, in order to give a precise meaning to the series, it is necessary to admit it in a weak sense and to work on the sphere at infinity. By this we mean the following. Consider the asymptotic behavior of the Hankel functions [7]:

$$H_n^{(1)}(kr) \sim_{r\to\infty} \sqrt{\frac{2}{\pi}} e^{-i\pi/4} \frac{e^{ikr}}{\sqrt{kr}} (-i)^n. \tag{7}$$

Then, at least formally, we can write:

$$\sum_n i_n J_n(kr) e^{in\theta} \simeq \sqrt{\frac{2}{\pi}} e^{-i\pi/4} \frac{1}{\sqrt{kr}} \sum_n (-i)^n \left(\frac{i_n}{2} e^{ikr} + \frac{i_n}{2} e^{-ikr}\right) e^{in\theta}.$$

Obviously, the series $\sum_n (-i)^n i_n/2\, e^{in\theta}$ does not converge pointwise. However, provided the coefficients do not grow to rapidly, it has a meaning in the sense of Schwartz distributions. Indeed, given a function $\varphi(\theta) = \sum_n \varphi_n e^{in\theta} \in \mathcal{D}(S^1)$ (the Schwartz space of C^∞ functions with compact support in S^1) and $T \in \mathcal{D}'(S^1)$ (the dual space of Schwartz distributions), the duality $(\mathcal{D}, \mathcal{D}')$ can be defined by: $\langle T, \varphi \rangle = \sum_n i_n \varphi_n$, and the coefficients (φ_n) are such that $\varphi_n = O(1/n^p), \forall p > 0$. With this said, we can also consider the space of sequences $\mathbb{C}^{\mathbb{Z}}$ and define the scattering amplitude T as a linear operator relating the sequence of coefficients of the incident field to the sequence of coefficients of the scattered field: $(s_n)_n = T[(i_n)_n]$. Then, the scattering matrix relates the "incoming" field to the "outgoing" field as follows. The incoming coefficients are $1/2(i_n)_n$ and the outgoing coefficients are $(s_n)_n + 1/2(i_n)_n$. We obtain directly that: $(s_n)_n + 1/2(i_n)_n = 1/2(i_n)_n + T[(i_n)_n] = S[(i_n)]$. Thus: $S = 1 + 2T$. This gives a meaning of the splitting of the field, and a definition of the scattering matrix in terms of bilateral complex sequences. However the splitting does not hold in terms of fields, since it would require the series $\sum_n i_n H_n(kr)e^{in\theta}$ to be convergent and thus the coefficients $(i_n)_n$ to tend rapidly to 0. In fact, a necessary condition is that $i_n = o(e^{-n\log n})$. As for the integral representation, the question is the following. Consider a Jordan curve Γ. The question is to know whether it is possible to find σ such that:

$$u^{\text{reg}}(\mathbf{x}) = \int_\Gamma \sigma(\mathbf{x}')\Re(g^+(\mathbf{x}-\mathbf{x}')dx', \forall \mathbf{x} \in \mathbb{R}^3.$$

Let us make this integral equation explicit by choosing the unit circle as the curve Γ. We are led to considering the following integral equation:

$$K(\sigma) = \int_0^{2\pi} \sigma(\theta')J_0(k|e^{i\theta} - e^{i\theta'}|)d\theta' = u(\theta).$$

Lemma 1. *The operator K is not invertible, since 0 belongs to its essential spectrum.*

Proof. To see this, let us reformulate the equation in terms of Fourier coefficients, by using the expansion:

$$J_0(k|e^{i\theta} - e^{i\theta'}|) = \sum_n J_n^2(k)e^{in(\theta-\theta')}.$$

Then, upon inserting this expansion into the integral equation, we obtain:

$$K(\sigma) = \int \sum_n J_n^2(k)e^{in(\theta-\theta')}\sigma(\theta')d\theta' =$$

$$\sum_n J_n^2(k)e^{in\theta}\left(\int_0^{2\pi} \sigma(\theta')e^{-in\theta'}d\theta'\right) = \sum_n \sigma^n J_n^2(k)e^{in\theta}.$$

Consider now the sequence $\sigma_n(\theta) = e^{in\theta}$, then $\|\sigma_n\|_2 = 1$. Since $\sigma_n^m = \delta_n^m$, we see that $\|K(\sigma_n)\|_2 = J_n^2(k) \to 0$. Moreover, given $\varphi \in L^2(S^1)$, the scalar product $(\sigma_n, \varphi)_{L^2(S^1)}$ tends to 0, by Riemann-Lebesgue lemma. We thus see that (σ_n) is a Weyl sequence for K and 0. □

The possibility of representing the field as an integral over Γ amounts to solving the integral equation: $K(\sigma) = u(\theta)$. Expanding the field u in Fourier series, we obtain: $u(\theta) = \sum_n u_n e^{in\theta}$. Thus solving the integral equation gives:

$$\sigma(\theta) = \sum_n \frac{u_n}{J_n^2(k)}e^{in\theta}.$$

Thereofore, the L^2 convergence of the series requires:

$$\sum_n \left|\frac{u_n}{J_n^2(k)}\right|^2 < +\infty.$$

This is the so-called Picard condition [4, th. 15.18,p.311]. Indeed, denoting $e_n(\theta) = e^{in\theta}$, we remark that: $K(e_n) = J_n^2(k)e_n$. The integral operator K is self-adjoint with eigenvalues $\mu_n = J_n^2(k)$ and eigenvectors e_n. The eigenvalues are thus also the singular values of K.

Consider finally, as an counter-example, the possibility of an integral representation for $u(r,\theta) = e^{ikr\sin\theta}$. Since $e^{ikr\sin\theta} = \sum_n J_n(kr)e^{in\theta}$, we find that $\sigma(\theta) = \sum_n \frac{1}{J_n(k)}e^{in\theta}$ which diverges exponentially. This shows that a simple plane wave is not amenable to such an integral representation.

3. Local Splitting and Integral Representation

3.1. The Far-Field Operator

A regular field satisfying $\Delta u^{\text{reg}} + k^2 u^{\text{reg}} = 0$ in Ω can be represented in the following integral form, for $\mathbf{x} \in \mathbb{R}^3 \setminus \Omega$:

$$u^{\text{reg}}(\mathbf{x}) = \int_\Gamma \sigma(\mathbf{x}')g^+(k|\mathbf{x}-\mathbf{x}'|)ds(\mathbf{x}').$$

Asymptotically this reads as:

$$u^{\text{reg}}(\mathbf{x}) \sim \frac{-k}{4\pi}\frac{e^{ikx}}{kx}\int_\Gamma \sigma(\mathbf{x}')e^{-ik\hat{x}\cdot\mathbf{x}'}ds(\mathbf{x}').$$

In the following we denote J_Γ the "far-field operator" [4, p.24] defined by:

$$H^{-1/2}(\Gamma) \ni \sigma \to \left(\hat{x} \in S^2 \to J_\Gamma[\sigma](\hat{x}) = \int_\Gamma \sigma(\mathbf{x}')e^{-ik\hat{x}\cdot\mathbf{x}'}ds(\mathbf{x}') \in L^2(S^2).\right)$$

We derive a few properties of this operator that will be useful later on.

Proposition 1. *The adjoint J_Γ^+ of J_Γ is given by:*

$$J_\Gamma^+[v](\mathbf{x}') = \int_{S^2} e^{ik\hat{x}\cdot\mathbf{x}'}v(\hat{x})d\hat{x}. \tag{8}$$

The adjoint is therefore of the form $J_\Gamma^+[v](\mathbf{x}') = J_{S^2}[v](-\mathbf{x}')$.

Proof. The adjoint J_Γ^+ of J_Γ is defined by:

$$(J_\Gamma[\sigma], v)_{S^2} = \langle \sigma, J_\Gamma^+[v]\rangle_\Gamma,$$

where $\langle \cdot,\cdot\rangle_\Gamma$ denotes the duality $(H^{-1/2}(\Gamma), H^{1/2}(\Gamma))$. This gives:

$$(J_\Gamma[\sigma], v)_{S^2} = \int_{S^2} d\hat{x}\left(\int_\Gamma d\mathbf{x}'\overline{\sigma(\mathbf{x}')}e^{ik\hat{x}\cdot\mathbf{x}'}\right)v(\hat{x}) = \int_\Gamma d\mathbf{x}'\overline{\sigma(\mathbf{x}')}\int_{S^2} d\hat{x}e^{ik\hat{x}\cdot\mathbf{x}'}v(\hat{x}).$$

The proposition follows. □

Proposition 2. *The self-adjoint operator $J_\Gamma^+ J_\Gamma$ has the following expression:*

$$J_\Gamma^+ J_\Gamma[\sigma](\mathbf{x}) = (4\pi)^{3/2}\int_\Gamma d\mathbf{x}' j_0(k|\mathbf{x}-\mathbf{x}'|)\sigma(\mathbf{x}') = (4\pi)^{5/2}\sum_{nm}\sigma_{nm}j_n(kx)Y_n^m(\hat{x})$$

where

$$\sigma_{nm} = \int_\Gamma \sigma(\mathbf{x}')j_n(kx')\overline{Y_n^m(\hat{x'})}d\mathbf{x}'.$$

Proof. By a direct computation, we find that:

$$J_\Gamma^+ J_\Gamma[\sigma](\mathbf{x}) = \int_{S^2} d\hat{x}e^{ik\hat{x}\cdot\mathbf{x}}\int_\Gamma d\mathbf{x}'\sigma(\mathbf{x}')e^{-ik\hat{x}\cdot\mathbf{x}'} = \int_\Gamma d\mathbf{x}'\sigma(\mathbf{x}')\int_{S^2} d\hat{x}e^{ik\hat{x}\cdot(\mathbf{x}-\mathbf{x}')}.$$

Inserting the following expansions:

$$e^{ik\hat{x}\cdot\mathbf{x}'} = 4\pi\sum_{n,m} i^n j_n(kx')\overline{Y_n^m(\hat{x})}Y_n^m(\hat{x}'), \tag{9}$$

$$j_0(k|\mathbf{x}-\mathbf{x}'|) = 4\pi\sum_{n,m} j_n(kx')j_n(kx)Y_n^m(\hat{x}')\overline{Y_n^m}(\hat{x}),$$

the proposition follows. □

 doi:10.20944/preprints202507.2310.v1

Two natural questions arise about operator J_Γ: is it injective and how to characterize its image? The first question is answered in the following proposition.

Proposition 3. *Assume that k^2 is not an eigenvalue of $-\Delta$ inside Ω with homogeneous Dirichlet boundary conditions on Γ. If $J_\Gamma[\sigma](\hat{x}) = 0$ a.e. then $\sigma = 0$.*

Proof. Consider the function defined by $v(\mathbf{x}) = \int_\Gamma g^+(\mathbf{x} - \mathbf{x}')\sigma(\mathbf{x}')ds(\mathbf{x}')$. Then, from potential theory [4], v is the unique function satisfying:

$$\Delta v + k^2 v = 0 \text{ in } \Omega \cup (\mathbb{R}^p \setminus \overline{\Omega}),$$

with an outgoing wave condition and the boundary conditions:

$$[v]_\Gamma = 0,\ [\partial_n v]_\Gamma = \sigma.$$

The unicity follows from the following argument: assume there exists a function $u \neq v$ satisfying the same problem. Then $w = u - v$ satisfies $\Delta w + k^2 w = 0$ in $\mathbb{R}^p$ and a radiation condition. Therefore it is null thanks to Rellich lemma. Outside B_e, the field can be expanded in spherical harmonics in the form:

$$v(\mathbf{x}) = \sum_{nm} v_{nm} h_n^{(1)}(kx) Y_n^m(\hat{x}).$$

Explicitly, the coefficients (v_{nm}) are obtained by use of formula (9):

$$v(\mathbf{x}) = -ik \sum_{nm} \left(\int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m}(\hat{x}') d\mathbf{x}' \right) h_n^{(1)}(kx) Y_n^m(\hat{x}).$$

Using the asymptotic forms of the Hankel functions (7), we obtain the following asymptotics behavior:

$$v(\mathbf{x}) \sim \frac{e^{ikx}}{kx} w(\hat{x}),$$

with:

$$w(\hat{x}) = \sum_{nm} v_{nm} e^{-i(n+1)\pi/2} Y_n^m(\hat{x}).$$

Besides, using the asymptotic form of the Green function:

$$g^\pm(\mathbf{x} - \mathbf{x}') \sim_{x\to\infty} -\frac{e^{\pm ikx}}{4\pi x} e^{\mp kx \cdot x'},$$

we obtain:

$$v(x) \sim_{x\to\infty} -\frac{k}{4\pi} \frac{e^{ikx}}{4\pi x} J_\Gamma[\sigma](\hat{x}),$$

and thus $J_\Gamma[\sigma](\hat{x}) = -\frac{4\pi}{k} w(\hat{x})$. Consequently, the nullity of $J_\Gamma[\sigma](\hat{x}) = 0$ implies that of w and thus that of v in $\mathbb{R}^3 \setminus \Omega$. Therefore $\gamma^-(v) = 0$ and $\gamma^-(\partial_n v)$ is non zero iff v is a solution of the Dirichlet problem, therefore $v = 0$ in Ω by the hypothesis on k^2, and thus $\sigma = 0$. □

Corollary 1. *It holds that:*

$$\int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m(\hat{x}')} d\mathbf{x}' = 0 \textit{ if and only if } \sigma = 0.$$

Therefore $\left(\gamma\left(j_n(kx')Y_n^m(\hat{x}')\right)\right)_{(n,m)}$ *is dense in* $L^2(\Gamma)$.

Proof. Note that:

$$J_\Gamma[\sigma](\hat{x}) = 4\pi \sum_{n,m} (-i)^n \sigma_{nm} Y_n^m(\hat{x}),$$

where:

$$\sigma_{nm} = \int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m(\hat{x'})} d\mathbf{x}'.$$

Let us proof the "only if" direction: assume $\int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m(\hat{x'})} d\mathbf{x}'$, $\forall (n,m)$. Then $J[\sigma](\hat{x}) = 0$ and thus $\sigma = 0$ by the proposition above. The converse is obvious. □

Finally, we have that :

Proposition 4. *Provided that k^2 is not an eigenvalue of $-\Delta$ in the ball of radius 1, J_Γ has a dense range in $L^2(S^2)$.*

Proof. take $v \in L^2(S^2)$ in the range of J_Γ, therefore there is $\sigma \in H^{-1/2}(\Gamma)$ such that: $J_\Gamma[\sigma] = v$. To show that the range is dense, we show that if $\int_{S^2} \overline{v(\hat{x})} \phi(\hat{x}) d\hat{x} = 0, \forall v \in \mathrm{Ran}(J_\Gamma)$ then $\phi = 0$. Assume then that it holds, then we have

$$0 = (J_\Gamma[\sigma], \phi)_{S^2} = \langle \sigma, J_\Gamma^*[\phi] \rangle_\Gamma.$$

Hence, explicitly:

$$\int_{S^2} d\hat{x}\, e^{ik\hat{x}\cdot\mathbf{x}'} \phi(\hat{x}) = 0, \text{ for a.e. } \mathbf{x}' \in \Gamma.$$

Therefore, we are reduced to a special case of Proposition 3. We conclude that $\phi(\mathbf{x}) = 0$ a.e., provided k^2 is not an eigenvalue of $-\Delta$ in the ball of radius 1. □

On this topic, see also [4, p.76].

We can now remark the following. When solving the equation :$J_\Gamma[\sigma] = \phi$, it holds $J_\Gamma[\sigma](\hat{x}) = 4\pi \sum_{n,m} (-i)^n \sigma_{nm} Y_n^m(\hat{x})$, where $\phi = \sum_{nm} \phi_{nm} Y_n^m$. Hence we conclude that: $|\sigma_{nm}| = |\phi_{nm}|$ hence $\phi_{nm} = O(j_n)$ since $\sigma_{nm} = \int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m(\hat{x'})} d\mathbf{x}'$.

3.2. Integral Representation of a Regular Field

At this stage, we address the problem of the possibility of representing a regular field as an integral over Γ that would be valid outside Ω. For an incident field satisfying the Helmholtz equation in all $\mathbb{R}^3$, we cannot impose an outgoing wave condition and the kernel of the integral representation should be regular. This leads to a natural representation in the following form:

$$u^{\text{reg}}(\mathbf{x}) = \int_\Gamma \sigma(\mathbf{x}') \Im(g^+(\mathbf{x} - \mathbf{x}')) dr', \ \mathbf{x} \in \mathbb{R}^3.$$

As we shall see, it turns out that this representation imposes strong constraints on the field u^{reg}. This is somewhat similar to the possibility of splitting, at infinity, a regular field defined over $\mathbb{R}^3$ into an incoming and an outgoing field. The possibility of this decomposition was analyzed in [3], where it was shown that this was possible provided $u^{\text{reg}}(\mathbf{x}) = \int A(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}$ where $A(\mathbf{k})$ is twice continuously differentiable. If the field $u^{\text{reg}}(\mathbf{x})$ admits the integral representation over Γ then it can be decomposed at infinity into an outgoing field and an incoming field as can be infered by noting that $\Im(g^+(\mathbf{x} - \mathbf{x}'))$ is the sum of two functions satisfying an outgoing wave condition and an incoming wave condition respectively. However, the converse is not true, as the integral representation implies that the splitting is always true, i.e. for all $\mathbf{x}$, and not only at infinity. This is described in the following result.

Theorem 1. *Let u^{reg} satisfy $\Delta u^{\text{reg}} + k^2 u^{\text{reg}} = 0$, $\mathbf{x} \in \mathbb{R}^3$. The following statements are equivalent.*

1. *The field u^{reg} can be represented in the form:*

$$u^{\text{reg}}(\mathbf{x}) = \int_\Gamma \sigma(\mathbf{x}') \Im(g^+(\mathbf{x} - \mathbf{x}')) dr', \ \mathbf{x} \in \mathbb{R}^3$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 28 July 2025 doi:10.20944/preprints202507.2310.v1

where σ belongs to $H^{-1/2}(\Gamma)$.

2. *The field u^{reg} is the sum of two pointwise convergent series in the form:*

$$u^{\text{reg}}(\mathbf{x}) = \sum_{nm} \frac{1}{2} i_{nm} h_n^{(1)}(kx) Y_n^m(\hat{x}) + \sum_{nm} \frac{1}{2} i_{nm} h_n^{(2)}(kx) Y_n^m(\hat{x}). \tag{10}$$

3. *At infinity, the field can be split into an outgoing field and an incoming field in the form:*

$$u^{\text{reg}}(x) \sim \frac{e^{ikx}}{kx} u_\infty^+(\hat{x}) - \frac{e^{-ikx}}{kx} u_\infty^+(-\hat{x}),$$

where $u_\infty^+(\hat{x})$ is such that $J[\sigma](\hat{x}) = u_\infty^+(\hat{x})$ and therefore it satisfies a Picard condition.

Proof. Assume the integral form. Writing that:

$$\Im(g^+(\mathbf{x} - \mathbf{x}')) = 1/2(g^+(\mathbf{x} - \mathbf{x}') + g^-(\mathbf{x} - \mathbf{x}')),$$

it is obtained that:

$$u^{\text{reg}}(\mathbf{x}) = 1/2 \int_\Gamma \sigma(\mathbf{x}') g^+(\mathbf{x} - \mathbf{x}') d\mathbf{x}' + 1/2 \int_\Gamma \sigma(\mathbf{x}') g^-(\mathbf{x} - \mathbf{x}') d\mathbf{x}'.$$

Then, upon expanding $g^\pm(\mathbf{x} - \mathbf{x}')$ in series using (1), it is obtained that:

$$\begin{aligned} u^{\text{reg}}(\mathbf{x}) = k \sum_{n,m} \frac{1}{2} \left(\int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m}(\hat{x}') \right) h_n^{(1)}(kx) Y_n^m(\hat{x}) + \\ k \sum_{n,m} \frac{1}{2} \left(\int_\Gamma \sigma(\mathbf{x}') j_n(kx') \overline{Y_n^m}(\hat{x}') \right) h_n^{(2)}(kx) Y_n^m(\hat{x}) \end{aligned}$$

Let us now assume the representation of u^{reg} as a sum of two series. Then, upon using the asymptotic forms of the spherical Hankel functions (7), we obtain the existence of two functions $u_\infty^\pm(\hat{x})$ defined on S^2 and such that:

$$u^{\text{reg}}(x) \sim \frac{e^{ikx}}{kx} u_\infty^+(\hat{x}) + \frac{e^{-ikx}}{kx} u_\infty^-(\hat{x}).$$

Explicitly, these functions are given by:

$$u_\infty^+(\hat{x}) = \frac{1}{2} \sum_{nm} i_{nm} e^{-i(n+1)\pi/2} Y_n^m(\hat{x}), \; u_\infty^-(\hat{x}) = \frac{1}{2} \sum_{nm} i_{nm} e^{i(n+1)\pi/2} Y_n^m(\hat{x}).$$

Since:

$$e^{-i(n+1)\pi/2} = (-1)^{n+1} e^{i(n+1)\pi/2} \text{ and } Y_n^m(-\hat{x}) = (-1)^n Y_n^m(\hat{x}),$$

we have that:

$$\begin{aligned} u_\infty^-(-\hat{x}) = \frac{1}{2} \sum_{nm} i_{nm} e^{i(n+1)\pi/2} Y_n^m(-\hat{x}) &= \frac{1}{2} \sum_{nm} i_{nm} (-1)^n e^{i(n+1)\pi/2} Y_n^m(\hat{x}) \\ &= -\frac{1}{2} \sum_{nm} i_{nm} e^{-i(n+1)\pi/2} Y_n^m(\hat{x}) = -u_\infty^+(\hat{x}). \end{aligned}$$

The existence of σ satisfying: $u_\infty^+(\hat{x}) = \int_\Gamma \sigma(\mathbf{x}') e^{-ik\hat{x}\cdot\mathbf{x}'} d\mathbf{x}'$ follows from corollary (1) and lemma (4). The relation:

$$u_\infty^-(\hat{x}) = -\int_\Gamma \sigma(\mathbf{x}') e^{ik\hat{x}\cdot\mathbf{x}'} d\mathbf{x}'$$

is fulfilled thanks to:

$$u_\infty^-(\hat{x}) = -u_\infty^+(-\hat{x}) = -\int_\Gamma \sigma(\mathbf{x}') e^{ik\hat{x}\cdot\mathbf{x}'} d\mathbf{x}'.$$

Finally the Picard condition is a consequence of [4, th. 15.18,p.311] and is in fact encoded in the convergence of the series (10).

□

4. Conclusion

We have established that for a regular field, the splitting into an incoming field and an outgoing field is in general not possible in the vicinity of a given surface. We have obtained a characterization of the fields that can be split that way. These results constitute a generalization of the results already obtained for the fields at infinity [3,4].

References

1. Felbacq, D.; Rousseau, E. *Wave Motion* **2025**, *132*, 103422.
2. Maystre, D. *PIER* **2006**, *57*, 55-84.
3. Martin, P. A. *Math. Meth. Appl. Sci.*,**2018**, *41*, 2961-2970.
4. Colton D.; Kress, R. *SIAM J. Math. Anal.* **1995**, *26*, 601-615.
5. Felbacq, D; Tayeb, G. ;Maystre, D *J. Opt. Soc. Am. A* **1995**,*11*, 2526-2538..
6. Gumerov, N. ; Duraiswami, R *Fast Multipole Methods for the Helmholtz Equation in Three Dimensions*, Elsevier: Amsterdam, 2004.
7. Abramowitz M.; Stegun, I. *Handbook of Mathematical Functions*, Dover: London, 1964.
8. Cessenat, M. *Mathematical Methods in Electromagnetism*, World Scientific: Singapore, 1996.
9. Colton, D. ; Kress, R. *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Nature: Switzerland, 1996.