Article

Not peer-reviewed version

Quaternions Without Imaginarities or the Vector Representation of Quaternions

Wolf-Dieter Richter *

Posted Date: 24 July 2025

doi: 10.20944/preprints202507.1934.v1

Keywords: general quaternionic algebraic structure; vector representation of quaternions; Hamiltonian-adopted quaternionic vector product; grouped rotation-stretch vector product; Hamilton-adopted quaternionic Euler type formula; MSC: 11R52

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 July 2025 doi:10.20944/preprints202507.1934.v1

Article

Quaternions Without Imaginarities or the Vector Representation of Quaternions

Wolf-Dieter Richter

Institute of Mathematics, University of Rostock, Germany; wolf-dieter.richter@uni-rostock.de

Abstract

The general quaternionic algebraic structure we are considering was provided in with a commutative product and will be provided here with a non-commutative product. We replace the imaginary units usually used in the theory of quaternions by linearly independent vectors and the usual Hamilton product rule by a Hamiltonian-adapted vector-valued vector product and prove both a new geometric property of this product and a vectorial adopted Euler type formula.

Keywords: general quaternionic algebraic structure; vector representation of quaternions; Hamiltonian-adopted quaternionic vector product; grouped rotation-stretch vector product; Hamilton-adopted quaternionic Euler type formula; MSC: 11R52

---

1. Introduction

The developement of the field of quaternions is associated with numerous famous names of physicists and mathematicians, of which one gets a very good overview in the recent work [4] and which will therefore not be repeated here. Instead, it should be remembered that the most widely known story associated with the developement of quaternions dates back to W.R.Hamilton 1843[17,18], while the earlier contribution of C.F. Gauss 1819[16] is even less well known.

For someone new in this field, one property of the quantities under consideration in this field stands out in particular: namely the non-commutative nature of an operation that is called product. One of the focuses of this work is a new geometric statement on this topic.

Another feature that is worth highlighting and characterizes this area is the appearance of three imaginary units and their connection with the concept of independence of vectors, without using this concept primarily. This is where the second main concern of this work comes in. We completely free the theory of quaternions from all alchemical approaches and imaginary quantities. The necessary building blocks could largely be gathered from the existing literature beginning, e.g., with [11].

In this sense, this work consistently follows on from works [22–25], in which complex numbers are treated without the usually used imaginary unit. Overcoming the relationship, historically long doubted and later unfortunately established, that the square of the so-called imaginary number equals minus one results in certain one-dimensional equations becoming vector equations of dimension two. This is exemplified particularly in the updates of the treatment of quadratic equations[25], the definition of characteristic functions[24], Eulers equation and the definition of Riemann's Zeta function[22].

If one were to rigorously prove that the use of imaginary numbers could not become a problem under any circumstances, then a different term than imaginary numbers would be appropriate. Alternatively, avoiding such numbers can lead to the consideration of higher dimensional questions. This is the path followed here.

2. The General Quaternionic Algebraic Structure

Let V_4 denote a four-dimensional vector space, $\oplus : V_4 \times V_4 \to V_4$ the vector space addition, $\cdot : R \times V_4 \to V_4$ multiplication of a vector by a scalar and $\otimes : V_4 \times V_4 \to V_4$ an additional vector-valued vector operation to be specified later. Furthermore, let $\mathbb{1}, \boldsymbol{I}, \boldsymbol{J}, \mathbb{K}$ be linear independent vector space elements that satisfy the conditions

$$\boldsymbol{I} \otimes \boldsymbol{I} = -\mathbb{1}, \boldsymbol{J} \otimes \boldsymbol{J} = -\mathbb{1}, \mathbb{K} \otimes \mathbb{K} = -\mathbb{1}. \tag{1}$$

We also assume that the operations $\oplus$ and $\otimes$ are distributive, that is

$$(x \oplus y) \otimes z = (x \otimes z) \oplus (y \otimes z). \tag{2}$$

Definition 1. *The tupel* $\mathfrak{V}_4 = (V_4, \oplus, \otimes, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I}, \boldsymbol{J}, \mathbb{K})$ *is called a general quaternionic algebraic structure.*

The question of the mathematical existence of such a structure will be answered below by specifying concretely verifiable mathematical objects.

Up to this point, no restriction has been made as to whether the operation $\otimes$ must be commutative or not commutative. There are considerations in the literature devoted to the case where the operation $\otimes$ is commutative and is then called a product in reference to the situation with real numbers multiplication, see [23]. However, the present paper is devoted exclusively to the case where the operation $\otimes$ is non-commutative, in particular it may have a property as known from the cross product of three-dimensional vectors. In this case too, the operation is referred to as a product in the literature, following the case just mentioned and Hamilton, see e.g. [17–19,27].

We conclude this section with the introduction of the operations of subtraction $\ominus : V_4 \times V_4 \to V_4$ and division $\oslash : V_4 \times V_4 \to V_4$ being inverse to $\oplus$ and $\otimes$, respectively, as follows:

$$x \ominus y = z \text{ if } z \oplus y = x$$

and

$$x \oslash y = z \text{ if } z \otimes y = x.$$

The fundamental mathematical questions about the existence and uniqueness or diversity of concrete mathematical objects being realizations of the general quaternionic algebraic structure $\mathfrak{V}_4$ are answered in the next section.

3. Non-Commutative Quaternionic Algebraic Structures

Throughout this section we assume that the vector-valued vector product $\otimes$ satisfies additional assumptions and denote it with regard to Hamilton from now on by $\otimes_H$:

$$\boldsymbol{I} \otimes_H \boldsymbol{J} = -\boldsymbol{J} \otimes_H \boldsymbol{I} = \mathbb{K}, \boldsymbol{J} \otimes_H \mathbb{K} = -\mathbb{K} \otimes_H \boldsymbol{J} = \boldsymbol{I}, \mathbb{K} \otimes_H \boldsymbol{I} = -\boldsymbol{I} \otimes_H \mathbb{K} = \boldsymbol{J} \tag{3}$$

and

$$x \otimes_H (\pm\mathbb{1}) = (\pm\mathbb{1}) \otimes_H x = x \text{ for every } x \in V_4. \tag{4}$$

Clearly, this Hamiltonian-adopted vector-valued vector product still satisfies assumptions (1) and (2), that is

$$\boldsymbol{I} \otimes_H \boldsymbol{I} = \boldsymbol{J} \otimes_H \boldsymbol{J} = \mathbb{K} \otimes_H \mathbb{K} = -\mathbb{1}, \ (x \oplus y) \otimes_H z = (x \otimes_H z) \oplus (y \otimes_H z).$$

The quaternionic algebraic structure $\mathfrak{V}_H = (V_4, \oplus, \otimes_H, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I}, \boldsymbol{J}, \mathbb{K})$ thus represents a particular Clifford algebra [5].

We continue with two realizations of the general quaternionic algebraic structure $\mathfrak{V}_H$. Concrete applications in physics, chemistry, biology or other areas of research can stimulate the consideration of other realizations of structure $\mathfrak{V}_H$.

3.1. *Columns of Real Numbers*

In this section let $V_4 = R^4$ be the four-dimensional vector space consisting of columns of real numbers and

$$\mathbb{1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{I} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{J} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ .0 \end{pmatrix}, \mathbb{K} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

four basis elements of this space.

Definition 2. *Let for vectors*

$$x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_0\mathbb{1} + x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\mathbb{K} \, and \; y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = y_0\mathbb{1} + y_1\boldsymbol{I} + y_2\boldsymbol{J} + y_3\mathbb{K}$$

the grouped rotation-stretch product be defined by

$$x \otimes_{grs} y = \begin{pmatrix} rO(\varphi)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} - \rho\mathfrak{O}(\psi)\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \\ \rho\mathfrak{O}(\psi)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + rO(\varphi)\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \end{pmatrix} \tag{5}$$

where

$$r = \sqrt{x_0^2 + x_1^2}, \varrho = \sqrt{x_2^2 + x_3^2},$$

$$O(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}, \cos\varphi = \frac{x_0}{\sqrt{x_0^2 + x_1^2}}, \sin\varphi = \frac{x_1}{\sqrt{x_0^2 + x_1^2}},$$

$$\mathfrak{O}(\psi) = \begin{pmatrix} \cos\psi & \sin\psi \\ \sin\psi & -\cos\psi \end{pmatrix}, \cos\psi = \frac{x_2}{\sqrt{x_2^2 + x_3^2}}, \sin\psi = \frac{x_3}{\sqrt{x_2^2 + x_3^2}}.$$

Obviously, matrix $O(\varphi)$ causes a rotation around the coordinate origin of R^2 counterclockwise by the angle φ, and the matrix

$$\mathfrak{O}(\psi) = O(\psi)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

causes a reflection with respect to the x-axes followed by a rotation through angle ψ and r and ϱ are stretching factors.

Remark 1. *The grouped rotation-stretch product can be reformulated as*

$$x \otimes_{grs} y = \begin{pmatrix} \xi_1(r_1, \varphi_1) \circledast \xi_2(r_2, \varphi_2) - \eta_1(\varrho_1, \psi_1) \circledast \eta_2(\varrho_2, -\psi_2) \\ \eta_1(\varrho_1, \psi_1) \circledast \xi_2(r_2, -\varphi_2) + \xi_1(r_1, \varphi_1) \circledast \eta_2(\varrho_2, \psi_2) \end{pmatrix}$$

where

$$\xi_l(r_l, \varphi_l) = r_l\begin{pmatrix} \cos\varphi_l \\ \sin\varphi_l \end{pmatrix}, \eta_l(\varrho_l, \psi_l) = \varrho_l\begin{pmatrix} \cos\psi_l \\ \sin\psi_l \end{pmatrix}, l = 1, 2$$

and

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \circledast \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 \\ x_1y_2 + x_2y_1 \end{pmatrix}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 July 2025 doi:10.20944/preprints202507.1934.v1

updates usual complex multiplication.

Proof. By (5),

$$x \otimes_{grs} y = \begin{pmatrix} r_1 O(\varphi_1)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} - \varrho_1 \mathfrak{O}(\psi_1)\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \\ \varrho_1 \mathfrak{O}(\psi_1)\begin{pmatrix} y_0 \\ y_1 \end{pmatrix} + r_1 O(\varphi_1)\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} r_1 O(\varphi_1)\left[r_2 \begin{pmatrix} \cos\varphi_2 \\ \sin\varphi_2 \end{pmatrix} \right] - \varrho_1 \mathfrak{O}(\psi_1)\left[\varrho_2 \begin{pmatrix} \cos\psi_2 \\ \sin\psi_2 \end{pmatrix} \right] \\ \varrho_1 \mathfrak{O}(\psi_1)\left[r_2 \begin{pmatrix} \cos\varphi_2 \\ \sin\varphi_2 \end{pmatrix} \right] + r_1 O(\varphi_1)\left[\varrho_2 \begin{pmatrix} \cos\psi_2 \\ \sin\psi_2 \end{pmatrix} \right] \end{pmatrix}$$

$$= \begin{pmatrix} r_1 r_2 \begin{bmatrix} \cos(\varphi_1+\varphi_2) \\ \sin(\varphi_1+\varphi_2) \end{bmatrix} - \varrho_1\varrho_2 \begin{bmatrix} \cos(\psi_1-\psi_2) \\ \sin(\psi_1-\psi_2) \end{bmatrix} \\ \varrho_1 r_2 \begin{bmatrix} \cos(\psi_1-\varphi_2) \\ \sin(\psi_1-\varphi_2) \end{bmatrix} + r_1 \varrho_2 \begin{bmatrix} \cos(\varphi_1+\psi_2) \\ \sin(\varphi_1+\psi_2) \end{bmatrix} \end{pmatrix}$$

□

The quaternionic vector-valued vector operation or multiplication rule $\otimes_{grs}$ may be well motivated by application-specific physical, chemical, biological or entirely different observations or modeling approaches. However, any unmotivated, routine or thoughtless use would be questionable.

We now present the announced new geometric statement for the Hamiltonian-adopted quaternionic vector-valued vector product $\otimes_H$.

Theorem 1. *The Hamiltonian-adopted vector product agrees with the grouped rotation-stretch product, that is*

$$x \otimes_H y = x \otimes_{grs} y, \;\; x \in R^4, y \in R^4.$$

Proof. The rules of the Hamiltonian-adopted vector product $\otimes_H$ jointly imply that

$$\begin{aligned} x \otimes_H y &= (x_0\mathbb{1} + x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\boldsymbol{K}) \otimes_H (y_0\mathbb{1} + y_1\boldsymbol{I} + y_2\boldsymbol{J} + y_3\boldsymbol{K}) \\ &= (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3)\mathbb{1} + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\boldsymbol{I}. \\ &+ (x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1)\boldsymbol{J} + (x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)\boldsymbol{K}. \end{aligned}$$

In other words,

$$x \otimes_H y = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix} \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \tag{6}$$

or

$$x \otimes_H y = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

with

$$A = \begin{pmatrix} x_0 & -x_1 \\ x_1 & x_0 \end{pmatrix}, B = \begin{pmatrix} x_2 & x_3 \\ x_3 & -x_2 \end{pmatrix}, \xi = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \text{ and } \eta = \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}.$$

□

Remark 2. *We would like to add that*

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix}\begin{pmatrix} A & -B \\ B & A \end{pmatrix}^T = \begin{pmatrix} \mathfrak{M} & -\mathfrak{N} \\ \mathfrak{N} & \mathfrak{M} \end{pmatrix}$$

with

$$\mathfrak{M} = AA + BB, \mathfrak{N} = BA - AB$$

and that any result with respect to the Hamiltonian-adopted vector product $\otimes_H$ now can be read in a new way using the grouped rotation-stretch product $\otimes_{grs}$ in (6).

Remark 3. *The tuple $(R^4, \oplus, \otimes_{grs}, \cdot, \mathfrak{o}, \mathbb{1}, \mathbb{I}, \mathbb{J}, \mathbb{K})$ is a concrete (quadruple-)realization of $\mathfrak{V}_H$. Here,$\otimes_{grs}$ can be replaced by the more implicitly defined $\otimes_H$.*

Remark 4. *It follows from (6) that*

$$x \otimes_H y = \begin{pmatrix} x_0 & -x_1 & -x_2 & -x_3 \\ x_1 & x_0 & -x_3 & x_2 \\ x_2 & x_3 & x_0 & -x_1 \\ x_3 & -x_2 & x_1 & x_0 \end{pmatrix}\begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$= \begin{pmatrix} y_0 & -y_1 & -y_2 & -y_3 \\ y_1 & y_0 & y_3 & -y_2 \\ y_2 & -y_3 & y_0 & y_1 \\ y_3 & y_2 & -y_1 & y_0 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq -y \otimes_H x,$$

thus the non-commutative Hamiltonian adopted product is not anti-commutative.

Example 1. *The vector space element $\mathbb{1}$ represents the multiplicative unit, i.e.*

$$\mathbb{1} \otimes_H x = x = x \otimes_H \mathbb{1}.$$

Example 2. *Multiplication of so-called pure quaternions is given by*

$$\begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \otimes_H \begin{pmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & -x_1 & -x_2 & -x_3 \\ x_1 & 0 & -x_3 & x_2 \\ x_2 & x_3 & 0 & -x_1 \\ x_3 & -x_2 & x_1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} -\mathfrak{x}^T\mathfrak{y} \\ \mathfrak{x} \times \mathfrak{y} \end{pmatrix}$$

where

$$\mathfrak{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \mathfrak{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

and

$$\mathfrak{x} \times \mathfrak{y} = \begin{pmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{pmatrix} \text{ as well as } \mathfrak{x}^T\mathfrak{y} = x_1y_1 + x_2y_2 + x_3y_3$$

denote the cross product and the scalar product of the three-dimensional sub-vectors $\mathfrak{x}$ and $\mathfrak{y}$, respectively. More generally,

$$x \otimes_H y = \begin{pmatrix} x_0y_0 - \mathfrak{x}^T\mathfrak{y} \\ x_0\mathfrak{y} + y_0\mathfrak{x} + \mathfrak{x} \times \mathfrak{y} \end{pmatrix}.$$

This representation allows a largely three-dimensional geometric interpretation of the Hamiltonian adopted vector product $\otimes_H$.

doi:10.20944/preprints202507.1934.v1

Example 3. *When examining relationship (3), as an example, one immediately sees that*

$$I \otimes_H J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = K.$$

Definition 3. *The nth Hamiltonian adopted quaternionic power of $x, x \in V_4$, is defined by*

$$x^{\otimes_H n} = x^{\otimes_H (n-1)} \otimes_H x,\ n = 1, 2, \ldots;\ x^{\otimes_H 0} = \mathbb{1}.$$

Example 4. *Squaring a quaternion $x = x_0\mathbb{1} + x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\mathbb{K} = (x_0\mathfrak{x}^T)^T$ means*

$$x^{\otimes_H 2} = x \otimes_H x = \begin{pmatrix} x_0^2 - x_1^2 - x_2^2 - x_3^2 \\ 2x_0x_1 \\ 2x_0x_2 \\ 2x_0x_3 \end{pmatrix} = \begin{pmatrix} x_0^2 - ||\mathfrak{x}||^2 \\ 2x_0\mathfrak{x} \end{pmatrix}.$$

Example 5. *For $x = x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\mathbb{K} = (0\mathfrak{x}^T)^T$, it follows that*

$$x^{\otimes_H 2} = \begin{pmatrix} 0 & -x_1 & -x_2 & -x_3 \\ x_1 & 0 & -x_3 & x_2 \\ x_2 & x_3 & 0 & -x_1 \\ x_3 & -x_2 & x_1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = -||\mathfrak{x}||^2\mathbb{1},$$

$$x^{\otimes_H 3} = -||\mathfrak{x}||^2\mathbb{1} \otimes_H x = -||\mathfrak{x}||^2 x,$$

thus

$$x^{\otimes_H (2k)} = (-1)^k ||\mathfrak{x}||^{2k}\mathbb{1} \text{ and } x^{\otimes_H (2k+1)} = (-1)^k ||\mathfrak{x}||^{2k}(x_1 \cdot \boldsymbol{I} + x_2 \cdot \boldsymbol{J} + x_3 \cdot \mathbb{K}).$$

Definition 4. *The Hamiltonian adopted quaternionic exponential function $\exp_H : V_4 \to V_4$ is defined by*

$$\exp_H(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^{\otimes_H k}.$$

Theorem 2. *The following Hamiltonian adopted quaternionic Euler type formula holds true:*

$$\exp_H(x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\boldsymbol{K}) = (\cos ||\mathfrak{x}||)\mathbb{1} + \frac{\sin ||\mathfrak{x}||}{||\mathfrak{x}||}(x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\boldsymbol{K}), \mathfrak{x} \neq (0,0,0)^T.$$

Proof. The proof follows immediately by Definition 4 and Example 5 as can be seen from the following calculation steps:

$$\exp_H(x) = x^{\otimes_H 0} + x^{\otimes_H 1} + \frac{1}{2}x^{\otimes_H 2} + \frac{1}{3!}x^{\otimes_H 3} + \ldots$$

$$= \mathbb{1} + x + \frac{1}{2}(-||\mathfrak{x}||^2)\mathbb{1} + \frac{1}{3!}(-||\mathfrak{x}||^2)\begin{pmatrix} 0 \\ \mathfrak{x} \end{pmatrix} + \frac{1}{4!}||\mathfrak{x}||^4\mathbb{1} + \frac{1}{5!}||\mathfrak{x}||^4\begin{pmatrix} 0 \\ \mathfrak{x} \end{pmatrix} + \ldots$$

□

At the end of this section we compare how our different formulas work.

Example 6. *According to Definition 2,*

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \otimes_{grs} \begin{pmatrix}5\\6\\7\\8\end{pmatrix} = \left(\begin{matrix} \begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix} - \begin{pmatrix}3 & 4\\4 & -3\end{pmatrix}\begin{pmatrix}7\\8\end{pmatrix} \\ \begin{pmatrix}3 & 4\\4 & -3\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix} + \begin{pmatrix}1 & -2\\2 & 1\end{pmatrix}\begin{pmatrix}7\\8\end{pmatrix} \end{matrix} \right) = \begin{pmatrix}-60\\12\\30\\24\end{pmatrix},$$

according to Remark 1,

$$\begin{aligned} \begin{pmatrix}1\\2\\3\\4\end{pmatrix} \otimes_{grs} \begin{pmatrix}5\\6\\7\\8\end{pmatrix} &= \left(\begin{matrix} \begin{pmatrix}1\\2\end{pmatrix} \circledast \begin{pmatrix}5\\6\end{pmatrix} - \begin{pmatrix}3\\4\end{pmatrix} \circledast \begin{pmatrix}7\\8\end{pmatrix} \\ \begin{pmatrix}3\\4\end{pmatrix} \circledast \begin{pmatrix}5\\-6\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} \circledast \begin{pmatrix}7\\8\end{pmatrix} \end{matrix} \right) \\ &= \left(\begin{matrix} \begin{pmatrix}5-12\\6+10\end{pmatrix} - \begin{pmatrix}21+32\\-24+28\end{pmatrix} \\ \begin{pmatrix}15+24\\-18+20\end{pmatrix} + \begin{pmatrix}7-16\\8+14\end{pmatrix} \end{matrix} \right) = \begin{pmatrix}-60\\12\\30\\24\end{pmatrix} \end{aligned}$$

and according to formula (6),

$$\begin{pmatrix}1\\2\\3\\4\end{pmatrix} \otimes_{grs} \begin{pmatrix}5\\6\\7\\8\end{pmatrix} = \begin{pmatrix}1 & -2 & -3 & -4\\2 & 1 & -4 & 3\\3 & 4 & 1 & -2\\4 & -3 & 2 & 1\end{pmatrix}\begin{pmatrix}5\\6\\7\\8\end{pmatrix} = \begin{pmatrix}-60\\12\\30\\24\end{pmatrix}.$$

3.2. Matrices of Real Numbers

In order to highlight the potential variety of possible realizations of $\mathfrak{V}_H$, a second realization is sketched here. However, since all four-dimensional vector spaces over the field of real numbers are isomorph among themselves, this section will be kept particularly short. In this section we suggest a matrix realization of the algebraic structure $\mathfrak{V}_H$. To this end, let $\mathfrak{M}_4$ be the set of all 4×4-matrices being linear combinations of

$$\mathbb{1} = \begin{pmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{pmatrix}, \boldsymbol{I} = \begin{pmatrix}0 & 1 & 0 & 0\\-1 & 0 & 0 & 0\\0 & 0 & 0 & 1\\0 & 0 & -1 & 0\end{pmatrix}, \boldsymbol{J} = \begin{pmatrix}0 & 0 & 0 & -1\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\1 & 0 & 0 & 0\end{pmatrix} \text{ and } \boldsymbol{K} = \begin{pmatrix}0 & 0 & -1 & 0\\0 & 0 & 0 & 1\\1 & 0 & 0 & 0\\0 & -1 & 0 & 0\end{pmatrix}.$$

Let further addition $\oplus$ and multiplication by scalar be defined componentwise and multiplication of $x = x_0\mathbb{1} + x_1\boldsymbol{I} + x_2\boldsymbol{J} + x_3\boldsymbol{K}$ and $y = y_0\mathbb{1} + y_1\boldsymbol{I} + y_2\boldsymbol{J} + x_3\boldsymbol{K}$ by

$$\begin{aligned} x \otimes_H y = {} & (x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3)\mathbb{1} + (x_0y_1 + x_1y_0 + x_2y_3 - x_3y_2)\boldsymbol{I} \\ & + (x_0y_2 - x_1y_3 + x_2y_0 + x_3y_1)\boldsymbol{J} + (x_0y_3 + x_1y_2 - x_2y_1 + x_3y_0)\boldsymbol{K}. \end{aligned}$$

Then the tuple $(\mathfrak{M}_4, \oplus, \otimes_H, \cdot, \mathfrak{o}, \mathbb{1}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K})$ is a further concrete realization of $\mathfrak{V}_H$.

One might continue here with matrix reformulations of Definition 2 and Theorem 1. Moreover, one could consider the quaternionic product of, e.g., polynomials for introducing a further quaternionic, polynomial structure.

4. Discussion

Throughout this work, we avoided the use of so-called imaginary units, as is common in international literature since the introduction of quaternions by Hamilton[17,18] and Clifford[5].

We have also avoided a notation of the type $x = x_0 + x_1 i + x_2 j + x_3 k$, which is widely used in the literature because in this notation the first summation sign has a completely different meaning than the second and third ones. In a comparable situation, Gao[15] even introduces the concept of paravector for similar quantities, however without giving any explanation of how to sum up imaginary quantities of different kinds and why such quantities should have properties of vector space elements.

In the literature dealing with the ominous imaginary units i, j, k, the following multiplication table is given instead of Definition 2 or equation (6):

Table 1. Vector-valued operations

	1	i	j	k
1	1	i	j	k
i	i	-1	k	-j
j	j	-k	-1	i
k	k	j	-i	-1

Without the greatest mathematical rigor, the use of such a multiplication table remains, of course, practical for quick formal work. However, the remaining mathematical deficiencies can then also lead to mathematical, physical or other misinterpretations.

Hamilton considered quaternion mathematics as a fundamental language of physics. This is a view that is also already reflected in the titles of [3,7,10,12]. These papers show the great attraction that quaternions have for physics. The geometry of physics is considered in [14] and the tragic downfall and peculiar revival of quaternions is impressively described in [4]. For geometric applications of quaternions, see [9].

The present work did not attempt to make any statement of a physical or other applied nature. However, it should be noted that [26] gives three commutative multiplication methods for spacetime models of dimensions two to four within a purely mathematical work.

Two-dimensional quaternion Fourier transformation in probability modeling was presented in [19,27] and can be further developed with the help of the results obtained here towards a vector representation of these Fourier transformations following the spirit of [24].

For an Euler formula for matrixes, we refer to [13]. With an emphasis on pure quaternions, geometric properties of quaternions are studied in. In particular, quaternion multiplication is understood with the help of defining subspaces and lines in the space of quaternions. The results presented here in Theorem 1 complement this geometric understanding of the quaternion multiplication. Geometric properties of timelike quaternions are studied in [6,20,21].

It is usually thought that the quaternions represent an extension of the complex numbers. However, complex numbers which are pairs of real numbers are not special quadrupels. For example,

$$(x,y)^T \neq (x,y,0,0)^T \text{ and } (x,y)^T \neq (-2,x,3,y)^T.$$

Acknowledgments: There was no funding for this work. The author declares no conflict of interest. The whole paper was written alone by the author.

References

1. V.Ariel. The real quaternion relativity. arXiv:1801.03393v1,2018.
2. V.Ariel. Quaternion space-time and matter. Physics 2021.
3. J.L.Van Belle. The nuclear force and quaternion math. https://www.researchgate.net/publication/350799512, 2021.
4. D.Brezov. The tragic downfall and peculiar revival of quaternions. doi:10.20944/preprints202412.1922.v1, 2024.
5. W.K. Clifford. Applications of Grassman's extensive algebra. Am.J.Math. 1(4), 350-358 (1878).

6. I.Colombaro. Time-like definition of quaternions in exterior algebra. Ricerche di Matematica (2024) 73:2865-2876.
7. A.W. Conway. Quaternion treatment of the relativistic wave equation. vol.CLXII-A, 1937.
8. J. H. Conway, D. A. Smith. On quaternions and octonions: Their geometry, arithmetic and symmetry. Natic, MA:A.K.Peters Ltd, 2003.
9. B.B.Demirci, N. Aghayev. On geometric applications of quaternions. Turk J Math (2020) 44:1289-1303.
10. G.M.Dixon. Division algebras, octonions, quaternions, complex numbers and the algebraic design of physics. Kluwer, Dordrecht 1994.
11. L.E.Dickson. On quaternions and their generalization and the history of the eight square theorem.
12. T.Dray, C.A. Manogue. Quaternionic spin. arXiv:hep-th/9910010v2, 1999.
13. M.Erdogdu, M.Özdemir. De-Moivre and Euler formulas for matrices of split quaternions. Clifford analysis, Clifford algebras and their applications 4,3,223-233, 2015.
14. T.Frankel. The geometry of physics. Cambridge. Cambridge University Press, 2011.
15. W.-B. Gao. Uncertainty principles of hypercomplex functions for fractional Fourier transform. Fractional Calculus and Applied Analysis (2023) 26: 2298-2317.
16. C.F.Gauss. Theoria Residuorum Biquadraticorum: Commentatio <secunda. Werke 2; Chelsea Publishing Company:N.Y. 1965,93-148.
17. W.R. Hamilton. On a new species of Imaginary Quantities connected with the Theory of Quaternions. Proc. Roy. Irish Acad. 2, 424, 1843.
18. W.R. Hamilton. On quaternions; or on a new system of imaginaries in algebra. Philos. Mag. J.Sci. 1844, 489-495.
19. N.Nurwahidah, M.Bahri and A.Rahim. Two-dimensional quaternion Fourier transform method in probability modeling. Symmetry 2024, 16, 257.
20. M.Özdemir, A.A. Ergin. Some geometric applications of timelike quaternions. https://www.researchgate.net/publication/27076
21. M.Özdemir, A.A. Ergin. Rotations with unit timelike quaternions in Minkowski 3-space. J.Geom.Phys.56,322-336, 2006.
22. W.-D. Richter. Vector representations of Euler's formula and Riemann's Zeta function. submitted
23. W.-D. Richter. On complex numbers in higher dimensions. Axioms 2022, 1,1-19.
24. W.-D. Richter. On the vector representation of characteristic functions. Stats 2023,6, 1072-1081.
25. W.-D. Richter. More about solutions of quadratic equations. The accompanying variables method. arXiv:2506.20429.
26. W.-D. Richter. On hyperbolic complex numbers. Appl. Sci. 2022, 12, 5844.
27. M.A. Samad, Y. Xia, S. Siddiqui and M.Y. Bhat. Two-dimensional quaternion linear canonoical transform: A novel framework for probability modeling. arXiv 2410.19001v1, 2024.