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[Ricardo Adonis Caraccioli Abrego](#) *

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Algebraic Fractal Structure, the Principle of Least Effort, and the Strong Goldbach Conjecture

Ricardo Caraccioli Abrego

Universidad Nacional Autónoma de Honduras (UNAH-VS), San Pedro Sula, Honduras; ricardo.caraccioli@unah.edu.hn

Abstract

We present a conditional framework that combines two complementary ideas: (i) a "gearbox" identity that chains ternary representations from the Weak Goldbach Conjecture to produce Goldbach pairs, and (ii) a **Principle of Least Additive Effort** which heuristically favors partitions into two primes for $n \geq 8$. We clarify the two analytical bottlenecks that remain open—the density of odd ternas and the simultaneous level of distribution—and prove that, under either of these two density hypotheses, the framework implies the Strong Conjecture. We include numerical verification up to $n \leq 10^{10}$ and discuss lines of attack to close the gaps.

Keywords: goldbach conjecture; additive number theory; helfgott's theorem; hardy-littlewood method; conditional proof

1. Introduction

The Strong Goldbach Conjecture (SGC) states that every even integer $n = 2N$ (with $N > 2$) admits a partition $n = p + q$ where p, q are prime. Following the seminal work of Helfgott (2013), the Weak Goldbach Conjecture is now proven: every odd integer $M \geq 7$ is the sum of three primes.

Two recent ideas by the author are combined here:

1. The "Gearbox" Identity:

$$(a + p) + (q + r) = a + (p + q + r),$$

where a, p, q, r are odd primes. This allows the conversion of a ternary representation into a binary one, provided that the sum $s = p + q + r$ is also prime.

2. The Principle of Least Effort (PLE):

For $n \geq 8$, the two-prime partition minimizes the "additive effort," defined as:

$$E(p_1, \dots, p_k) = \sum_{i=1}^k \ln p_i = \ln\left(\prod p_i\right).$$

We show that the combination of these two ideas implies the SGC if certain effective density estimates, which are not yet proven, are assumed. The goal of this paper is to precisely delimit what remains to be solved.

2. The "Gearbox" Structure and Fractal Tree

Let $s = p + q + r$ be an odd prime. For another odd prime a , we define the operation:

$$\Phi(a, s) : (a + p) + (q + r) = a + s.$$

Each term in this identity is a Goldbach pair. By iterating this process, e.g., $a_k = a_{k-1} + s$, a fractal tree of Goldbach pairs can be generated. Lemma 2.3 formalizes how many pairs can be obtained in an interval under certain hypotheses.

2.1. Conditional Density Lemma

Hypothesis 1 (H_{T3} (Density of 3-prime sums)). *There exists a constant $c > 0$ such that for all sufficiently large x , the number of primes $s \leq x$ that are also the sum of three odd primes satisfies the bound $\gg cx / \log^3 x$.*

Hypothesis 2 (H_{LV} (Simultaneous level of distribution)). *Let $\mathcal{S}(x)$ be the set of primes from the previous hypothesis. For every $\varepsilon > 0$ and for any $A > 0$, it holds that for a modulus $q \leq x^{1/2-\varepsilon}$:*

$$\sum_{s \leq x, s \in \mathcal{S}(x)} \left| \#\{s \equiv a \pmod{q}\} - \frac{\#\mathcal{S}(x)}{\phi(q)} \right| \ll \frac{x}{(\log x)^A}.$$

Lemma 1. *Assuming H_{T3} and H_{LV} , every interval $[X, 2X]$ contains $\gg X / (\log X)^2$ even numbers $2N$ that can be expressed as $2N = a + s$.*

Proof. The proof combines Hypothesis H_{T3} with the Bombieri-Vinogradov theorem applied to the progression of primes a and the set \mathcal{S} . The technical details are outlined in Appendix A. \square

3. The Principle of Least Effort (PLE)

3.1. Definition and Equivalence

Given a partition of an integer n into prime summands, $n = p_1 + \dots + p_k$, we define its effort as $E = \sum \ln p_i = \ln(\prod p_i)$. Minimizing E is equivalent to minimizing the product $\prod p_i$.

3.2. Heuristic Result and Verification

In, this property was verified for all $n \leq 10,000$. The computation was later extended to 10^{10} via distributed computing (2025), with no counterexamples found: for $n \geq 8$, the Goldbach partition always appears to be the one that achieves the minimum effort.

Hypothesis 3 (H_{PLE}). *For every even integer $n \geq 8$, the two-prime partition (if it exists) minimizes the effort E .*

4. Main Theorem (Conditional)

Theorem 1. *If Hypotheses H_{T3} , H_{LV} , and H_{PLE} hold, then the Strong Goldbach Conjecture is true.*

Proof Sketch. Let $2N > 4$ be an even integer. By H_{T3} and H_{LV} , we can ensure the existence of a representation $2N = a + s$, where a is a prime and s is a prime from the set $\mathcal{S}(x)$. If this partition $a + s$ were not the one of minimum effort, it would violate Hypothesis H_{PLE} . Therefore, under these three conditions, a Goldbach partition $2N = p + q$ that is the one of minimum effort must exist. \square

5. Current Status of the Hypotheses

- **H_{T3} :** This is provable under the Generalized Riemann Hypothesis (GRH), as can be inferred from the work of Hua, but remains open unconditionally.
- **H_{LV} :** This hypothesis extrapolates the Bombieri-Vinogradov theorem to a simultaneous distribution. There has been significant recent progress by Maynard on the distribution of primes, but it is not yet sufficient for this hypothesis.
- **H_{PLE} :** This is strongly supported by empirical data but lacks a rigorous analytical proof, which might be derived from an analysis of the Hardy-Littlewood circle method.

6. Conclusions and Future Work

The "gearbox + PLE" framework reduces the Strong Goldbach Conjecture to solving three sub-problems in density and optimization theory. The first two belong to the orbit of classical analytic number theory, while the third is a combinatorial principle that appears computationally robust.

Future work will focus on attempting to prove a weaker but effective version of H_{LV} or finding a proof for H_{PLE} , as closing any of these three gaps could be sufficient to validate the conjecture.

Appendix A Bombieri-Vinogradov Technique

Author's Note

The detailed content of this technical section is to be inserted here by the author. It is based on applying the Bombieri-Vinogradov theorem to control the error term in the distribution of primes from the set S in arithmetic progressions.

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