Communication

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Triviality of the Limit Distributions Class for Sums of Random Numbers of Positive Random Variables

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Abstract

We prove that any positive random variable Y represents a limit $Y = \lim_{m\to\infty} \sum_{k=1}^{\nu_m} X_k$ of a suitably chosen sequence of random sums $\{\sum_{k=1}^{\nu_m} X_k, m \geq 1\}$ of independent identically distributed positive random variables X_k with a finite mean. For this to be true, the probability generating function $\mathcal{P}_m(z)$ of integer-valued non-negative random variable ν_m must be expressible as $\mathcal{P}_m(z) = \psi(m(1-z))$, where $\psi(s)$ is a Laplace transform of a probability distribution on the positive semiaxis.

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1. Introduction

Limit theorems in probability theory probably originated with the result now called the De Moivre-Laplace theorem, originally obtained by De Moivre in 1733. Further developments in this area led to the emergence of a certain class of theorems under the general name "Central Limit Theorem" and numerous generalizations of these theorems. The main results of this direction are presented in the monograph by B.V. Gnedenko and V.Yu. Korolev [1]. More recent results are given in [2].

For the sums of random numbers of random variables, the topics of this communication, many results are similar to classical limit theorems of probability theory. However, there are, nevertheless, fundamental differences, one of which we would like to point out. It turns out that in the case of the summation of positive random variables with finite mean, a suitably chosen sequence of random sums yields a class of limit laws that is large and coincides with all probability distributions concentrated on the positive semiaxis.

2. Main Result

Theorem 2.1. *Let $X = (X_1, X_2, \ldots, X_n, \ldots)$ be a sequence of independent identically distributed positive random variables having unit first moment, i.e. $\mathbb{E}X_i = 1, i = 1, 2, \ldots, n, \ldots$. Suppose that Y is an arbitrary positive random variable. Then there exists a family $\{\nu_m,\ m \geq 1\}$ of nonnegative integer-valued random variables ν_m, depending on the distribution of Y only, and such that*

1. *The family $\{\nu_m,\ m \geq 1\}$ and X are independent;*
2. *ν_m tends in probability to infinity as $m \to \infty$;*
3. *The random sum $\sum_{k=1}^{\nu_m} X_k$ tends to Y in distribution as $m \to \infty$.*

For the Proof, we need the following two Lemmas.

Lemma 2.1. *Let $\psi(s)$ be the Laplace transform of a distribution on the positive semiaxis. Then $\mathcal{P}(z) = \psi(1-z)$ is a probability generating function.*

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Proof. (see [3]). Suppose that
$$\psi(s) = \int_0^\infty \exp(-sx) d\mathcal{A}(x),$$
where $\mathcal{A}(x)$ is a probability distribution function. Therefore
$$\mathcal{P}(z) = \psi(1-z) = \int_0^\infty \exp(-(1-z)x) d\mathcal{A}(x) = \sum_{k=0}^\infty \frac{z^k}{k!} \int_0^\infty x^k \exp(-x) d\mathcal{A}(x).$$
The series on the right-hand side has non-negative coefficients, and their sum is 1. □

Lemma 2.2. *Let $\mathcal{Q}(z)$ be a probability generating function. The function $\mathcal{Q}(1-A+Az)$ is probability generating function for all $A > 1$ if and only if $\mathcal{Q}(z) = \psi(1-z)$, where $\psi(s)$ is Laplace transform of a probability distribution on positive semiaxis. While the transformation $\mathcal{Q}(z) \to \mathcal{Q}(1-A+Az)$ with $0 < A < 1$, can be applied to every pgf, the case with $A > 1$ is limited to scalable pgfs only. As shown in [4], the corresponding probability mass functions satisfy the relation $(n+1)p_{n+1} = p_n g(n)$. The class of scalable distributions consists, besides others, of negative binomial, shifted logarithmic, and discrete stable distributions.*

Proof. 1. Suppose that $\mathcal{Q}(z) = \psi(1-z)$, where $\psi(s)$ is a Laplace transform of a probability distribution on the positive semiaxis. We have
$$\mathcal{Q}(1-A+Az) = \psi(A(1-z)) = \int_0^\infty \exp(-(1-z)x) d\mathcal{A}(x/A)$$
and the statement follows from Lemma 2.1.

2. Suppose now the function $\mathcal{Q}(1-A+Az)$ is probability generating function for all $A > 1$. Let $\xi(s)$ be the Laplace transform of a distribution with unit mean. Then $\mathcal{Q}(1-A(1-\xi(s/A)))$ is the Laplace transform of a distribution on the nonnegative semiaxis for all $A > 1$. Its limit as $A \to \infty$ equals $\mathcal{Q}(1-s)$ and is continuous at the point $s = 0$. Therefore, $\psi(s) = \mathcal{Q}(1-s)$ is the Laplace transform of a distribution on the nonnegative semiaxis. □

Proof of Theorem 2.1. Denote by $\varphi(s)$ the Laplace transform of the positive random variable Y. According to Lemma 2.1, define a family of probability generating functions
$$\mathcal{P}_m(z) = \varphi(m(1-z)), \quad m > 1, \tag{2.1}$$
and let $\{\nu_m,\ m > 1\}$ be a family of corresponding random variables taking non-negative integer values. Clearly, we can construct the family to be independent of X.

Let $B > 0$ be an arbitrary positive number and calculate the probability $\mathbb{P}\{\nu_m \geq B\}$. We have
$$\mathcal{P}_m(z) = \varphi(m(1-z)) = \sum_{k=0}^\infty \frac{m^k z^k}{k!} \int_0^\infty x^k \exp(-mx) d\mathcal{A}(x),$$
where $\mathcal{A}$ is probability distribution function of Y. Therefore,
$$\mathbb{P}\{\nu_m = k\} = \frac{m^k}{k!} \int_0^\infty x^k \exp(-mx) d\mathcal{A}(x).$$
For any fixed $k \geq 0$ and $x > 0$ we have
$$\lim_{m\to\infty} \frac{m^k x^k \exp(-(m-1)x)}{k!} = 0.$$
Therefore,
$$\sum_{k=0}^{[B]} \mathbb{P}\{\nu_m = k\} \longrightarrow 0 \text{ as } m \to \infty,$$

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where $[B]$ denotes integer part of B. It means $\mathbb{P}\{\nu_m \geq B\} \to 1$ as $m \to \infty$, that is ν_m converges to infinity in probability.

Consider the following random sum $\sum_{k=1}^{\nu_m} X_k/m$. The Laplace transform of its distribution is

$$\mathcal{P}_m(f(s/m)) = \varphi(m(1 - f(s/m))) \longrightarrow \varphi(s).$$

Here $f(s)$ is the Laplace transform of the distribution of X_1 and we have used the fact $\mathbb{E}X_1 = 1$. □

Note that Theorem 2.1 gives possible limit distributions for the random sums, which are not necessarily stable. The finding analogues of stable distributions for sums of a random number of random variables is more complicated. Some results in this direction are given in [5]

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