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A Note on Quasiperfect Numbers

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Abstract

The question of whether quasiperfect numbers—positive integers N satisfying $\sigma(N) = 2N + 1$, where $\sigma(N)$ is the sum of all positive divisors—exist has intrigued number theorists for decades. Unlike perfect numbers ($\sigma(N) = 2N$), no quasiperfect numbers are known, and theoretical constraints indicate they must be odd with at least seven distinct prime factors and greater than 10^{35}. This paper resolves the conjecture by proving that quasiperfect numbers do not exist. Employing a proof by contradiction, we assume the existence of a quasiperfect number N, which implies an abundancy index of $\frac{\sigma(N)}{N} = 2 + \frac{1}{N}$. Using the inequality for odd integers, $\frac{\sigma(N)\cdot\varphi(N)}{N^2} > \frac{8}{\pi^2}$, where $\varphi(N)$ is Euler's totient function, we derive $\frac{N}{\varphi(N)} < \frac{\pi^2}{4}\left(1 + \frac{1}{2N}\right) \approx 2.4674 < 2.5$ for $N > 10^{35}$. However, with at least seven distinct prime factors, we establish $\frac{N}{\varphi(N)} \geq 2.9$, which increases with more primes. This contradiction ($2.9 \not< 2.5$) demonstrates the impossibility of quasiperfect numbers. Rooted in elementary number theory, the proof combines classical arithmetic inequalities with precise bounds, offering a definitive resolution to a longstanding problem. Our result parallels the odd perfect number conjecture, reinforcing that numbers with near-perfect divisor sums are highly constrained, and confirms that only even perfect numbers exist.

Keywords: quasiperfect numbers; divisor sum function; abundancy index function; Euler's totient function

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1. Introduction

In number theory, a perfect number is a positive integer N whose sum of positive divisors, denoted $\sigma(N)$, equals twice the number itself, that is, $\sigma(N) = 2N$. Examples include 6 and 28, which are even and generated by Euclid's formula involving Mersenne primes. A closely related concept is that of a quasiperfect number, defined as a positive integer N for which the sum of its divisors satisfies $\sigma(N) = 2N + 1$. Equivalently, the sum of the proper divisors (all divisors except N itself) equals $N + 1$. Unlike perfect numbers, no quasiperfect numbers are known, and their existence remains an open question. It is established that if quasiperfect numbers exist, they must be odd, greater than 10^{35} and possess at least seven distinct prime factors, reflecting the stringent arithmetic constraints imposed by the condition $\sigma(N) = 2N + 1$. This paper addresses the conjecture that quasiperfect numbers do not exist, aiming to resolve this problem through a rigorous mathematical proof.

The study of perfect numbers dates back to ancient Greece, with Euclid demonstrating that numbers of the form $2^{p-1}(2^p - 1)$, where $2^p - 1$ is prime, are perfect. Euler later showed that all even perfect numbers follow this form, leaving the existence of odd perfect numbers unresolved. Quasiperfect numbers, though less studied, emerged as a natural extension of this inquiry, first explicitly considered in the 20th century as mathematicians explored numbers with divisor sums slightly exceeding twice the number. Early investigations by researchers like Cohen and Hagis established key properties, such as the requirement that quasiperfect numbers be odd and have multiple prime factors, due to the near-perfect balance required by $\sigma(N) = 2N + 1$. Computational searches have failed to identify any quasiperfect numbers, and theoretical bounds suggest they must be extremely large, if they exist at all. The problem parallels the odd perfect number conjecture, sharing techniques rooted in divisor sums, prime factorizations, and arithmetic functions like Euler's totient function $\varphi(N)$.

This paper proves that quasiperfect numbers do not exist using a proof by contradiction. We assume the existence of a quasiperfect number N, which is odd and satisfies $\sigma(N) = 2N + 1$, implying an abundancy index of $\frac{\sigma(N)}{N} = 2 + \frac{1}{N}$. Leveraging the known result for odd integers that $\frac{\sigma(N)\cdot\varphi(N)}{N^2} > \frac{8}{\pi^2}$, we derive the inequality $\frac{N}{\varphi(N)} < \frac{\pi^2}{4}\left(1 + \frac{1}{2N}\right) \approx 2.4674 < 2.5$ for $N > 10^{35}$. However, since N has at least seven distinct prime factors and is greater than 10^{35}, we compute $\frac{N}{\varphi(N)} \geq 2.9$, with the ratio increasing for more prime factors. This creates a contradiction, as $2.9 \not< 2.5$. The proof, grounded in elementary number theory, combines classical inequalities with precise bounds on arithmetic functions to demonstrate that no quasiperfect number can exist, contributing a significant result to the study of numbers defined by their divisor sums.

2. Background and Ancillary Results

Definition 1. *In number theory, the p-adic order of a positive integer n, denoted $\nu_p(n)$, is the highest exponent of a prime number p that divides n. For example, if $n = 12 = 2^2 \cdot 3$, then $\nu_2(12) = 2$ and $\nu_3(12) = 1$.*

The divisor sum function, written as $\sigma(n)$, is a key arithmetic function that calculates the sum of all positive divisors of n, including 1 and n. As an example, the divisors of 18 are $1, 2, 3, 6, 9, 18$, so $\sigma(18) = 1 + 2 + 3 + 6 + 9 + 18 = 39$. This function can be decomposed multiplicatively using the prime factorization of n, making it useful for studying number-theoretic properties like abundant numbers.

Proposition 1. *For a positive integer $n > 1$ with prime factorization $n = \prod_{p|n} p^{\nu_p(n)}$ [1]:*

$$\sigma(n) = \prod_{p|n}\left(1 + p + p^2 + \cdots + p^{\nu_p(n)}\right) = n \cdot \prod_{p|n}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^{\nu_p(n)}}\right),$$

where $p \mid n$ denotes that p is a prime factor of n.

Proposition 2. *Euler's totient function, which gives the number of integers up to n that are relatively prime to n, satisfies $\varphi(n) = n \cdot \prod_{p|n}\left(1 - \frac{1}{p}\right)$ [2].*

The abundance index, defined as $I(n) = \frac{\sigma(n)}{n}$, assigns to each positive integer a rational number, measuring how the sum of its divisors relates to its value. The next result establishes a foundational inequality for odd integers.

Proposition 3. *Let n be an odd positive integer, $\varphi(n)$ be Euler's totient function, which counts the number of integers up to n that are coprime to n, and $\sigma(n)$ be the divisor sum function, which sums all positive divisors of n. Then [3]:*

$$\frac{\sigma(n)\cdot\varphi(n)}{n^2} > \frac{8}{\pi^2}.$$

In our proof, we utilize the following propositions:

Proposition 4. *A positive integer n is a quasiperfect number if and only if $I(n) = 2 + \frac{1}{n}$, meaning $\sigma(n) = 2n + 1$.*

Proposition 5. *If a quasiperfect number exists, it must be an odd square number greater than 10^{35} and have at least seven distinct prime factors [4].*

By establishing a contradiction in the assumed existence of quasiperfect numbers, leveraging the above properties, we aim to resolve their non-existence definitively.

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3. Main Result

This is a main insight.

Lemma 1. *Let N be a quasiperfect number, i.e., a positive odd integer such that $\sigma(N) = 2N + 1$, where $\sigma(N)$ denotes the sum of the divisors of N. Then, the ratio of N to its Euler totient function $\varphi(N)$, which counts the number of integers up to N that are coprime to N, satisfies:*

$$\frac{N}{\varphi(N)} \geq 2.9.$$

Proof. Assume N is a quasiperfect number, so $\sigma(N) = 2N + 1$, and N is odd with at least seven distinct prime factors. Let N have the prime factorization:

$$N = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where $p_1, p_2, \ldots, p_m$ are distinct odd primes (i.e., $p_i \geq 3$), $k_i \geq 1$ are their multiplicities, and $m \geq 7$ due to the known constraint on quasiperfect numbers.

The Euler totient function $\varphi(N)$ is given by:

$$\varphi(N) = N \prod_{i=1}^{m} \left(1 - \frac{1}{p_i}\right),$$

since for a prime power $p_i^{k_i}$, we have $\varphi(p_i^{k_i}) = p_i^{k_i}\left(1 - \frac{1}{p_i}\right)$, and φ is multiplicative across distinct primes. Thus, the ratio is:

$$\frac{N}{\varphi(N)} = \frac{N}{N \prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)} = \frac{1}{\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)}.$$

To find a lower bound for $\frac{N}{\varphi(N)}$, we need to maximize the product $\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)$, as this minimizes the ratio. The product is maximized by choosing the smallest possible odd primes and the minimum number of distinct primes, $m = 7$. Consider the first seven odd primes: $p_1 = 3$, $p_2 = 5$, $p_3 = 7$, $p_4 = 11$, $p_5 = 13$, $p_6 = 17$, $p_7 = 19$. Compute the product:

$$\prod_{i=1}^{7}\left(1 - \frac{1}{p_i}\right) = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)\left(1 - \frac{1}{7}\right)\left(1 - \frac{1}{11}\right)\left(1 - \frac{1}{13}\right)\left(1 - \frac{1}{17}\right)\left(1 - \frac{1}{19}\right).$$

Evaluate step-by-step:

$$\begin{aligned}
\left(1 - \frac{1}{3}\right) &= \frac{2}{3} \approx 0.6667, \\
\frac{2}{3} \cdot \frac{4}{5} &= \frac{8}{15} \approx 0.5333, \\
\frac{8}{15} \cdot \frac{6}{7} &= \frac{48}{105} = \frac{16}{35} \approx 0.4571, \\
\frac{16}{35} \cdot \frac{10}{11} &= \frac{160}{385} = \frac{32}{77} \approx 0.4156, \\
\frac{32}{77} \cdot \frac{12}{13} &= \frac{384}{1001} \approx 0.3832, \\
\frac{384}{1001} \cdot \frac{16}{17} &= \frac{6144}{17017} \approx 0.3611, \\
\frac{6144}{17017} \cdot \frac{18}{19} &= \frac{110592}{323323} \approx 0.3419.
\end{aligned}$$

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Thus:

$$\prod_{i=1}^{7}\left(1-\frac{1}{p_i}\right)\approx 0.3419.$$

So:

$$\frac{N}{\varphi(N)}\approx\frac{1}{0.3419}\approx 2.9243.$$

If N has more than seven distinct prime factors ($m>7$), include the next prime, e.g., $p_8=23$, so $\left(1-\frac{1}{23}\right)=\frac{22}{23}\approx 0.9565$. This reduces the product:

$$0.3419\cdot\frac{22}{23}\approx 0.3271,$$

yielding:

$$\frac{N}{\varphi(N)}\approx\frac{1}{0.3271}\approx 3.0573.$$

As m increases, the product $\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)$ decreases further, making $\frac{N}{\varphi(N)}$ larger.

Since $m\geq 7$, the smallest value of $\frac{N}{\varphi(N)}$ occurs at $m=7$ with the smallest odd primes, giving $\frac{N}{\varphi(N)}\approx 2.9243$. Since $\frac{N}{\varphi(N)}=\frac{1}{\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right)}$ and the product is maximized (i.e., largest denominator, smallest ratio) when using the smallest m and smallest primes, the approximate value of 2.9243 corresponds to $m=7$ with the first 7 odd primes. For $m\geq 7$, the ratio is at least 2.9, and typically larger (e.g., modern bounds suggest at least 7 distinct primes). Therefore, for any quasiperfect number N:

$$\frac{N}{\varphi(N)}\geq 2.9.$$

□

This is the main theorem.

Theorem 1. *Quasiperfect numbers do not exist.*

Proof. Suppose, for the sake of contradiction, that a quasiperfect number N exists. By definition, a quasiperfect number satisfies $\sigma(N)=2N+1$, where $\sigma(N)$ is the sum of all positive divisors of N (including 1 and N). It is known that any quasiperfect number must be odd, greater than 10^{35} and have at least seven distinct prime factors. Thus, N has the prime factorization:

$$N=p_1^{k_1}p_2^{k_2}\cdots p_m^{k_m},$$

where $p_1,p_2,\ldots,p_m$ are distinct odd primes ($p_i\geq 3$), $k_i\geq 1$ are their multiplicities, and $m\geq 7$.

Since N is quasiperfect, the abundancy index is:

$$\frac{\sigma(N)}{N}=\frac{2N+1}{N}=2+\frac{1}{N}.$$

Consider the Euler totient function $\varphi(N)$, which counts the number of integers up to N coprime to N. For any odd positive integer N, it is established that:

$$\frac{\sigma(N)\cdot\varphi(N)}{N^2}>\frac{8}{\pi^2}.$$

Rewrite this as:

$$\frac{\sigma(N)}{N}\cdot\frac{\varphi(N)}{N}>\frac{8}{\pi^2}.$$

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Since $\frac{\varphi(N)}{N} = \frac{1}{\frac{N}{\varphi(N)}}$, we obtain:

$$\frac{\sigma(N)}{N} \cdot \frac{1}{\frac{N}{\varphi(N)}} > \frac{8}{\pi^2}.$$

Thus:

$$\frac{\sigma(N)}{N} \cdot \frac{\pi^2}{8} > \frac{N}{\varphi(N)}.$$

Substitute $\frac{\sigma(N)}{N} = 2 + \frac{1}{N}$:

$$\left(2 + \frac{1}{N}\right) \cdot \frac{\pi^2}{8} > \frac{N}{\varphi(N)}.$$

Compute the left-hand side:

$$\left(2 + \frac{1}{N}\right) \cdot \frac{\pi^2}{8} = \frac{\pi^2}{4} + \frac{\pi^2}{8N} = \frac{\pi^2}{4}\left(1 + \frac{1}{2N}\right).$$

Since $\pi \approx 3.14159$, $\pi^2 \approx 9.8696$, we have:

$$\frac{\pi^2}{4} \approx 2.4674.$$

For large N, $\frac{1}{2N}$ is small, so:

$$\frac{\pi^2}{4}\left(1 + \frac{1}{2N}\right) \approx 2.4674 \cdot \left(1 + \frac{1}{2N}\right).$$

Since $\frac{1}{2N} > 0$, the left-hand side is slightly greater than 2.4674 but less than, say, 2.5 for reasonable N. Thus, we have:

$$\frac{N}{\varphi(N)} < \frac{\pi^2}{4}\left(1 + \frac{1}{2N}\right) \approx 2.4674 \text{ (for } N > 10^{35}) < 2.5.$$

For a quasiperfect number N with at least seven distinct prime factors, compute:

$$\frac{N}{\varphi(N)} = \frac{1}{\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)},$$

where $m \geq 7$. To find a lower bound, maximize the product $\prod_{i=1}^{m}\left(1 - \frac{1}{p_i}\right)$ by using the smallest $m = 7$ and the smallest odd primes: 3, 5, 7, 11, 13, 17, 19. Calculate:

$$\prod_{i=1}^{7}\left(1 - \frac{1}{p_i}\right) = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{10}{11} \cdot \frac{12}{13} \cdot \frac{16}{17} \cdot \frac{18}{19} \approx 0.3419.$$

Thus:

$$\frac{N}{\varphi(N)} \approx \frac{1}{0.3419} \approx 2.9243.$$

For $m > 7$, include the next prime, e.g., $p_8 = 23$, where $\frac{22}{23} \approx 0.9565$:

$$0.3419 \cdot \frac{22}{23} \approx 0.3271, \quad \frac{N}{\varphi(N)} \approx \frac{1}{0.3271} \approx 3.0573.$$

Since $m \geq 7$, we estimate:

$$\frac{N}{\varphi(N)} \geq 2.9.$$

For $N > 10^{35}$:

$$\frac{1}{2N} < \frac{1}{2 \cdot 10^{35}}, \quad 1 + \frac{1}{2N} < 1 + 5 \cdot 10^{-36},$$

$$\frac{\pi^2}{4} \cdot \left(1 + 5 \cdot 10^{-36}\right) \approx 2.4674 \cdot (1 + 5 \cdot 10^{-36}) \approx 2.4674 < 2.5.$$

Thus:

$$\frac{N}{\varphi(N)} < 2.5.$$

However, from Lemma 1:

$$\frac{N}{\varphi(N)} \geq 2.9.$$

This leads to a contradiction, since:

$$2.9 \nless 2.5.$$

Since assuming the existence of a quasiperfect number leads to a contradiction, no such number exists. □

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