Article

Not peer-reviewed version

Witt-like Operators For Deriving Virasoro Algebra with Central Charge

Huber Nieto-Chaupis *

Posted Date: 2 July 2025

doi: 10.20944/preprints202506.2080.v1

Keywords: Witt algebra; Virasoro algebra; quantum mechanics; commutators

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

Article

Witt-like Operators For Deriving Virasoro Algebra with Central Charge

Huber Nieto-Chaupis

Universidad Autónoma del Perú; hubernietochaupis@gmail.com

Abstract

Operators in the form of integrals containing the Dirac-delta function and the first derivative with respect to coordinate have been constructed on the grounds of quantum mechanics formalism have demonstrated to fulfill Witt algebra. Such operators appear to be connected to total angular momentum and momentum operators, and to some extent to well-known free particle Hamiltonian. Such operators have allowed to develop a noteworthy semi-classical formalism that involve classical definitions. These objects called Witt-like operators when are expressed as polynomials have exhibited to be dependent on quantum mechanics 1-dimension momentum operator. In this manner, Witt-operators have been redefined to explore their involvement in the derivation of Virasoro algebra. After of working out in the closed-form derivation, it was found that these Witt-like operators have reproduced a kind of deformed Virasoro algebra with an interesting connection between the central charge a Grassmann operators that have inherently emerged from the redefinition of Wiit-like operators. The results of this paper are clearly demonstrating that would exist a direct link between theoretical methodologies that extraordinarily describe string theories and non relativistic quantum mechanics, dictated by scalar-based constructions and Schrödinger equation

---

1. Introduction

1.1. Motivation

Once Veneziano [1] proposed an elegant methodology to treat four point amplitude at the context of a newborn strong interaction theory, through beta functions, the field was boosted substantially. In this manner, new methods of perturbation for Feynman amplitudes based in unitarity were developed by Kikkawa and others in [2]. Guided by the advance in the field, Virasoro in [3] introduced new strategies to study dual-resonance models through operators from quantum mechanics operators known later as gauge operators. Because this evident progress, the Virasoro's ideas were worked out by Brower in [4] over the line of algebraic structures. In this manner the annihilation and creation operators were attained to novel definitions such as $L_0 = \frac{1}{2}p_0^2 + \sum_n a_n^\dagger a_n$. In 1972 Frampton and Nielsen [5] have designed a novel model of string theory by which new gauge operators L_m and L_n (suggested by Virasoro in its paper [3]) based strictly in annihilation and operators supports the commutator defined as: $[L_m, L_n] = (n-m)L_{m+n} + s\delta_{m,-n}$ with $s = \frac{1}{3}(n^3 - n)$. Such operators were defined as: $L_n = -\sqrt{2}pa^{(n)\dagger}$-$\frac{1}{2}\sum_{r=1}^{n-1}\sqrt{r(n-r)}a^{(r)\dagger}a^{(n-r)\dagger} - \sum_{r=1}^{\infty}\sqrt{r(n+r)}a^{(r+n)\dagger}a^{(r)}$. In [6] Veneziano specifies steps towards no-ghost states based at algebra built with gauge operators. The constant $\frac{1}{12}$ that multiplies central charge in algebra as accepted nowadays was determined in [7]. The centerless Virasoro algebra also known as Witt algebra received some attention in a territory dominated by the mathematical rigorousness as seen in [8]. Kaplansky in [9] studied the validity and applicability range of Virasoro algebra in physics in the context of string theory. It important that algebra in this investigation was rigorously characterized by three theorems, being the most important the one that states Lie algebra is isomorphic to Virasoro algebra without central charge. The central charge in its role in gravity models was studied in [10] in order to demonstrate that Poisson brackets is a central extension of conformal group algebra. Redlich in [11] followed the debate of the relevance of central

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

charge in the context of manifolds in supergravity and supersymmetry. In 1987, Murthy [12] provides a first non-relativistic quantum mechanics interpretation to gauge operators in the sense of calling them conformal transformations such as: $L_n = z^{n+1}\frac{d}{dz}$ fulfilling $[L_m, L_n] = (n-m)L_{n+m}$, with L_0 identified as angular momentum related to gauge operators. In [13] Akhoury and Okada, have employed Bjorken-Johnson-Low (BJL) procedure based at the amplitude $\langle A|\,[L_n^{\mathcal{H}}(0), L_m^{\mathcal{H}}(0)]\,|B\rangle$ (being the bra and ket arbitrary states and $\mathcal{H}$ denoting the Heisenberg operators) to arrive commutation relations encompassed to Virasoro algebra. Indeed, in an inspiring manner they have linked evolution operator and Schrödinger equation through the usage of Heisenberg operators. In [14-16] further capabilities of algebra under study were examined as to its consistency and feasibility to tackle down weakness in string theory. The association of Virasoro and Kac-Moody algebra by means the commutator $[L_m, T_n^a] = -nT_{m+n}^a$ was boarded by N. Sakai and P. Suranyi in [17]. A rather motivational work was that of Embacher [18] in which defines at the form of covariant Virasoro operators to inspect string states embedded in an action functional. In [19] Curtright and Zachos have treated deformed Virasoro algebra to derive important rules of quantum commutators. As noted by Kato in [20], string theory as well particle physics can be limited to certain minimum distances and as consequence that would valid this conjecture the Virasoro algebra emerges. Under this approach the gauge operators acquire the form as: $L_n = \frac{\kappa}{2}\int_{-\pi}^{\pi} d\tau \exp[in\tau p_\mu(\tau)p^\mu(\tau)]$ with κ having dimension of length square. In [21-23] further realizations and studies on the central charge were done. The idea that algebra can play a notable role in efective theory of field was developed in [24-29], as for example the work of Chuo at 1994. The fact that classical variables are closely related to algebra generators was boarded by Anderson in [30]. The version holomorephic for $\mathcal{N} = 2$ in a supersymmetric conformal theory as an extension of Virasoro algebra was studied by Blumenhagen in [31]. A lot of progress was seen in [32-38] later emphasizing diverse approaches for the study of central charge. Virasoro algebra was directly linked to study of black holes as can be seen in the work of Bañados [39] where it is linked the Ramon-Virasoro and Virasoro operators Q_n and L_n, respectively through the relation $L_n = \frac{1}{c}Q_{cn}$ with c central charge. Here, the Virasoro algebra is recovered by mean of $[L_m, L_n] = \frac{1}{c^2}[Q_{cn}, Q_{cm}]$ yielding $\frac{1}{c^2}\left[c(n-m)Q_{c(m+n)} + \frac{c^3n^3}{12}\delta_{c(m+n)}\right]$. Associations of algebra to quantum mechanics realizations can be seen in [40-47]. Pedagogical demonstrations about the exact derivation of Virasoro algebra can be read in the work of Zwiebach [48]. Here, the factor $\frac{1}{12}$ is derived from mathematical induction $\sum_{k=1}^{m} k^2 = \left[\frac{1}{6}(2m^3 + 3m^2 + m)\right]$ that would led elegantly to term $\frac{1}{12}(m^3 - m)$. Kojima and Sairaichi in [49] have presented various propositions, definitions and theorems regarding realizations of deformed Virasoro algebra in a rigorous manner to construct elliptic deformation of integrals of motion in the arena xof conformal field theory. Other realizations no less relevant can be found in [50-60]. In [61] Gao and collaborators, have used the probabilistic theory of Rota-Baxter to generate operators fulfilling Virasoro algebra. Thus, the Rota-Baxter homogeneous operator is given by: $R_k^{\ell,\gamma}(L_m) = \frac{k}{m+2k}\gamma\delta_{m+k,\ell}L_{m+k}$ with m, z, ℓ integer numbers whereas $k$andγ complex numbers. In [62] Bagchi and collaborators have faced Bose-Einstein condensation by using tensile closed string theory in which it is seen a prominent role varied versions of Virasoro algebra such as Bondi-Metzner-Sachs algebra given by $[L_m, M_n]$=$(m-n)L_{m+n}$+$\frac{c_L}{12}m(m^3-1)\delta_{m+n,0}$. In [63-65] an interesting progress focused at th Heisenberg-Virasoro was noted. Recently, Virasoro algebra studies has taken an important boost when it is linked to quantum mechanics as seen in the work of Kim and collaborators [68]. In fact, they have arrived to novel definition $\langle\Psi|\,[[\tilde{L}_n, L_0], \tilde{L}_{-n}]\,|\Psi\rangle$=$\frac{\pi c}{48}n(n^2-1)\tan\frac{\pi}{2n}$. In [69], the Witt algebra was derived through the usage of simple scalar formulation. It yielded that algebra might to enclose the quantum mechanics fundamental commutation relation fact that have left to conclude whether quantum mechanics might be a theory entirely or partially derived from string theory. The present paper is to some extent a continuation of it.

1.2. Structure of Paper

In second section, the Witt algebra is presented in association to the derivation of Witt-like operators. I was seen that these entities might be closely related to total angular momentum and

 doi:10.20944/preprints202506.2080.v1

momentum operators. These called Witt-like operators are studied in a territory of Hamiltonians. In third section the operational properties of Witt-like operators are presented. In fourth section, the polynomial Wiit algebra is presented, by emphasizing the connotation of integer numbers function, fact that would be a solid reason to investigate new forms of Witt-like operators. Therefore, in fifth section the deformed Witt-like and Virasoro algebra have been derived. Finally, in sixth section the commutator involving new representation of Witt algebra is briefly treated. Finally, the conclusion of paper is presented.

2. Witt Algebra in Quantum Mechanics

2.1. Witt-like Operators and Quantum Mechanics Operators

Consider the Schrödinger equation written in the following manner:

$$\mathbf{H}\Psi(x,t) = \mathbf{L}f(x,t), \tag{1}$$

being $\mathbf{L}$ the total momentum angular and $f(x,t)$ a coordinate-dependent arbitrary function being it proportional to wave function $\Psi(x,t)$ such as:

$$f(x,t) = \mathbf{S}\Psi(x,t), \tag{2}$$

with $\mathbf{S}$ an arbitrary operator that allows to write Equation (1) again as:

$$\mathbf{H}\Psi(x,t) = \mathbf{L}\cdot\mathbf{S}\Psi(x,t). \tag{3}$$

For example when $\mathbf{S}$ denoting spin operator then Equation (3) is easily recognized as spin-orbit coupling Hamiltonian [70]. On the other side consider for example in Equation (1) that $f(x)$ is only polynomial x^m with $m \in \mathbb{Z}$. Indeed with the well-known Schrödinger equation in Equation (1) one gets:

$$i\hbar\frac{d}{dt}\Psi_m(x,t) = x^m\mathbf{L}, \tag{4}$$

with $\hbar = \frac{h}{2\pi}$ and h the Planck's constant [71]. Consider the commutation relationship:

$$[\mathbf{L}, x^m] = 0. \tag{5}$$

It should be noted that order of wave function coincides with power of polynomial in right-side of Equation (4) above. In this way, one can define the Witt-like operator as follows:

$$\mathcal{L}_m = \frac{d}{dt}\Psi_m(x,t), \tag{6}$$

establishing a direct link of operators $\mathcal{L}_m$ with ordinary quantum mechanics wave function. Because this, it would led to Equation (4) to be rewritten as:

$$i\hbar\mathcal{L}_m = x^m\mathbf{L}. \tag{7}$$

It is clear that under the previous equations, the total angular momentum can be now written as:

$$\mathbf{L} = i\hbar x^{-m}\mathcal{L}_m. \tag{8}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

Although Equation (7) reflects the fact that there is a linear correspondence between a physical operator and one of arbitrary nature, it is feasible to work out in a nonlinear level at the sense of opting for the quadratical scheme such as:

$$-\hbar^2 x^{-2m}\mathcal{L}_m^2 = \mathbf{L}^2 \Rightarrow -\hbar^2\mathcal{L}_m^2 = x^{2m}\mathbf{L}^2. \tag{9}$$

Under the argument as stated above $f(x) = x^{2m}$ as well as Equation (5) then commutator $[\mathbf{L}, x^{2m}] = 0$. It is evident from Equation (9) right-side one gets:

$$-\hbar^2\mathcal{L}_m^2 = \mathbf{L}^2 x^{2m}. \tag{10}$$

One can treat right-side of Equation (10) as a part of the eigenvalues equation of total angular momentum in the following sense:

$$\mathbf{L}^2 x^{2m} = \hbar^2(m+1)x^{2m}. \tag{11}$$

By inserting Equation (11) into Equation (10) one gets

$$-\hbar^2\mathcal{L}_m^2 = \hbar^2(m+1)x^{2m}, \tag{12}$$

that allows to write the square of Witt-like operators in the following way:

$$\mathcal{L}_m^2 = -(m+1)x^{2m}. \tag{13}$$

It is obvious that Witt-like operators are inherently complex so that one can write them as:

$$\mathcal{L}_m = i\sqrt{m+1}x^m. \tag{14}$$

Aside, one can claim that Equation (14) can be seen as a particular case of a most general definition of root square $\sqrt{m+1}$ by which this can be rewritten as $\sqrt{m+r+1}$, with $r \in \mathbb{Z}$. Therefore, Equation (14) is valid for $r = 0$, nevertheless for all other cases one can rewrite Equation (14) as [69]:

$$\mathcal{L}_m = \frac{i}{\sqrt{m+r+1}}(m+1)x^m. \tag{15}$$

It is easy to note that Equation (15) can also be expressed as a derivative respect to x such as:

$$\mathcal{L}_m = \frac{i}{\sqrt{m+r+1}}\frac{d}{dx}x^{m+1}. \tag{16}$$

While multiplying both sides by $\frac{\hbar}{i}$ the right-side emerges in an explicit manner the well-known momentum operator in the representation of coordinate:

$$\mathbf{p} = \frac{\hbar}{i}\frac{d}{dx}, \tag{17}$$

so that one arrives to a clear dependence of Witt-like operator of quantum mechanics 1-dimension momentum operator written as:

$$\frac{\hbar}{i}\mathcal{L}_m = \frac{i}{\sqrt{m+r+1}}\frac{\hbar}{i}\frac{d}{dx}x^{m+1} = \frac{i}{\sqrt{m+r+1}}\mathbf{p}x^{m+1}, \tag{18}$$

by which Witt-like operator can be written also as:

$$\mathcal{L}_m = -\frac{1}{\hbar\sqrt{m+r+1}}\mathbf{p}x^{m+1}, \tag{19}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

and subsequently one can test its validity by fulfilling the Witt-algebra:

$$[\mathcal{L}_m, \mathcal{L}_n] = \mathcal{L}_m\mathcal{L}_n - \mathcal{L}_n\mathcal{L}_m = \frac{1}{\hbar^2(m+r+1)}\left[\mathbf{p}x^{m+1}, \mathbf{p}x^{n+1}\right], \tag{20}$$

and with the usage of identity: $[AB, CD]{=}A[B, C]D{+}[A, C]BD + CA[B, D] + C[A, D]B$ applied in a straightforward manner in Equation (20) one arrives directly to Witt algebra, such as:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\frac{m+n+1}{m+r+1}x^{m+n} = (m-n)x^{m+n} \Leftrightarrow n = r. \tag{21}$$

The quantum mechanics total angular momentum operator can be tested in the framework of Witt algebra. For example, from Equation (7) the angular momentum operator can be expressed as function of operator Witt-like operator $\mathcal{L}_m$ as follows:

$$\mathbf{L}_m = \frac{i\hbar\mathcal{L}_m}{x^m}. \tag{22}$$

The subindex "m" in physical operator (total angular momentum) is clearly valid because the definition of $\mathcal{L}_m$ in right-side. Taking into account Equation (22) then one can assume that physical operator satisfies Witt algebra written as:

$$[\mathbf{L}_m, \mathbf{L}_n] = (m-n)\mathbf{L}_{m+n} = (m-n)\frac{i\hbar\mathcal{L}_{m+n}}{x^{m+n}}. \tag{23}$$

As noted in [69] and from Equation (21) one can assume that $\mathcal{L}_{m+n}{=}x^{m+n}$, so that one gets:

$$[\mathbf{L}_m, \mathbf{L}_n] = i\hbar(m-n). \tag{24}$$

From the quantum mechanics commutator [72]:

$$[\mathbf{L}_m, \mathbf{L}_n] = i\epsilon_{mnq}\mathbf{L}_q, \tag{25}$$

with ϵ_{mnq} Levi-Cevita tensor and from Equation (24) one can naively to arrive in the following relationship:

$$\mathbf{L}_q = \frac{\hbar(m-n)}{\epsilon_{mnq}}. \tag{26}$$

By knowing the well-known fundamental commutation relation for momentum and position operators: $[x, \mathbf{p}_x] = i\hbar$, under the present context argued by Equation (23) and equations above it is easy to arrive to:

$$[\mathbf{L}_m, \mathbf{L}_n] = [x, \mathbf{p}](m-n). \tag{27}$$

Also, by comparing Equation (25) and Equation (27) the fundamental commutation relation can be now written as:

$$[x, \mathbf{p}_x] = \mathbf{L}_q\frac{i\epsilon_{mnq}}{m-n}. \tag{28}$$

Reader should be aware that subindex as given in Equation (25) could be replaced by coordinates by following the congruent content of commutator of Equation (28) left-side. Nevertheless, such subindex at the order "mnq" as well as integer numbers appearing in denominator "$m-n$" above

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

are kept because them come from Witt algebra. In this way with the change $\mathbf{p}_x \rightarrow \mathbf{p}_y$ then one can tentatively to validate consistently the commutator write down as:

$$[x, \mathbf{p}_y] = \mathbf{L}_z \frac{i\epsilon_{xyz}}{m-n}, \tag{29}$$

or also in a manner encompassing the context based in integer numbers as defined at the beginning of this document:

$$[x_m, \mathbf{p}_n] = \mathbf{L}_q \frac{i\epsilon_{mnq}}{m-n}. \tag{30}$$

2.2. *Witt Algebra and Hamiltonians*

From Equation (6) the case for $m+n$ term can be written as:

$$\mathcal{L}_{m+n} = \frac{d}{dt}\Psi_{m+n}(x,t), \tag{31}$$

in conjunction to Witt algebra so that one can write down:

$$\left[\frac{d}{dt}\Psi_m(x,t), \frac{d}{dt}\Psi_n(x,t)\right] = (m-n)\frac{d}{dt}\Psi_{m+n}(x,t). \tag{32}$$

Operating the commutator:

$$\left[\frac{d}{dt}\Psi_m(x,t), \frac{d}{dt}\Psi_n(x,t)\right] = \frac{d}{dt}\Psi_m(x,t)\frac{d}{dt}\Psi_n(x,t) - \frac{d}{dt}\Psi_n(x,t)\frac{d}{dt}\Psi_m(x,t) \tag{33}$$

$$= \frac{d}{dt}\Psi_m(x,t)\frac{d}{dt}\Psi_n(x,t) + \Psi_m(x,t)\frac{d^2}{dt^2}\Psi_n(x,t) - \frac{d}{dt}\Psi_n(x,t)\frac{d}{dt}\Psi_m(x,t) - \Psi_n(x,t)\frac{d^2}{dt^2}\Psi_m(x,t) \tag{34}$$

$$\Rightarrow \Psi_m(x,t)\frac{d^2}{dt^2}\Psi_n(x,t) - \Psi_n(x,t)\frac{d^2}{dt^2}\Psi_m(x,t) = (m-n)\frac{d}{dt}\Psi_{m+n}(x,t). \tag{35}$$

Although the second derivative of wave function is not a genuine quantum mechanics definition, this simply is solved by using chain's rule so that one gets below:

$$\frac{d^2}{dt^2}\Psi(x,t) = \frac{d}{dt}\left[\frac{d}{dx}\Psi(x,t)\frac{dx}{dt}\right] = \frac{d^2}{dx^2}\Psi(x,t)\frac{d^2x}{dt^2}, \tag{36}$$

by which one can see that a kind of "acceleration" emerged. Roughly speaking, it provides a semi classical character to present formalism. By introducing Equation (36) into Equation (35) one obtains the following:

$$\Psi_m(x,t)\frac{d^2}{dx^2}\Psi_n(x,t)\frac{dx}{dt^2} - \Psi_n(x,t)\frac{d^2}{dx^2}\Psi_m(x,t)\frac{dx}{dt^2} = (m-n)\frac{d}{dt}\Psi_{m+n}(x,t)$$
$$= \left[\Psi_m(x,t)\frac{d}{dx^2}\Psi_n(x,t) - \Psi_n(x,t)\frac{d}{dx^2}\Psi_m(x,t)\right]\frac{d^2x}{dt^2} = (m-n)\frac{d}{dt}\Psi_{m+n}(x,t). \tag{37}$$

By multiplying both sides by $i\hbar$ one can see that right-side of Equation (37) acquires the following rather familiar form given by:

$$i\hbar\frac{d}{dt}\Psi_{m+n}(x,t), \tag{38}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

by suggesting the existence of type Schrödinger equation as follows:

$$i\hbar(m-n)\frac{d}{dt}\Psi_{m+n}(x,t) = i\hbar\left[\Psi_m(x,t)\frac{d^2}{dx^2}\Psi_n(x,t) - \Psi_n(x,t)\frac{d^2}{dx^2}\Psi_m(x,t)\right]\frac{d^2x}{dt^2}. \tag{39}$$

One can include the factor $\frac{\hbar^2}{i^2}$ aside $\frac{d^2}{dx^2}$ in order to arrive to:

$$i\hbar\frac{d}{dt}\Psi_{m+n}(x,t) = \frac{i^2 i\hbar}{\hbar^2(m-n)}\frac{d^2x}{dt^2}\left[\Psi_m(x,t)\frac{\hbar^2}{i^2}\frac{d^2}{dx^2}\Psi_n(x,t) - \Psi_n(x,t)\frac{\hbar^2}{i^2}\frac{d^2}{dx^2}\Psi_m(x,t)\right]. \tag{40}$$

Previous step was done to write in an explicit manner the momentum operator in the coordinate representation given by:

$$\frac{\hbar^2}{i^2}\frac{d}{dx^2} \equiv \frac{\hbar^2}{i^2}\nabla_x^2 \equiv \mathbf{p}_x^2, \tag{41}$$

that is included in Equation (40) yielding:

$$i\hbar\frac{d}{dt}\Psi_{m+n}(x,t) = \frac{i^2 i\hbar}{\hbar^2(m-n)}\frac{d^2x}{dt^2}\left[\Psi_m(x,t)\mathbf{p}_x^2\Psi_n(x,t) - \Psi_n(x,t)\mathbf{p}_x^2\Psi_m(x,t)\right]. \tag{42}$$

On the other side, because the free particle Hamiltonian:

$$\mathbf{H}_F = \frac{\mathbf{p}_x^2}{2M}, \tag{43}$$

into Equation (42) can be incorporated also mass M so that one arrives to:

$$\begin{aligned} i\hbar\frac{d}{dt}\Psi_{m+n}(x,t) &= \frac{2Mi}{\hbar(n-m)}\frac{d^2x}{dt^2}\left[\Psi_m(x,t)\frac{\mathbf{p}_x^2}{2M}\Psi_n(x,t) - \Psi_n(x,t)\frac{\mathbf{p}_x^2}{2M}\Psi_m(x,t)\right] \\ &= \frac{2i}{\hbar(n-m)}M\frac{d^2x}{dt^2}[\Psi_m(x,t)\mathbf{H}_F\Psi_n(x,t) - \Psi_n(x,t)\mathbf{H}_F\Psi_m(x,t)]. \end{aligned} \tag{44}$$

In this manner it is easy to recognize that emerged the classical (fact that would cause confusion in the present document, however it¿s suggested to reader see [73] where concepts can be encompassing to) force given by [74]:

$$F_C = M\frac{d^2x}{dt^2}, \tag{45}$$

so that Equation (44) can be rewritten again as:

$$i\hbar\frac{d}{dt}\Psi_{m+n}(x,t) = \frac{2iF_C}{\hbar(n-m)}[\Psi_m(x,t)\mathbf{H}_F\Psi_n(x,t) - \Psi_n(x,t)\mathbf{H}_F\Psi_m(x,t)]. \tag{46}$$

The well-know Schrödinger equation and Equation (46) allows to write that

$$\mathbf{H}\Psi_{m+n}(x,t) = \frac{2iF_C}{\hbar(n-m)}[\Psi_m(x,t)\mathbf{H}_F\Psi_n(x,t) - \Psi_n(x,t)\mathbf{H}_F\Psi_m(x,t)], \tag{47}$$

with $\mathbf{H} \neq \mathbf{H}_F$. In right-side Equation (47) one can apply in a straightforward manner the following eigenvalues equations written as:

$$\mathbf{H}_F\Psi_n(x,t) = \mathcal{E}_n\Psi_n(x,t), \tag{48}$$

$$\mathbf{H}_F\Psi_m(x,t) = \mathcal{E}_m\Psi_m(x,t). \tag{49}$$

With these eigenvalues equations Equation (47) acquires the form as:

$$\mathbf{H}\Psi_{m+n}(x,t) = \frac{2iF_C}{\hbar(n-m)}[\Psi_m(x,t)\mathcal{E}_n\Psi_n(x,t) - \Psi_n(x,t)\mathcal{E}_m\Psi_m(x,t)], \tag{50}$$

and with the condition of commutation $[\Psi_m(x,t), \Psi_n(x,t)] = 0$ then one arrives to:

$$\mathbf{H}\Psi_{m+n}(x,t) = \frac{2iF_C}{\hbar(m-n)}[\mathcal{E}_m - \mathcal{E}_n]\Psi_m(x,t)\Psi_n(x,t), \tag{51}$$

$$= \frac{[\mathcal{E}_m - \mathcal{E}_n]}{m-n}\frac{2iF_C}{\hbar}\Psi_m(x,t)\Psi_n(x,t). \tag{52}$$

It should be noted to fulfill physics units in both sides the wave functions must have units as

$$\Psi(x,t) \propto \sqrt{\frac{1}{\text{length} \times \text{time}}}. \tag{53}$$

It is easy to see in Equation (52) that case $m = n$ is discarded. However, one can illustrate Equation (52) for $m = -n$

$$\mathbf{H}\Psi_0(x,t) = \frac{[\mathcal{E}_m - \mathcal{E}_{-m}]}{m}\frac{iF_C}{\hbar}\Psi_m(x,t)\Psi_{-m}(x,t). \tag{54}$$

By looking into Equation (54), the factor $\frac{iF_C}{\hbar}$ is rewritten as $\frac{iF_C\ell\tau}{\hbar\ell\tau}$ so that follows:

$$\mathbf{H}\Psi_0(x,t) = \frac{[\mathcal{E}_m - \mathcal{E}_{-m}]}{m}\frac{i\ell\tau F_C}{\hbar\ell\tau}\Psi_m(x,t)\Psi_n(x,t). \tag{55}$$

On the other side, the classical force can be approximated as:

$$F_C \approx \frac{M\ell}{t^2}. \tag{56}$$

Consider the approximations: $[\mathcal{E}_m - \mathcal{E}_{-m}] \approx \hbar\mathbf{\Omega}$ (for example to see in [75]) that cancels $\hbar$ at denominator and would convert the analysis in one of classical nature. Also $\frac{1}{\tau} = \mathbf{\Omega}$, and $m = 2$ (and $n = -2$) that yields:

$$\mathbf{H}\Psi_0(x,t) = \frac{\hbar\mathbf{\Omega}^2}{2}\frac{i\ell^2\tau M}{\hbar\ell t^2}\Psi_2(x,t)\Psi_{-2}(x,t) = \frac{1}{2}M\ell^2\mathbf{\Omega}^2\frac{i\tau}{\ell t^2}\Psi_2(x,t)\Psi_{-2}(x,t)$$

$$= \left[\frac{1}{2}M\ell^2\mathbf{\Omega}^2\right]\frac{i\tau}{\ell t^2}\Psi_2(x,t)\Psi_{-2}(x,t), \tag{57}$$

where one recognize the potential energy of harmonic oscillator the term in brackets. While "τ" might to be expressing a kind of time due to quantum transitions, "t" a classical time associated to classical force because Equation (56).

3. Properties of Scalar Witt-like Operators

As seen in Equation (15) these Witt-like operators can be examined for example at the case of $r = 0$ leading to $\mathcal{L}_m = \frac{i}{\sqrt{m+1}}(m+1)\,x^m$. Interestingly for $m = 0$ it is evident $\mathcal{L}_0 = i$, for example. Their general form includes an extra integer r so that is reminded to reader the Witt-like operators as:

$$\mathcal{L}_m = \frac{i}{\sqrt{m+r+1}}(m+1)x^m, \tag{58}$$

 doi:10.20944/preprints202506.2080.v1

by which has been probed that expressed as function of quantum mechanics operator momentum satisfy directly the Witt algebra:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}. \tag{59}$$

Independently of Equation (15) these Witt-like operators can be expressed as a convolution integral. Furthermore, as seen in Equation (16) them can be written as a first derivative respect to coordinate with $r \to n$ in the following manner:

$$\mathcal{L}_m = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int_{-\infty}^{\infty} x^m y\delta(x-y)dy, \tag{60}$$

$$\mathcal{L}_n = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int_{-\infty}^{\infty} x^n z\delta(x-z)dz. \tag{61}$$

The term $\frac{1}{\sqrt{m+n+1}}$ emerges as a kind of normalization factor that would led to fulfill algebra Equation (59). As seen above, the variables y and z inside integration was done in order to omit redundancy along the subsequent integrations. Unlike previous definitions the ones given above contain the Dirac delta function. One can see the similarity of Equation (60) and Equation (61) with Laguerre's polynomials written as: $\mathcal{P}_n$=$e^x\delta_{1,k}\frac{d^k}{dx^k}\int_{-\infty\infty} x^n e^y\delta(y-x)dy$ [76]. A comparative analysis between these approaches is beyond the scope of this paper. By operating left-side of Equation (59), then one has below

$$\mathcal{L}_m \otimes \mathcal{L}_n = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int_{-\infty}^{\infty} x^m y\delta(x-y)dy\frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int_{-\infty}^{\infty} x^n z\delta(x-z)dz, \tag{62}$$

$$= \frac{-1}{m+n+1}\left(\frac{d}{dx}\int_{-\infty}^{\infty} x^m y\delta(x-y)dy\right)\left(\frac{d}{dx}\int_{-\infty}^{\infty} x^n z\delta(x-z)dz\right). \tag{63}$$

One can see that integral of right-side is trivial and result to be operated by left-side integration, so that one has below:

$$\mathcal{L}_m \otimes \mathcal{L}_n = \frac{-1}{m+n+1}\left(\frac{d}{dx}\int_{-\infty}^{\infty} x^m y\delta(x-y)dy\right)\left(\frac{d}{dx}x^{n+1}\right) \tag{64}$$

$$= \frac{-(n+1)}{m+n+1}\left(\frac{d}{dx}\int_{-\infty}^{\infty} x^{m+n} y\delta(x-y)dy\right) \tag{65}$$

$$= \frac{-(n+1)(m+n+1)}{m+n+1}x^{m+n} = -(n+1)x^{m+n}. \tag{66}$$

By applying a similar procedure as done from Equation (62) to 66, one can verify that product $\mathcal{L}_n \otimes \mathcal{L}_m$ turns out to be:

$$\mathcal{L}_n \otimes \mathcal{L}_m = \frac{-(m+1)(m+n+1)}{m+n+1}x^{m+n} = -(m+1)x^{m+n}. \tag{67}$$

When left-side of Equation (66) and Equation (67) are inserted into Equation (59) one gets in a straightforward manner:

$$[\mathcal{L}_m, \mathcal{L}_n] = -(n+1)x^{m+n} + (m+1)x^{m+n} = (m-n)x^{m+n}. \tag{68}$$

For example in [69] some attempts to demonstrate $\mathcal{L}_{m+n}$=x^{m+n} were done. However, from Equation (60) one can work out in the sense that:

$$\mathcal{L}_m = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int_{-\infty}^{\infty} x^m y\delta(x-y)dy,$$

$$\Rightarrow \mathcal{L}_{m+n} = \frac{i}{\sqrt{m+2n+1}}\frac{d}{dx}\int_{-\infty}^{\infty} x^{m+n} y\delta(x-y)dy. \tag{69}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

The whole operation is done in a straightforward manner, so that one arrives to:

$$\mathcal{L}_{m+n} = \frac{i}{\sqrt{m+2n+1}}\frac{d}{dx}x^{m+n+1} = \frac{i(m+n+1)}{\sqrt{m+2n+1}}x^{m+n}. \tag{70}$$

3.1. Custodial Role of Complex Part in Witt Algebra Operators

It's interesting to note that $\mathcal{L}_{m+n}$ "protects itself" of being a pure real number in the sense when root square in denominator acquires a complex value. For instance reader can check out for $m+2n=-2$, the complex part is canceled. However, it produces changes such as $\mathcal{L}_{m+n} \to \mathcal{L}_{-2-n}$=$-(n+1)x^{-(2+n)}$. The case of $n=-2$ yields the inconsistency $\mathcal{L}_0 = 1$ as seen above in Equation (15) when $r=0$ one gets for $m=0$ $\mathcal{L}_{m=0} = i$. In this manner the only path that determines right-side Equation (68) is same than right-side Equation (50) is given by the condition that establishes:

$$m+n = \frac{i(m+n+1)}{\sqrt{m+2n+1}}. \tag{71}$$

3.2. Examples

From $[\mathcal{L}_m, \mathcal{L}_n]$=$(m-n)x^{m+n}$, some examples can illustrate the behavior of main commutator $[\mathcal{L}_m, \mathcal{L}_n]$ as for example the arbitrary case of $m=-n$ one gets:

$$[\mathcal{L}_m, \mathcal{L}_n]_{m=-n} = [\mathcal{L}_{-n}, \mathcal{L}_n] = (m-n)x^{-n+n} = -2n \tag{72}$$
$$[\mathcal{L}_m, \mathcal{L}_n]_{n=-m} = [\mathcal{L}_m, \mathcal{L}_{-m}] = (m-n)x^{m-m} = 2m, \tag{73}$$

that implies an property of Witt algebra for sum of two commutators given by:

$$[\mathcal{L}_m, \mathcal{L}_{-m}] + [\mathcal{L}_{-n}, \mathcal{L}_n] = 2(m-n). \tag{74}$$

Now it's focused in the case of $\mathcal{L}_{m+n}$=x^{m+n} and Witt algebra $[\mathcal{L}_m, \mathcal{L}_n]$=$(m-n)\mathcal{L}_{m+n}$. When Equation (74) is multiplied by x^{m+n} both sides one arrives to:

$$[\mathcal{L}_m, \mathcal{L}_{-m}]x^{m+n} + [\mathcal{L}_{-n}, \mathcal{L}_n]x^{m+n} = 2(m-n)x^{m+n}. \tag{75}$$

The right-side above Equation (75) is just Witt algebra Equation (59), and with $x^{m+n} = \mathcal{L}_{m+n}$ one arrives to:

$$[\mathcal{L}_m, \mathcal{L}_{-m}]\mathcal{L}_{m+n} + [\mathcal{L}_{-n}, \mathcal{L}_n]\mathcal{L}_{m+n} = 2[\mathcal{L}_m, \mathcal{L}_n], \tag{76}$$

and solving for $[\mathcal{L}_m, \mathcal{L}_n]$ one gets:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{2}\{[\mathcal{L}_m, \mathcal{L}_{-m}]\mathcal{L}_{m+n} + [\mathcal{L}_{-n}, \mathcal{L}_n]\mathcal{L}_{m+n}\}, \tag{77}$$

and inserting results from Equation (72) and Equation (73) one obtains the following:

$$= [\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{2}\{2m\mathcal{L}_{m+n} - 2n\mathcal{L}_{m+n}\} = (m-n)\mathcal{L}_{m+n}. \tag{78}$$

In this manner, one can see that $\mathcal{L}_{m+n}$=x^{m+n}, fact that corroborates the fulfilling of Witt algebra in the sense of:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} = (m-n)x^{m+n}. \tag{79}$$

4. Polynomial Witt Algebra

4.1. Witt-like Operators in Braket Notation

The gauge operators, as defined in the centerless Virasoro algebra can also be expressed in the language of braket (as for example to see [68]). Consider for instance Equation (61) with the incorporation of Plank constant, such as:

$$\mathcal{L}_n = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}\int_{-\infty}^{\infty} x^n y\delta(x-y)dy. \tag{80}$$

Now, the variable y can be expressed as a function given by $\Phi_N(y)$ in turn it can be introduced inside integration as the scalar product of "bra" and "ket" (usually called in quantum mechanics as state vectors belonging to Hilbert space of $\mathcal{N}$-dimensions) such as: $\Phi_N(y) = \langle y|\Phi_N\rangle$. In this same sense one can do exactly equal with Dirac delta function as: $\delta(x-y) = \langle x|y\rangle$. then one gets for $\mathcal{L}_n$

$$\mathcal{L}_n = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}\int_{-\infty}^{\infty} x^n\Phi_N(y)\langle x|y\rangle\, dy = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}\int_{-\infty}^{\infty} x^n \langle y|\Phi_N\rangle \langle x|y\rangle\, dy, \tag{81}$$

by rearranging the products inside integration one gets:

$$\mathcal{L}_n = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}\int_{-\infty}^{\infty} x^n \langle x|y\rangle \langle y|\Phi_N\rangle\, dy, \tag{82}$$

and with "bra" left out the integration one has below:

$$\mathcal{L}_n = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}x^n \langle x| \int_{-\infty}^{\infty} |y\rangle\langle y|\, dy\, |\Phi_N\rangle. \tag{83}$$

On the other hand, one can employ the completeness relationship given by:

$$\int_{-\infty}^{\infty} |y\rangle\langle y|\, dy = \mathbb{I}, \tag{84}$$

with $\mathbb{I}$ unitary operator inside Equation (83) in the sense that: $\mathbb{I}\,|\Phi\rangle_N = |\Phi_N\rangle$ so that:

$$\mathcal{L}_n = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}x^n \langle x|\,\mathbb{I}\,|\Phi_N\rangle = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}x^n \langle x|\Phi_N\rangle. \tag{85}$$

By following the arguments above on the usage of "bra" and "ket" it is the allowed to define:

$$\langle x|\Phi_N\rangle = \Phi_N(x), \tag{86}$$

and therefore, in conjunction of momentum operator: $\mathbf{p}=\frac{\hbar}{i}\frac{d}{dx}$ one arrives to:

$$\mathcal{L}_n = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}x^n\Phi_N(x) \equiv \frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}[x^n\Phi_N(x)], \tag{87}$$

and same procedure can be exactly applied to partner $\mathcal{L}_n$ so therefore one gets the following:

$$\mathcal{L}_m = \frac{-1}{\hbar\sqrt{m+n+1}}\frac{\hbar}{i}x^n\Phi_M(x) \equiv \frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}[x^m\Phi_M(x)]. \tag{88}$$

 doi:10.20944/preprints202506.2080.v1

With these alternative definitions of these so-called Witt-like operators Equation (87) and Equation (88), it is relevant to see their effect at the commutator $[\mathcal{L}_m, \mathcal{L}_n]$ as follows $[\mathcal{L}_m, \mathcal{L}_n] =$

$$\begin{aligned}\mathcal{L}_m\mathcal{L}_n - \mathcal{L}_n\mathcal{L}_m &= \frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}(x^m\Phi_M(x))\frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}(x^n\Phi_N(x)) \\ &\quad -\frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}(x^n\Phi_N(x))\frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}(x^m\Phi_M(x)) \\ &= \frac{1}{\hbar^2(m+n+1)}\{\mathbf{p}(x^m\Phi_M(x))\mathbf{p}(x^n\Phi_N(x)) - \mathbf{p}(x^n\Phi_N(x))\mathbf{p}(x^m\Phi_M(x))\},\end{aligned} \tag{89}$$

being Equation (89) clearly a commutator given by:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{\hbar^2(m+n+1)}[\mathbf{p}(x^m\Phi_M(x)), \mathbf{p}(x^n\Phi_N(x))]. \tag{90}$$

4.1.1. Special Cases

As exercise one can opt by the case when $M = N$ in Equation (90) so that commutator $[\mathcal{L}_m, \mathcal{L}_n]$ acquires the form:

$$\frac{1}{\hbar^2(m+n+1)}\{\mathbf{p}(x^m\Phi_N(x))\mathbf{p}(x^n\Phi_N(x)) - \mathbf{p}(x^n\Phi_N(x))\mathbf{p}(x^m\Phi_N(x))\}. \tag{91}$$

One can note that $\mathbf{p}$ and $\Phi_N(x)$ can be factorized at the extremes of left and right sides, fact that simplifies Equation (91) to one given by:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{\hbar^2(m+n+1)}\mathbf{p}\{x^m\Phi_N(x)\mathbf{p}x^n - x^n\Phi_N(x)\mathbf{p}x^m\}\Phi_N(x). \tag{92}$$

Furthermore, with the commutator $[\Phi_N(x), x] = 0$, it allows to arrive to:

$$\begin{aligned}[\mathcal{L}_m, \mathcal{L}_n] &= \frac{1}{\hbar^2(m+n+1)}\mathbf{p}\{\Phi_N(x)x^m\mathbf{p}x^n - \Phi_N(x)x^n\mathbf{p}x^m\}\Phi_N(x) \\ &= \frac{1}{\hbar^2(m+n+1)}\mathbf{p}\Phi_N(x)\{x^m\mathbf{p}x^n - x^n\mathbf{p}x^m\}\Phi_N(x).\end{aligned} \tag{93}$$

Indeed, by adding and subtracting inside brackets the pairs $\mathbf{p}x^m x^n$, and $\mathbf{p}x^n x^m$ one gets:

$$\begin{aligned}[\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{\hbar^2(m+n+1)}&\mathbf{p}\Phi_N(x)(x^m\mathbf{p}x^n + \mathbf{p}x^m x^n - \mathbf{p}x^m x^n \\ &-x^n\mathbf{p}x^m + \mathbf{p}x^n x^m - \mathbf{p}x^n x^m)\Phi_N(x).\end{aligned} \tag{94}$$

Guided by the fundamental commutation relation $[x, \mathbf{p}]$ then one arrives to:

$$\begin{aligned}[\mathcal{L}_m, \mathcal{L}_n] &= \frac{1}{\hbar^2(m+n+1)}\mathbf{p}\Phi_N(x)([x^m, \mathbf{p}]x^n + \mathbf{p}x^m x^n - [x^n, \mathbf{p}]x^m - \mathbf{p}x^n x^m)\Phi_N(x) \\ &= \frac{1}{\hbar^2(m+n+1)}\mathbf{p}\Phi_N(x)([x^m, \mathbf{p}]x^n - [x^n, \mathbf{p}]x^m + \mathbf{p}x^m x^n - \mathbf{p}x^n x^m)\Phi_N(x).\end{aligned} \tag{95}$$

Since third and fourth terms are same because $x^m x^n = x^n x^m = x^{m+n}$, then both are canceled and subsequently one gets:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{1}{\hbar^2(m+n+1)}\mathbf{p}\Phi_N(x)([x^m, \mathbf{p}]x^n - [x^n, \mathbf{p}]x^m)\Phi_N(x). \tag{96}$$

 doi:10.20944/preprints202506.2080.v1

Taking into account the identity

$$[x^k, \mathbf{p}] = k[x, \mathbf{p}]x^{k-1}, \tag{97}$$

and by applying this in Equation (96), reader can check that Witt algebra's commutator acquires the form:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{m-n}{\hbar^2(m+n+1)}\mathbf{p}[x, \mathbf{p}]x^{m+n-1}\Phi_N^2(x), \tag{98}$$

indicating clearly that $[\mathcal{L}_m, \mathcal{L}_n]$ is directly proportional to the canonical commutation relation $[x, \mathbf{p}]$=$i\hbar$, as found in [69]. By applying this then one gets from Equation (98):

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{i(m-n)}{\hbar(m+n+1)}\mathbf{p}\Phi_N^2(x)x^{m+n-1}. \tag{99}$$

It is noteworthy while $\Phi_N(x) = 1$ taking the value of unity, then by plugging this above Equation (99) one gets:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{i(m-n)}{\hbar(m+n+1)}\mathbf{p}x^{m+n-1} = \frac{i(m-n)}{\hbar(m+n+1)}\frac{\hbar}{i}\frac{d}{dx}x^{m+n-1} = \frac{(m-n)}{(m+n+1)}\frac{d}{dx}x^{m+n-1}$$
$$= \frac{(m-n)}{(m+n+1)}(m+n-1)x^{m+n-2} = \left[\left(\frac{1}{x^2}\right)\frac{m+n-1}{m+n+1}\right](m+n)x^{m+n}, \tag{100}$$

with the term without brackets is Witt algebra exactly, under the condition of $\Phi_N(x) = 1$. The fact that Equation (100) is valid, one can see that exists there different approaches demonstrating that Witt-like operators as written in Equation (58) satisfy Witt algebra. On the other hand, Equation (100) also confirms that usage of integration and derivative respect to coordinate, are intrinsic to Witt-like operators. Thus, additional polynomial scenarios for $\Phi_N(x)$ are explored as follows:

4.1.2. Case I: $\Phi_N(x) = x^m$

Thus, introducing this polynomial form into Equation (99) one arrives to:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{i(m-n)}{\hbar(m+n+1)}\mathbf{p}x^{2m}x^{m+n-1} = \frac{i(m-n)}{\hbar(m+n+1)}\mathbf{p}x^{3m+n-1}. \tag{101}$$

Again, by using the coordinate representation of momentum operator, it is evident that:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{(m-n)(3m+n-1)}{(m+n+1)}x^{3m+n-2} = \frac{(m-n)(3m+n-1)}{(m+n+1)}x^{m+n}x^{2m-2}. \tag{102}$$

4.1.3. Case II: $\Phi_N(x) = x^L$

In this case, the notation is changed and it is employed the integer number L:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{i(m-n)}{\hbar(m+n+1)}\mathbf{p}x^{2L}x^{m+n-1} = \frac{i(m-n)}{\hbar(m+n+1)}\mathbf{p}x^{m+n+2L-1} \tag{103}$$

$$\frac{(m-n)(m+n+2L-1)}{(m+n+1)}x^{m+n+2L-2} = \frac{(m-n)(m+n+2L-1)}{(m+n+1)}x^{m+n}x^{2L-2}. \tag{104}$$

In this manner, one can establish that the momentum representation of Witt commutator for any integer L (not related to integer numbers of commutator) has the following form $[\mathcal{L}_m, \mathcal{L}_n] =$

$$\frac{(m-n)(m+n+2L-1)}{(m+n+1)}x^{m+n}x^{2L} = \frac{x^{2L-2}(m+n+2L-1)}{(m+n+1)}(m-n)\mathcal{L}_{m+n}. \tag{105}$$

 doi:10.20944/preprints202506.2080.v1

Evidently, Equation (105) fulfill the Witt commutator only if:

$$\frac{x^{2L-2}(m+n+2L-1)}{(m+n+1)} = 1 \Rightarrow x = \left[\frac{m+n+1}{m+n+2L-1}\right]^{\frac{1}{2L}}. \tag{106}$$

Interestingly, the case when $L = 1$ yields directly $x = 1$ or also by operating in a straightforward in right-side of Equation (105) so that one arrives to:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{x^0(m+n+2-1)}{(m+n+1)}(m-n)\mathcal{L}_{m+n} = \frac{(m+n+1)}{(m+n+1)}(m-n)\mathcal{L}_{m+n} = (m-n)\mathcal{L}_{m+n}. \tag{107}$$

Consider now right-side Equation (105) with $x = 1$ solely, so that:

$$[\mathcal{L}_m, \mathcal{L}_n] = \frac{(m+n+2L-1)}{(m+n+1)}(m-n)\mathcal{L}_{m+n}, \tag{108}$$

that can be rewritten again as:

$$\begin{aligned} [\mathcal{L}_m, \mathcal{L}_n] = \frac{(m+n+2L-1)}{(m+n+1)}(m-n)\mathcal{L}_{m+n} &= \left[1 + \frac{2L-2}{m+n+1}\right](m-n)\mathcal{L}_{m+n} \\ &= (m-n)\mathcal{L}_{m+n} + \left[\frac{(2L-2)(m-n)}{m+n+1}\right]\mathcal{L}_{m+n}, \end{aligned} \tag{109}$$

exhibiting the fact that exists a clear surplus beyond the term $(m-n)\mathcal{L}_{m+n}$ and that can be seen as an additional term such as:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \left[\frac{2(L-1)}{m+n+1}\right](m-n)\mathcal{L}_{m+n}, \tag{110}$$

by which only the value of $L = 1$ returns again the Witt algebra. In any case one can adjudicate the name of deformed Witt algebra by accepting the following structure:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + H(m,n,L)(m-n)\mathcal{L}_{m+n}, \tag{111}$$

with $H(m,n,L) = \frac{2(L-1)}{m+n+1}$. Speculatively, one can see that second term of right-side Equation (111) might be a form to express the Virasoro term in the sense that:

$$H(m,n,L)(m-n)\mathcal{L}_{m+n} = \frac{L}{12}(m^3 - m), \tag{112}$$

by which L now would take the role of being the central charge. In the next section, just attention shall be paid on this point. Indeed one should take into account that from Equation (112) one also arrives to:

$$\mathcal{L}_{m+n} = \frac{L}{12}\frac{m^3 - m}{m-n}\frac{1}{H(m,n,L)}. \tag{113}$$

In this manner, Equation (113) is to some extent contradictory because there is not dependence on the variable x, fact that invalid completely this speculative scenario. Furthermore, from right-side Equation (110) then Equation (113) can be explicitly written as:

$$\mathcal{L}_{m+n} = \left(\frac{L}{12}\right)\left(\frac{m^3 - m}{m-n}\right)\left(\frac{m+n+1}{2L-1}\right), \tag{114}$$

indicating the fact that Virasoro algebra protects itself of trivial assumptions as well as from arbitrary comparisons.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

5. Derivation of a Deformed Witt Algebra and Virasoro Algebra

The fact that Equation (111) contains a kind of surplus as to be associated to the term $\frac{c}{12}(m^3 - m)$, would be suggesting the existence of a realistic window to work out on an upgrade version of Equation (60) and Equation (61), that define the Witt-like operators. In this manner, one can embark over the grounds achieved in previous sections. Thus, a practical path is the redefinition of Equation (60) and Equation (61) through the proposal given below by:

$$\mathcal{L}_m^{\mathrm{R}} = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int x^m(\mathbb{I}y - \mathbf{G}_A)\delta(y-x)dy, \tag{115}$$

$$\mathcal{L}_n^{\mathrm{R}} = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}\int x^n(\mathbb{I}z - \mathbf{G}_B)\delta(z-x)dz. \tag{116}$$

with $\mathbb{I}$ the identity operator,and $\mathbf{G}_A$ and $\mathbf{G}_B$ are arbitrary operators and so far have not any declared relationship among them. If one defines $\mathbf{\Lambda}(y) = \mathbb{I}y - \mathbf{G}_B$ and $\mathbf{\Lambda}(z) = \mathbb{I}z - \mathbf{G}_B$ then Equation (115) and Equation (116) are rewritten in a more compact way as:

$$\mathcal{L}_m^{\mathrm{R}} = \frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}\int x^m\mathbf{\Lambda}(y)\delta(y-x)dy = \frac{-1}{\sqrt{m+n+1}}\int\frac{\mathbf{p}\mathbf{\Lambda}(y)x^m}{\hbar}\delta(y-x)dy, \tag{117}$$

$$\mathcal{L}_n^{\mathrm{R}} = \frac{-1}{\hbar\sqrt{m+n+1}}\mathbf{p}\int x^n\mathbf{\Lambda}(z)\delta(z-x)dz = \frac{-1}{\sqrt{m+n+1}}\int\frac{\mathbf{p}\mathbf{\Lambda}(z)x^n}{\hbar}\delta(z-x)dz. \tag{118}$$

Turning back to Equation (115) and Equation (116) are indicating a kind of proportionality of Witt operator $\mathcal{L}_{m,n}$ and product: $\mathbf{p}\mathbf{\Lambda}$. Reader can note that this product has units of $\hbar$. One can operate both Equation (115) and Equation (116) so that one obtains for both partners:

$$\mathcal{L}_m^{\mathrm{R}} = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}x^{m+1} - \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}x^m\mathbf{G}_A, \tag{119}$$

$$\mathcal{L}_n^{\mathrm{R}} = \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}x^{n+1} - \frac{i}{\sqrt{m+n+1}}\frac{d}{dx}x^n\mathbf{G}_B. \tag{120}$$

With redefinitions Equation (119) and Equation (120) the cases of Witt's commutators shall be explored as follows:

5.1. Determination of $[\mathcal{L}_m^{\mathrm{R}}, \mathcal{L}_n^{\mathrm{R}}]$

In order to reduce the notation, it is applied the change: $\alpha = \frac{i}{\sqrt{m+n+1}}$ above Equation (119) and Equation (120) acquires the form:

$$\mathcal{L}_m^{\mathrm{R}} = \alpha\frac{d}{dx}x^{m+1} - \alpha\frac{d}{dx}x^m\mathbf{G}_A, \tag{121}$$

$$\mathcal{L}_n^{\mathrm{R}} = \alpha(n+1)x^n - \alpha n x^{n-1}\mathbf{G}_B. \tag{122}$$

In this manner with Equation (121). and Equation (122) it is calculated the terms commutator $[\mathcal{L}_m^{\mathrm{R}}, \mathcal{L}_n^{\mathrm{R}}]$ = $\mathcal{L}_m^{\mathrm{R}}\mathcal{L}_n^{\mathrm{R}}$ - $\mathcal{L}_n^{\mathrm{R}}\mathcal{L}_m^{\mathrm{R}}$, so that the first term given by $\mathcal{L}_m^{\mathrm{R}}\mathcal{L}_n^{\mathrm{R}}$ yields:

$$\mathcal{L}_m^{\mathrm{R}}\mathcal{L}_n^{\mathrm{R}} = \left[\alpha\frac{d}{dx}x^{m+1} - \alpha\frac{d}{dx}x^m\mathbf{G}_A\right]\left[\alpha(n+1)x^n - \alpha n x^{n-1}\mathbf{G}_B\right] \tag{123}$$

$$= \alpha^2(n+1)\frac{d}{dx}x^{m+n+1} - \alpha^2 n\frac{d}{dx}x^{m+n}\mathbf{G}_B - \alpha^2(n+1)\frac{d}{dx}x^m\mathbf{G}_A x^n + \alpha^2 n\frac{d}{dx}x^m\mathbf{G}_A x^{n-1}\mathbf{G}_B \tag{124}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

By following same structure of Equation (121) and Equation (122) one gets for partners:

$$\mathcal{L}_n^{\mathrm{R}} = \alpha \frac{d}{dx} x^{n+1} - \alpha \frac{d}{dx} x^n \mathbf{G}_B, \tag{125}$$

$$\mathcal{L}_m^{\mathrm{R}} = \alpha(m+1) \frac{d}{dx} x^m - \alpha m x^{m-1} \mathbf{G}_A. \tag{126}$$

Now with Equation (125) and Equation (126) is calculated second term of commutator $\mathcal{L}_n^{\mathrm{R}} \mathcal{L}_m^{\mathrm{R}}$ yielding:

$$= \mathcal{L}_n^{\mathrm{R}} \mathcal{L}_m^{\mathrm{R}} = \left[\alpha \frac{d}{dx} x^{n+1} - \alpha \frac{d}{dx} x^n \mathbf{G}_B\right]\left[\alpha(m+1) x^m - \alpha m x^{m-1} \mathbf{G}_A\right] \tag{127}$$

$$= \alpha^2(m+1) \frac{d}{dx} x^{m+n+1} - \alpha^2 m \frac{d}{dx} x^{m+n} \mathbf{G}_A - \alpha^2(m+1) \frac{d}{dx} x^n \mathbf{G}_B x^m + \alpha^2 m \frac{d}{dx} x^n \mathbf{G}_B x^{m-1} \mathbf{G}_A. \tag{128}$$

Thus, Equation (124) and Equation (128) are grouped yielding:

$$[\mathcal{L}_m^{\mathrm{R}}, \mathcal{L}_n^{\mathrm{R}}] = \alpha^2(n+1) \frac{d}{dx} x^{m+n+1} - \alpha^2 n \frac{d}{dx} x^{m+n} \mathbf{G}_B - \alpha^2(n+1) \frac{d}{dx} x^m \mathbf{G}_A x^n + \alpha^2 n \frac{d}{dx} x^m \mathbf{G}_A x^{n-1} \mathbf{G}_B$$

$$-\alpha^2(m+1) \frac{d}{dx} x^{m+n+1} + \alpha^2 m \frac{d}{dx} x^{m+n} \mathbf{G}_A + \alpha^2(m+1) \frac{d}{dx} x^n \mathbf{G}_B x^m - \alpha^2 m \frac{d}{dx} x^n \mathbf{G}_B x^{m-1} \mathbf{G}_A. \tag{129}$$

By assuming that operators $\mathbf{G}_{A,B}$ commute with x^m, x^n, x^{m-1} and x^{n-1} then one can rewrite Equation (88) as:

$$[\mathcal{L}_m^{\mathrm{R}}, \mathcal{L}_n^{\mathrm{R}}] = \alpha^2(n+1) \frac{d}{dx} x^{m+n+1} - \alpha^2 n \frac{d}{dx} x^{m+n} \mathbf{G}_B - \alpha^2(n+1) \frac{d}{dx} x^{m+n} \mathbf{G}_A + \alpha^2 n \frac{d}{dx} x^{m+n-1} \mathbf{G}_A \mathbf{G}_B$$

$$-\alpha^2(m+1) \frac{d}{dx} x^{m+n+1} + \alpha^2 m \frac{d}{dx} x^{m+n} \mathbf{G}_A + \alpha^2(m+1) \frac{d}{dx} x^{m+n} \mathbf{G}_B - \alpha^2 m \frac{d}{dx} x^{m+n-1} \mathbf{G}_B \mathbf{G}_A. \tag{130}$$

By grouping properly then one has below that:

$$[\mathcal{L}_m^{\mathrm{R}}, \mathcal{L}_n^{\mathrm{R}}] = \alpha^2(n+1) \frac{d}{dx} x^{m+n+1} - \alpha^2(m+1) \frac{d}{dx} x^{m+n+1} - \alpha^2 n \frac{d}{dx} x^{m+n} \mathbf{G}_B + \alpha^2 m \frac{d}{dx} x^{m+n} \mathbf{G}_A$$
$$-\alpha^2(n+1) \frac{d}{dx} x^{m+n} \mathbf{G}_A + \alpha^2(m+1) \frac{d}{dx} x^{m+n} \mathbf{G}_B + \alpha^2 n \frac{d}{dx} x^{m+n-1} \mathbf{G}_A \mathbf{G}_B - \alpha^2 m \frac{d}{dx} x^{m+n-1} \mathbf{G}_B \mathbf{G}_A. \tag{131}$$

One can see that while α recover its initial value $\frac{i}{\sqrt{m+n+1}}$ then first 2 terms of Equation (131) (from left to right) becomes exactly $(m-n)x^{m+n}$ that in conjunction to commutator turns out to be Witt algebra. With respect to third and fourth terms one gets: $-\alpha^2(m+n)x^{m+n-1} n\mathbf{G}_B + \alpha^2 m(m+n)x^{m+n-1}\mathbf{G}_A = \frac{m+n}{m+n+1} x^{m+n-1}(n\mathbf{G}_B - m\mathbf{G}_A)$. Through a similar procedure applied to fifth and sixth terms follows: $\frac{(n+1)(m+n)}{m+n+1} x^{m+n-1}\mathbf{G}_A - \frac{(m+1)(m+n)}{m+n+1} x^{m+n-1}\mathbf{G}_B = [(n+1)\mathbf{G}_A - (m+1)\mathbf{G}_B]\frac{m+n}{m+n+1} x^{m+n-1}$.
Finally, commutator can be written as:

$$[\mathcal{L}_m^{\mathrm{R}}, \mathcal{L}_n^{\mathrm{R}}] = (m-n)x^{m+n} + \frac{m+n}{m+n+1} x^{m+n-1}(n\mathbf{G}_B - m\mathbf{G}_A) +$$
$$+[(n+1)\mathbf{G}_A - (m+1)\mathbf{G}_B] \frac{m+n}{m+n+1} x^{m+n-1} + 7\text{term} + 8\text{term}. \tag{132}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

Thus, Equation (132) is written below with the the calculated terms 7th and 8th, so that one gets below:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} + \frac{m+n}{m+n+1}x^{m+n-1}(n\mathbf{G}_B - m\mathbf{G}_A) + \\ +[(n+1)\mathbf{G}_A - (m+1)\mathbf{G}_B]\frac{m+n}{m+n+1}x^{m+n-1} + \\ +\frac{m(m+n-1)}{m+n+1}x^{m+n-2}\mathbf{G}_B\mathbf{G}_A - \frac{n(m+n-1)}{m+n+1}x^{m+n-2}\mathbf{G}_A\mathbf{G}_B, \tag{133}$$

that can be perceived as a deformed Witt algebra. From this one can wonder under what circumstances one can find a licit path that allows to arrive to Virasoro algebra with central charge. Moreover, under the present context of Witt-like algebra, the weakness of Equation (133) is clearly seen as the austerity of model due to the lack of more independent variables. Due to this, Equation (133) only allows us to associate the central charge to arbitrary operators $\mathbf{G}_{A,B}$.

5.2. *Case* $\mathbf{G}_A = \mathbf{G}_B$

By applying this equality into Equation (133) one gets a polynomial form with respect to the arbitrary operator $\mathbf{G}_A$ for example. In this manner one has below:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2\frac{(m-n)(m+n)}{m+n+1}x^{m+n-1}\mathbf{G}_A + (m-n)\left[\frac{(m+n-1)}{m+n+1}\right]x^{m+n-2}\mathbf{G}_A^2. \tag{134}$$

In order to accomplish Virasoro algebra the assumption that $\mathbf{G}_A$ is a Grassmann variable shall be done. Because this $\mathbf{G}_A^2 = 0$. This makes null third term of brackets in Equation (134), so that one arrives to:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2\frac{(m-n)(m+n)}{m+n+1}x^{m+n-1}\mathbf{G}_A. \tag{135}$$

In this way it is proposed an explicit form for arbitrary operators as follows:

$$\mathbf{G}_A = -\frac{cm^2}{24x^{m-2}}, \tag{136}$$

and with this is evident that Equation (135) can be written as:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2\frac{(m^2-n^2)}{m+n+1}x^{m+n-1}\left(\frac{-cm^2}{24x^{m-2}}\right). \tag{137}$$

When it is opted by value of $n = -1$ one arrives to:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2\frac{(m^2-1)}{m}x^{m-2}\left(\frac{-cm^2}{24x^{m-2}}\right). \tag{138}$$

Logically to cancel m^2 up an down in Equation (138), it is needed to multiply by m same places, so that one gets also the Virasoro's term "(m^3-m)" so that one gets:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2\frac{(m^3-m)}{m^2}\left(\frac{-cm^2}{24}\right). \tag{139}$$

After the cancellation, one arrives to:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} + (m^3-m)\left(\frac{c}{12}\right). \tag{140}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

Now it's clear that second term of right-side of Equation (137) must incorporate a Delta of Krönecker in the sense that:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} + \left(\frac{m^2-n^2}{m+n+1}\right)x^{m+n-1}\left(\frac{cm^2}{12x^{m-2}}\right)\delta_{n,-1}, \tag{141}$$

that returns Virasoro algebra with central charge c and that can be also written in a most pedagogical manner as:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} + \left(\frac{c}{12}\right)\left[\left(\frac{m^2-n^2}{m+n+1}\right)\left(\frac{m^2}{x^{m-2}}\right)x^{m+n-1}\right]\delta_{n,-1}. \tag{142}$$

The fact that arbitrary operator Equation (136) has a direct dependence on the coordinate x can be debatable, although it was incorporated in Equation (115) and Equation (116) as being proportional to variables y and z, respectively. However, from Equation (136) one can assume $m = 2$ to break any dependence on x. Because this choice one gets below:

$$\mathbf{G}_A = -\left|\frac{cm^2}{24x^{m-2}}\right|_{m=2} = -\frac{c}{6}. \tag{143}$$

It should be noted that option $m = 2$ cannot be incorporated in equations above since it breaks the second terms of right-side in Equation (140) invalidating the Virasoro algebra but keeping still Witt algebra.

5.3. *Infinitesimal Distances*

The derived relationship Equation (143) can be also derived through purely physical reasons as to the magnitude of coordinate x. To accomplish this, one can rewrite again Equation (134) as the Witt algebra and its surplus in brackets given by:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2(m-n)x^{m+n}\left\{\frac{m+n}{m+n+1}\frac{\mathbf{G}_A}{x} - \frac{m+n}{m+n+1}\frac{\mathbf{G}^2_A}{x^2} + \frac{1}{m+n+1}\frac{\mathbf{G}^2_A}{x^2}\right\}, \tag{144}$$

And the term $(m-n)$ is entered inside brackets yielding Equation (133) in another different way:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - 2x^{m+n}\left\{\frac{m^2-n^2}{m+n+1}\frac{\mathbf{G}_A}{x} - \frac{m^2-n^2}{m+n+1}\frac{\mathbf{G}^2_A}{x^2} + \frac{m-n}{m+n+1}\frac{\mathbf{G}^2_A}{x^2}\right\}. \tag{145}$$

Clearly, it is identified a function $F(m,n,x)$ given by:

$$F(m,n,x) = 2x^{m+n}\left\{\frac{m^2-n^2}{m+n+1}\frac{\mathbf{G}_A}{x} - \frac{m^2-n^2}{m+n+1}\frac{\mathbf{G}^2_A}{x^2} + \frac{m-n}{m+n+1}\frac{\mathbf{G}^2_A}{x^2}\right\}, \tag{146}$$

whose presence deforms the Witt algebra yielding the following:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_n] = (m-n)x^{m+n} - F(m,n,x). \tag{147}$$

Without appealing to Grassmann variables, second and third term of Equation (147) above can be considered null based at the argument that "x" is infinitesimal so that

$$\frac{\mathbf{G}^2_A}{x^2} \approx 0. \tag{148}$$

This approximation is applied at the sense that physical distances as seen in the redefined Witt-like operators Equation (115) and Equation (116) might be denoting a minimum observable length either particle or string, as first noted by M. Kato [20]. This assumption reduces substantially Equation (146) to:

$$F(m,n,x) = 2x^{m+n}\left\{\frac{m^2-n^2}{m+n+1}\frac{\mathbf{G}_A}{x}\right\}. \tag{149}$$

Now it is opted by $n=-1$ in order to convert Equation (147) in Virasoro algebra, so that with this one expects:

$$F(m,-1,x) = \frac{-c}{12}(m^3-m), \tag{150}$$

therefore by replacing $n=-1$ in Equation (149) one has in a straightforward manner that:

$$F(m,-1,x) = 2x^{m-1}\left\{\frac{m^2-1}{m}\frac{\mathbf{G}_A}{x}\right\} = 2x^{m-1}\left\{\frac{m^3-m}{m^2}\frac{\mathbf{G}_A}{x}\right\} = \frac{-c}{12}(m^3-m). \tag{151}$$

From above one can clearly see that:

$$2x^{m-1}\left\{\frac{m^3-m}{m^2}\frac{\mathbf{G}_A}{x}\right\} = -\frac{c}{12}(m^3-m), \tag{152}$$

canceling the Virasoro's term m^3-m in both sides. Henceforth, the dependence on coordinate x is broken when $m=2$ by yielding $\mathbf{G}_A$ and central charge are directly linked through the relation:

$$\mathbf{G}_A = -\frac{c}{6}.$$

that verifies Equation (143) without the assumption that $\mathbf{G}_A^2 = 0$ for being a Grassmann variable, however a negative central charge violated unitarity [78].

Under the validity of Equation (143) central charge is also a Grassmann variable. The fact why central charge can be a pure Grassmann number because $\mathbf{G}_A^2 \propto c^2$ can be questionable, nevertheless, it is beyond the scope of present document.

6. Implications of Equation (131)

Turning back again to Equation (131) when not any derivative is applied one can inspect its behavior at some special cases by which some assumptions can be done in involved integer numbers and not at the arbitrary operators $\mathbf{G}_{A,B}$ (inspired in Equation (33) of [37] for example).

6.1. Case of m=n

For this case $m=n$ the first and second term are each other canceled because the term $m-n$. For third and fourth terms one gets $\frac{2m^2}{2m+1}x^{2m-1}(\mathbf{G}_B-\mathbf{G}_A)$ and the fifth and sixth terms yields: $\frac{2m(m+1)}{2m+1}x^{2m-1}(\mathbf{G}_A-\mathbf{G}_B)$. Thus the sum of these results is given by (it is evident that $\alpha^2 = \frac{-1}{m+n+1}$ was used):

$$\begin{aligned}&\frac{2m^2}{2m+1}x^{2m-1}(\mathbf{G}_B-\mathbf{G}_A) - \frac{2m(m+1)}{2m+1}x^{2m-1}(\mathbf{G}_B-\mathbf{G}_A),\\ &= \left[\frac{2m^2}{2m+1} - \frac{2m(m+1)}{2m+1}\right]x^{2m-1}(\mathbf{G}_B-\mathbf{G}_A).\end{aligned} \tag{153}$$

$$= \left[\frac{2m^2}{2m+1} - \frac{2m^2}{2m+1} - \frac{2m}{2m+1}\right]x^{2m-1}(\mathbf{G}_B-\mathbf{G}_A) = -\left[\frac{2m}{2m+1}\right]x^{2m-1}(\mathbf{G}_B-\mathbf{G}_A). \tag{154}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 July 2025 doi:10.20944/preprints202506.2080.v1

Now last two terms of Equation (131) with $m = n$ can be worked out through the following manner:

$$\left|\frac{m(m+n-1)}{m+n+1}x^{m+n-2}\mathbf{G}_B\mathbf{G}_A - \frac{n(m+n-1)}{m+n+1}x^{m+n-2}\mathbf{G}_A\mathbf{G}_B\right|_{m=n} \tag{155}$$

$$\Rightarrow \frac{m(2m-1)}{2m+1}x^{2m-2}\mathbf{G}_B\mathbf{G}_A - \frac{m(2m-1)}{2m+1}x^{2m-2}\mathbf{G}_A\mathbf{G}_B \tag{156}$$

$$= \left[\frac{m(2m-1)}{2m+1}x^{2m-2}\right](\mathbf{G}_B\mathbf{G}_A - \mathbf{G}_A\mathbf{G}_B) = \left[\frac{m(1-2m)}{2m+1}x^{2m-2}\right][\mathbf{G}_A, \mathbf{G}_B]. \tag{157}$$

By gathering previous results as done by right-side Equation (154) and right-side Equation (157) one gets below that:

$$[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_m] = -\left[\frac{2m}{2m+1}\right]x^{2m-1}(\mathbf{G}_B - \mathbf{G}_A) + \left[\frac{m(1-2m)}{2m+1}x^{2m-2}\right][\mathbf{G}_A, \mathbf{G}_B], \tag{158}$$

that clearly should be null $[\mathcal{L}^{\mathrm{R}}_m, \mathcal{L}^{\mathrm{R}}_m] = 0$, and solving for commutator $[\mathbf{G}_A, \mathbf{G}_B]$ one obtains the following rule for commutator given by:

$$[\mathbf{G}_A, \mathbf{G}_B] = \frac{\left[\frac{2m}{2m+1}\right]x^{2m-1}(\mathbf{G}_B - \mathbf{G}_A)}{\left[\frac{m(1-2m)}{2m+1}x^{2m-2}\right]} = \frac{2x^{2m-1}(\mathbf{G}_B - \mathbf{G}_A)}{[(1-2m)x^{2m-2}]}. \tag{159}$$

An interesting case is when $m = 1$, so that:

$$[\mathbf{G}_A, \mathbf{G}_B] = 2x(\mathbf{G}_A - \mathbf{G}_B), \tag{160}$$

(that is related to structures of commutator of Equation (6) in [39]) result that is also obtained with $m = 0$. Furthermore, when $\mathbf{G}_A$ is a Grassmann number then $\mathbf{G}^2_A$, then solving for x one finally arrives to:

$$x = \frac{1}{2}\frac{\mathbf{G}_A\mathbf{G}_B\mathbf{G}_A}{\mathbf{G}_A\mathbf{G}_B}. \tag{161}$$

7. Conclusions

According to results achieved in this paper, it is evident that both Witt and Virasoro algebra are not exclusively derivations of a theory based strictly in oscillators satisfying relations such as $[\alpha_p, \alpha_q] = -q\delta_{p+q}$, as commonly employed in string theories. Instead of this, at present paper has been demonstrated that both algebra can also be fulfilled through scalar quantities or Witt-like operators by exhibiting not any compromise to those structures build on the basis of string-based operators. It is noteworthy that such Witt-like operators have their origin in non-relativistic quantum mechanics. Thus it was demonstrated that such operators are closely connected to total angular momentum and momentum operators, and as studied in previous paper, fully linked to fundamental commutation relation. Also, interestingly, Witt-like operators have turned out to be proportional to quantum mechanics wave function. Thus, Schröinger equations for free particle and harmonic oscillator have been also derived. Thus, up to two proposes have been presented. In on side the pair based at Equation (60) and Equation (61) that is translated in Witt-like operators yielding the Witt algebra, and on the other side the redefinitions given by Equation (115) and Equation (116) that under certain circumstances reproduces deformations of Witt and Virasoro algebra. In addition, such redefinitions have led to Virasoro algebra with central charge. In contrast to derivations done at the past not any mathematical induction in the involved integer numbers. Nevertheless, it is noteworthy that such redefinitions of Witt-like algebra have required to incorporate an extra arbitrary physical operator with units of distance. The closed-form derivation of Virasoro algebra with central charge has demanded to assume that such arbitrary operators would behave as Grassmann numbers (see Eq. 1.7 in [77]), nevertheless as noted in [78] fundamental unitary is violated. However, it might be explained in terms of Logarithm operators as seen in[79-80] . Also, it was seen that even when not any assumption about Grassmann numbers is applied, then the criterion given by Kato by which argues that a minimum length in physical theories of string or particles, might be permissible at the order of $< (\delta x)^2 > \approx \epsilon \lambda^2$ with $\epsilon \to 0$. Based on this criterion, the terms of order of x^{-2} were neglected. For both pictures (with or without Grassmann numbers), it was obtained that arbitrary operators turned out to be proportional to central charge.

References

1. Veneziano, G. Construction of a crossing-simmetric, Regge-behaved amplitude for linearly rising trajectories. Nuovo Cimento A (1965-1970) 57, 190–197 (1968).
2. K. Kikkawa, B. Sakita, and M. A. Virasoro, Feynman-Like Diagrams Compatible with Duality. I. Planar Diagrams, Phys. Rev. 184, 1701 - 1969.
3. M. A. Virasoro, Subsidiary Conditions and Ghosts in Dual-Resonance Models, Phys. Rev. D 1, 2933 – 1970.
4. Richard C. Brower, Spectrum -Generating Algebra and No -Ghost Theorem for the Dual Model, Phy. Rev. D, Vol 6, Number 6, 15 September 1972.
5. P. H. Frampton and H.B. Nielsen, Dual-model spectrum analysis using Virasoro algebra representations, Nucl. Phys. B45 (1972) 318-332.
6. G Veneziano, An introduction to dual models of strong interactions and their physical motivations, Physics Reports Volume 9, Issue 4, February 1974, Pages 199-242.
7. P. Van Alstine, Remark on the c-number anomaly in the Virasoro algebra, Phy. Rev. D, Vol 12, Number 12, 15 September 1975.
8. Helmut Strade, Representations of the Witt algebra, Journal of Algebra, Volume 49, Issue 2, 1977, Pages 595-605.
9. Irving Kaplansky, The Virasoro Algebra, Commun. Math. Phys. 86, 49-54 (1982).
10. Brown, J.D., Henneaux, M. Central charges in the canonical realization of asymptotic symmetries: An example from three dimensional gravity. Commun.Math. Phys. 104, 207–226 (1986).
11. A. N. Redlich, When is the central charge of the Virasoro algebra in string theories in curved space-time not a numerical constant? Phys. Rev. D 33, 1094 – 1986.
12. Ganpathy Murthy, Virasoro algebra in 2+1 dimensions and angular momentum fractionalization, Phys. Rev. Lett. 58, 755 (1987) - 1987.
13. Ratindranath Akhoury and Yasuhiro Okada, string in curved space-time: Virasoro algebra in the classical and quantum theory, Phy. Rev. D, Vol 35, Number 6, 15 March 1987.
14. Yoshihiko Saito, Nonexistence of a ground state and breaking of the no-ghost theorem on a squeezed string, Phy. Rev. D, Vol 36, Number 10, 15 November 1987.
15. Jonathan Bagger, Dennis Nemeschansky, and Shimon Yankielowicz, Virasoro algebras with central charge c>1, Phys. Rev. Lett. 60, 389 (1988) - 1988.
16. B. L. Feigin and D. B. Fuchs, Cohomology ofsomenilpotent subalgebras ofthe Virasoro and Kac-MoodyLie algebras, Volume 5, Issue 2, 1988, Pages 209-235, Journal of Geometry and Physics.
17. N. Sakai and P. Suranyi, Modification of Virasoro Generators by Kac-Moody Generators, Nuclear Physics B318 (1989) 655-668.
18. Franz Embacher, Covariant Virasoro operators and free bosonic string field theory, Phys. Rev. D 39, 1632 – 1989.
19. T. L. Curtright and C.K. Zachos, Deforming maps for quantum algebras, Physics Letters B, Vol 243, Number 3 (1990) 237.
20. Mitsuhiro Kato, Particle theories with minimum observable length and open string theory, Physics Letters B, Vol 245, Number 1, (1990) 245.
21. M. Chaichian D. Ellinas and Z. Popowicz, Quantum conformal algebra with central extension, Physics Letters B, Vol 248, Number 1,2 (1990) 95.
22. E. Ragoucy and P. Sorba, New realizations of the super Virasoro algebra at c= 3, Physics Letters B, Vol 256, Number 1, (1991) 30.
23. N. Aizawa and H. Sato, q-deformation of the Virasoro algebra with central extension, Physics Letters B, Vol 256, Number 2, (1991) 185.

24. Ch. Devchand and M.V. Saveliev, Comultiplication for quantum deformations of the centreless Virasoro algebra in the continuum formulation, Physics Letters B, Vol 258, Number 3,4 (1991) 364.
25. Noureddine Mohammedi, General super-Virasoro construction on affine F, Physics Letters B, Vol 260, Number 3,4 (1991) 317.
26. Earl J. Taft, Witt and Virasoro algebras as Lie bialgebras, Journal of Pure and Applied Algebra 87 (1993) 301-312.
27. L. Faddeev and A.Yu. Volkov, Abelian current algebra and the Virasoro algebra on the lattice, Physics Letters B 315 (1993) 311-318.
28. Ryuji Kemmoku and Satoru Saito, Discretization of Virasoro algebra, Physics Letters B 319 (1993) 471-477.
29. Y. M. Cho, Effective field theory of an internal string, Phys. Rev. D 49, 1007 – 1994.
30. Arlen Anderson, Quantum Backreaction on "Classical" Variables, Phys. Rev. Lett. 74, 621 – 1995.
31. Ralph Blumenhagen and Andreas Wikkirchen, Extension of the N = 2 Virasoro algebra by two primary fields of dimension 2 and 3, Physics Letters B 343 (1995) 168-180.
32. E.H. El Kinani and M. Zakkari, On the q-deformation of certain infinite dimensional Lie algebras, Physics Letters B 357 (1995) iO5-108 .
John Maddox, Classical and quantum physics mix, News and Views, Nature, Vol 373, 9 February 1995.
33. M. Chaichian and E Pregnajder, q-Virasoro algebra, q-conformal dimensions and free q-superstring, Nuclear Physics B 482 (1996) 466-478.
34. Sergei Lukyanov, A note on the deformed Virasoro algebra, Physics Letters B 367 (19
35. Haru-Tada Sato, OPE formulae for deformed super-Virasoro algebras, Nuclear Physics B 471 (1996) 553-569.
36. Michio Jimbo, Michael Lashkevich, Tetsuji Miwa, and Yaroslav Pugai, Lukyanov's screening operators for the deformed Virasoro algebra, Physics Letters A 229 (1997) 285-292.
37. Ham-Tada Sato, Deformation of super Virasoro algebra in noncommutative quantum superspace, Physics Letters B 415 (1997) 170-174.
38. J. Avan, L. Frappat, M. Rossi and P. Sorba, Central extensions of classical and quantum q-Virasoro algebras, Physics Letters A 251 (1999) 13-24.
39. Máximo Bañados , Embeddings of the Virasoro Algebra and Black Hole Entropy, Phys. Rev. Lett. 82, 2030 – 1999.
40. N. Mebarki, A. Boudine, H. Aissaoui, A. Benslama, A. Bouchareb, and A. Maasmi, No-ghost theorem for a q-deformed open bosonic string, Journal of Geometry and Physics 30 (1999) 103-112.
41. Patrick Marcel, Generalizations of the Virasoro algebra and matrix Sturm–Liouville operators, Journal of Geometry and Physics 36 (2000) 211–222.
42. V.P. Akulov, Sultan Catto, Oktay Cebecioglu, and A.Pashnev, Onthe time-dependent oscillator and the nonlinear realizations of the Virasoro group, Physics Letters B 575 (2003) 137–143.
43. E. H. El Kinani, Between Quantum Virasoro Algebra Lc and Generalized Clifford Algebras, math-ph arXiv:math-ph/0310044v1.
44. M. Chaichian and P.Presnajder, Discrete-time quantum field theory and the deformed super-Virasoro algebra, Physics Letters A 322 (2004) 156–165.
45. Jonas T. Hartwig, Daniel Larsson and Sergei D. Silvestrov, Deformations of Lie algebras using Sigma-derivations, ournal of Algebra 295 (2006) 314–361.
46. Thomas L. Curtrighta, David B. Fairlieb and Cosmas K. Zachos, Ternary Virasoro–Witt algebra, Physics Letters B 666 (2008) 386–39.
47. Guang Song, Yucai Su and Yuezhu Wu, Quantization of generalized Virasoro-like algebras, Linear Algebra and its Applications 428 (2008) 2888–2899.
48. Barton Zwiebach, A First Course in String Theory, Section 12.4, Second Edition, 2009, Cambridge University Press.
49. T.Kojimaa and J.Shiraishi, The integrals of motion for the elliptic deformation of the Virasoro and WN algebra, Nonlinear Analysis 71 (2009) e1915-e1921.
50. M. Rasettia and C.Marletto, A novel realization of the Virasoro algebra in number state space, Physics Letters A 373 (2009) 2901–2904.
51. Jeremie Unterberger, On vertex algebra representations of the Schrödinger–Virasoro Lie algebra, Nuclear Physics B 823 [PM] (2009) 320–371.
52. Luc Haine and Didier Vanderstichelen, A centerless representation of the Virasoro algebra associated with the unitary circular ensemble, Journal of Computational and Applied Mathematics 236 (2011) 19–2.

53. Shintarou Yanagida, Whittaker vectors of the Virasoro algebra in terms of Jack symmetric polynomial, Journal of Algebra 333 (2011) 273–294.
54. Azat M. Gainutdinov, Jesper Lykke Jacobsen, Hubert Saleur and Romain Vasseur, Aphysical approach to the classification of indecomposable Virasoro representations from the blob algebra, Nuclear Physics B 873 [PM] (2013) 614–681.
55. Yoshifumi Hyakutake, Super Virasoro algebra from supergravity, Phys. Rev. D 87, 045028 - 2013.
56. Biswajit Paul, Gauge symmetry and Virasoro algebra in quantum charged rigid membrane: A first order formalism, Phys. Rev. D 87, 045003 – 2013.
57. Itzhak Bars and Dmitry Rychkov, Is string interaction the origin of quantum mechanics? Physics Letters B 739 (2014) 451–456.
58. Bibhas Ranjan Majhi, Conformal transformation, near horizon symmetry, Virasoro algebra, and entropy, Phys. Rev. D 90, 044020 – 2014.
59. Hongyan Guo, Haisheng Li, Shaobin Tana and Qing Wang, q-Virasoro algebra and vertex algebras, Journal of Pure and Applied Algebra 219 (2015) 1258–1277.
60. Hyakutake, Y. Super Virasoro Algebras From Chiral Supergravity. Universe 2015, 1, 292-306. https://doi.org/10.3390/universe1020292.
61. XuGao, Ming Liu, Chengming Bai and Naihuan Jing, Rota–Baxter operators on Witt and Virasoro algebras, Journal of Geometry and Physics 108 (2016) 1–20.
62. Arjun Bagchi, Aritra Banerjee, and Pulastya Parekh, Tensionless Path from Closed to Open Strings, Phys. Rev. Lett. 123, 111601 - 2019.
63. Hongyan Guo, Haisheng Li, Shaobin Tan and Qing Wang, q-Virasoro algebra and affine Kac-Moody Lie algebras, Journal of Algebra 534 (2019) 168–189.
64. Yaohui Xue and Kaiming Zhao, Representations of the fermion-Virasoro algebras, math arXiv:2306.15250v1.
65. Chen, X. Quantization of the Rank Two Heisenberg–Virasoro Algebra. Axioms 2024, 13, 446. https://doi.org/10.3390/axioms13070446.
66. Guo, X.; Kaygorodov, I.; Tang, L. Maps on the Mirror Heisenberg–Virasoro Algebra. Mathematics 2024, 12, 802. https://doi.org/10.3390/math12060802.
67. Ivan Kaygorodov, Abror Khudoyberdiyev and Zarina Shermatova, Transposed Poisson structures on Virasoro-type algebras, math arXiv:2406.15532v1.
68. Isaac H. Kim, Xiang Li, Ting-Chun Lin, John McGreevy and Bowen Shi, Chiral Virasoro algebra from a single wavefunction, Annals of Physics 471 (2024) 169849.
69. Nieto-Chaupis, H. Canonical Commutation Relation Derived from Witt Algebra. Mathematics 2025, 13, 1910. https://doi.org/10.3390/math13121910.
70. ikhail I. Dyakonov, Optical orientation of spins in semiconductors, Editor(s): Tapash Chakraborty, Encyclopedia of Condensed Matter Physics (Second Edition), Academic Press, 2024, Pages 223-232.
71. C. Cohen-Tannoudji, B. Diu and F. Laloe, "Quantum Mechanics," John Wiley and Sons, New York, 1977, p.18.
72. J. J. Sakurai, San Fu Tuan. Modern Quantum Mechanics, Chap-II, Revised Edition. Pearson Education, 1994, p.196.
73. John Maddox, Classical and quantum physics mix, News and Views, Nature, Vol 373, 9 February 1995.
74. Goldstin H. Classical Mechanics, Third Edition, Addison Wesley 2002, p.2.
75. Astapenko, V.; Bergaliyev, T. Comparison of Harmonic Oscillator Model in Classical and Quantum Theories of Light-Matter Interaction. Foundations 2023, 3, 549-559. https://doi.org/10.3390/foundations3030031.
76. Kim, T., Dolgy, D.V., Kim, D.S. et al. A note on degenerate generalized Laguerre polynomials and Lah numbers. Adv Differ Equ 2021, 421 (2021). https://doi.org/10.1186/s13662-021-03574-8.
77. A.T. Filippov, A.P. Isaev, and A.B. Kurdikov, Para-Grassmann extensions of the Virasoro algebra, International Journal of Modern Physics AVol. 08, No. 28, pp. 4973-5003 (1993).
78. Shinsei Ryu and Junggi Yoon, Unitarity of Symplectic Fermions in Vacua with Negative Central Charge, Phys. Rev. Lett. 130, 241602 –2023.
79. Horst G. Kausch, Curiosities at c=-2 arXiv:hep-th/9510149.
80. Horst G. Kausch, Symplectic fermions, Nucl. Phys. B583, 513 (2000).