Article

Not peer-reviewed version

Pretopological Modal Logic for Local Reasoning and Uncertainty

Arturo Tozzi *

Posted Date: 24 June 2025

doi: 10.20944/preprints202506.1902.v1

Keywords: Non-normal modalities; observer-dependence; partial observability; informational locality; frame semantics

Article

Pretopological Modal Logic for Local Reasoning and Uncertainty

Arturo Tozzi

Center for Nonlinear Science, Department of Physics, University of North Texas, Denton, Texas, USA, 1155 Union Circle, #311427 Denton, TX 76203-5017 USA; tozziarturo@libero.it

Abstract

We introduce Pretopologically-Neighborhoood Modal Logic (PNML), a formal framework for reasoning about local knowledge and uncertainty based on pretopological neighborhood semantics. Unlike classical Kripke or general neighborhood models assuming global structural properties or arbitrary accessibility, PNML restricts neighborhood systems to satisfy the axioms of pretopological spaces, i.e., upward closure and self-inclusion, without requiring intersection stability or closure under arbitrary unions. This enables a finer-grained representation of agents' information in contexts where only partial or locally available knowledge is relevant. We define the truth conditions for modal operators in terms of pointwise neighborhood filters, introduce a basic axiomatic system and prove its soundness/completeness with respect to the full class of pretopological frames, ensuring that the syntactic and semantic components of the logic are aligned. Then, we examine the expressivity of PNML in relation to both normal and non-normal modal logics, arguing that pretopological constraints introduce structural distinctions not captured by standard neighborhood models, particularly under minimal closure conditions. We present examples illustrating the framework's utility in modeling observer-dependent scenarios, including epistemic uncertainty, context-sensitive reasoning and localized inference. Notably, PNML may accommodate settings in which traditional modal logic either overgeneralizes or underrepresents the dynamics of localized information. By grounding modal reasoning in pretopological structure, PNML may apply to distributed computing with local inputs, quantum mechanics involving contextual observation, cellular signaling and ecological systems in biology, modal update operations under information change, connections to fixpoint and coalgebraic semantic frameworks, proof systems for local inference and real-world modeling of belief revision and protocol verification.

Keywords: non-normal modalities; observer-dependence; partial observability; informational locality; frame semantics

---

1. Introduction

In modal logic, neighborhood semantics has long served as a flexible generalization of Kripke semantics, particularly in accommodating non-normal modal systems where the necessity operator does not obey standard axioms such as distribution or necessitation (Pacuit 2017; Chen et al., 2021; Ferenz and Tedder, 2023). The foundational approach, introduced by Scott (1970) and Montague (1970), allows modal truth conditions to depend on sets of sets of possible worlds, rather than fixed accessibility relations. This abstraction has proven especially useful in modeling epistemic states, deontic reasoning and belief revision, where the behavior of modal operators cannot be captured adequately by relational frames alone (Hansen 2003; Pacuit 2017). However, general neighborhood semantics remains structurally unconstrained: it allows any collection of sets to serve as neighborhoods, even when those collections violate intuitively desirable structural principles such as coherence or persistence (Fan 2020; Tedder and Ferenz 2022). As a result, it becomes difficult to interpret modal validity in terms of information flow or inferential stability, especially when the neighborhoods lack any formal closure properties. Recent efforts have explored more structured

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 June 2025 doi:10.20944/preprints202506.1902.v1

variations, such as topological semantics (McKinsey and Tarski 1944) and filter-based systems (Chellas 1980), but these often reintroduce the rigidity of relational frameworks or impose overly strong assumptions. This prompts the question of whether a middle ground is possible, one that retains the semantic richness of neighborhoods while enforcing minimal structural discipline.

To address this gap, we introduce a modal framework grounded in pretopological spaces, where each world is associated with a neighborhood filter satisfying upward closure and reflexivity, but not necessarily closed under finite intersection. This restriction preserves the flexibility of neighborhood semantics while incorporating formal constraints that enhance interpretability and consistency. By anchoring modal operators in pretopological filters, our system formalizes local information conditions without assuming full topological regularity or relational determinacy. This construction allows modal evaluation to be sensitive to locally available information and supports more precise reasoning about knowledge or belief that is grounded in structurally coherent, though incomplete, contexts.

We will proceed as follows. First, we provide the formal definitions of pretopological frames and the syntax and semantics of the proposed modal system. Next, we introduce a suitable axiom system and establish its soundness and completeness, examining the proof-theoretic properties that characterize and validate the logic. Finally, we explore structural and expressive features of the logic and conclude with a discussion of its implications.

2. Background

Modal logic has traditionally been framed using relational semantics, in which modal operators are interpreted via binary accessibility relations between possible worlds (Blackburn et al., 2002; Holliday 2024). While this Kripkean framework has proven powerful for normal modal systems, its limitations become apparent when modeling weaker or context-dependent forms of modality like default reasoning, obligation, belief or plausibility (Weiss and Birman, 2024). To address these cases, neighborhood semantics was introduced as a more general semantic foundation. Scott and Montague (1970) initiated this perspective by replacing relational frames with neighborhood functions, enabling each world to be associated with a collection of subsets of the domain considered admissible or relevant for modal evaluation. This abstraction removed the dependence on a binary accessibility relation and enabled the interpretation of modal logics beyond the scope of S5-like systems, including non-normal and subnormal modal logics. Later developments by Chellas (1980) and Hansen (2003) refined the technical properties of these models, leading to a robust semantics capable of capturing nonmonotonic reasoning, conditional obligations and informational vagueness.

Despite their generality, classical neighborhood models impose minimal structural constraints on the organization of neighborhoods across worlds (Hansen et al., 2009; Moniri and Maleki 2015). The sets of neighborhoods assigned to each world can vary arbitrarily and are not required to satisfy any closure properties or structural coherence. While this provides maximal flexibility, it also introduces interpretational ambiguity. For instance, the absence of minimal closure properties like upward containment or filter-like behavior can obscure the notion of informational robustness or local consistency. To address these issues, several authors have explored enriched semantics based on topological or algebraic constraints. Topological modal logic, for example, interprets modalities using open sets in topological spaces, with modalities reflecting interior or closure operations (McKinsey and Tarski, 1944). While this structure introduces regularity and continuity into modal interpretation, it also imposes strong global constraints, such as the requirement that open sets be closed under arbitrary unions and finite intersections. In many applications, particularly those involving uncertainty or observer-relative reasoning, these assumptions may be too strong or unrealistic. For instance, in environmental sensor networks, individual nodes collect data solely from their immediate surroundings. Since global access to all information is typically unrealistic, a logic such as PNML, grounded in locally accessible neighborhood structures, more accurately reflects how these systems function under uncertainty and partial, observer-relative conditions.

Pretopological spaces display a natural intermediate framework. Introduced in the context of generalized convergence theory (Kuratowski 1966), a pretopological space assigns to each point a filter of neighborhoods that satisfies self-inclusion and upward closure but does not require closure under finite intersections. This generalization retains the intuition of locally structured information without enforcing full topological regularity. It allows reasoning to be grounded in a notion of proximity or relevance that is internally coherent but only locally structured.

In this work, we build a modal semantic framework on pretopological spaces, interpreting necessity via membership in upward-closed neighborhood filters. We aim to provide a formally disciplined, semantically grounded middle path between unrestricted neighborhood semantics and fully topological models. By leveraging the minimal structure of pretopology, we aim to model local modalities that reflect constrained but flexible access to information, inference or belief.

3. Pretopological Modal Logic: Syntax and Semantics

We define our Pretopologically-Neighborhoood Modal Logic (PNML) by specifying a class of semantic models in which the modal operator is interpreted over a pretopological structure. Let W be a non-empty set of possible worlds. A **pretopological frame** is a pair $\langle W, N \rangle$, where $N: W \to P(P(W))$ assigns to each world $w \in W$ a **filter** $N(w)$ of neighborhoods. Each $N(w)$ must satisfy the following two conditions:

(1) every $N \in N(w)$ contains w;

(2) if $N \in N(w)$ and $N \subseteq M \subseteq W$, then $M \in N(w)$.

This ensures that each $N(w)$ is upward closed and reflexive, aligning with the classical definition of pretopological neighborhood systems (Kuratowski 1966). Importantly, closure under finite intersections is not assumed, distinguishing the structure from a topological base and allowing for a broader range of semantic behavior.

A **PNML model** is a triple $\langle W, N, V \rangle$, where $V: Prop \to P(W)$ is a valuation function assigning to each propositional variable a set of worlds at which it holds. The truth conditions for the modal language are defined inductively in the standard way, with atomic formulas evaluated according to V, Boolean connectives interpreted classically and the modal clause given as follows:

$$w \Vdash \Box\varphi iff \exists N \in N(w) \text{ such that } \forall v \in N, v \Vdash \varphi.$$

This semantic clause reflects the locality of necessity in pretopological settings: $\Box\varphi$ is true at w just in case there is a neighborhood of w where φ is uniformly true. Note that the use of existential quantification over neighborhoods aligns this clause with weaker modal systems, including monotonic and non-normal logics (Hansen 2003; Pacuit 2017). In contrast to Kripke semantics, there is no global accessibility relation dictating reachability between worlds. Instead, each world defines its own perspective through its neighborhood filter.

This formulation allows modal evaluation to reflect partial and context-sensitive knowledge. The minimal structural requirements on $N(w)$ preserve interpretive flexibility while avoiding the arbitrariness of fully general neighborhood systems. The absence of intersection closure allows different informational pathways to remain disjoint, preserving distinctions between incompatible or non-overlapping contexts. Consequently, PNML generalizes the semantics of many weak modal systems while grounding them in a mathematically coherent spatial structure. This model structure supports the definition of logical consequence, validity and frame definability, which will be examined in subsequent sections.

4. Axiom System and Semantic Correspondence

To formalize the deductive structure of PNML, we introduce a minimal axiomatic system reflecting the structural constraints of pretopological neighborhood semantics. The system extends classical propositional logic with modal rules and axioms that are valid in all PNML frames, as defined in the previous section. We denote this system by **PNML0_00**.

The axioms of PNML0_00 are as follows:

1. All tautologies of classical propositional logic are included in the system.
2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ (Monotonicity)
3. $\Box\varphi \to \varphi$ (Reflexivity)

The system includes a single rule of inference:

Modus Ponens: From φ and $\varphi \to \psi, infer\ \psi\backslash psi\psi$. Necessitation is omitted, consistent with the absence of normality.

The axiom of monotonicity corresponds to the upward closure of neighborhood filters, ensuring that if a formula holds in one neighborhood, it will also hold in all supersets of that neighborhood (Hansen 2003). The axiom of reflexivity reflects the requirement that every neighborhood of a world contains that world, hence guaranteeing that if a formula is necessary, then it must hold locally. This setup avoids commitment to stronger modal principles such as distribution or necessitation, which would not be valid in all pretopological frames.

To establish soundness, we verify that all axioms of PNML0_00 are valid in every PNML model and that the inference rule preserves validity. The reflexivity of neighborhoods ensures the validity of $\Box\varphi \to \varphi$, while monotonicity follows from the closure of neighborhoods under supersets. Since the semantics is Boolean and propositional tautologies are preserved under all valuations, classical reasoning is sound by construction.

Completeness is demonstrated via canonical model construction. We define the set of maximally consistent sets over the language and assign to each such set Γ a **neighborhood filter** $N(\Gamma)$ consisting of all sets of the form $\{\Delta \in Wc \mid \varphi \in \Delta\}$, where $\Box\varphi \in \Gamma$. This construction ensures that the neighborhood system satisfies the conditions of a pretopological space, including upward closure and self-inclusion, while avoiding intersection closure. Standard canonical model techniques (Chellas 1980; Pacuit 2017) are adapted to verify the truth lemma and derive the completeness theorem: every formula valid in all PNML frames is derivable in PNML0_00.

This establishes a sound and complete correspondence between the axiomatic system PNML0_00 and the class of pretopological modal frames, justifying the proposed semantic framework as a framework for reasoning with local and structurally constrained modalities.

5. Expressivity and Structural Properties

The expressive capacity of PNML can be examined in comparison with both standard Kripke-based modal logic and classical neighborhood semantics. Given that PNML permits the omission of necessitation and closure under intersection, it naturally aligns more closely with weak modal logics such as monotonic and subnormal systems. However, the introduction of pretopological structure introduces specific constraints that influence definability, bisimulation and frame conditions in nontrivial ways. We address these issues in the following paragraphs.

A primary structural feature of PNML is its alignment with monotonic modal logic, due to the enforcement of upward closure in all neighborhood filters. This ensures the validity of monotonicity $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, which distinguishes PNML from fully arbitrary neighborhood models, where this principle may fail (Hansen 2003). Furthermore, PNML's restriction to neighborhood systems that satisfy reflexivity $\Box\varphi \to \varphi$ allows it to capture the local realism often assumed in weak epistemic or observational frameworks. At the same time, the logic is strictly weaker than systems based on topological semantics, which validate stronger modal axioms such as closure under finite intersection or the idempotency of the interior operator $\Box\varphi \to \Box\Box\varphi$ (McKinsey and Tarski 1944).

Bisimulation in PNML requires modification from the standard relational and neighborhood settings (Hamal and Terziler, 2015). Since each world in a PNML frame carries a distinct pretopological filter rather than an accessibility relation or an unstructured neighborhood function, standard bisimulation invariance results do not carry over directly. A suitable notion of PNML-bisimulation must preserve the satisfaction of modal formulas by comparing not only pointwise valuations but also the structure of neighborhood filters. Two points w and w' are PNML-bisimilar if there exists a relation preserving propositional variables and respecting inclusion relations among neighborhoods in a way that reflects the structural constraints of pretopology. Establishing the bisimulation invariance of PNML formulas under this condition supports the adequacy of the semantic framework and bounds its expressive power with respect to definable frame properties (Blackburn et al., 2001).

Another relevant structural aspect is frame definability. In PNML, several frame properties correspond directly to specific modal axioms. For example, the axiom $\Box\varphi \rightarrow \varphi$ captures the reflexivity of the neighborhood filter, while the monotonicity axiom $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$ reflects the upward closure of neighborhoods. However, the lack of closure under intersection implies that axioms such as $\Box\varphi \land \Box\psi \rightarrow \Box(\varphi \land \psi)$ are not valid and frames satisfying stronger closure conditions are not definable within the base system PNML0_00. This sets clear boundaries for the modal definability of structural features within the logic and distinguishes PNML sharply from both topological and Kripkean modal systems.

Together, these features suggest that PNML occupies a distinct position in the modal landscape. It balances the generality of neighborhood semantics with the minimal order-theoretic structure of pretopology, providing a semantics that is weak enough to model uncertainty and locality, yet structured enough to permit rigorous logical treatment of modal inference.

6. Illustrative Example

To demonstrate the semantic mechanics of PNML and the interpretive distinctions it supports, we present a minimal model involving three possible worlds and a single propositional variable. The aim is to clarify how modal evaluation proceeds in a pretopological context and to contrast this with behavior in relational or fully topological models.

Let $W = \{w1, w2, w3\}$ be the set of worlds. Let the valuation V assign the propositional variable p as true at $w1$ and $w2$ and false at $w3$; that is, $V(p) = \{w1, w2\}$. The pretopological neighborhood filters are defined as follows:

- $N(w1) = \{\{w1\}, \{w1, w2\}, \{w1, w2, w3\}\}$
- $N(w2) = \{\{w2\}, \{w2, w3\}, \{w1, w2, w3\}\}$
- $N(w3) = \{\{w3\}, \{w2, w3\}, \{w1, w2, w3\}\}$

Each neighborhood filter satisfies upward closure and self-inclusion and none requires closure under intersection. Consider the evaluation of $\Box p$ at each world. At $w1$, the neighborhood $\{w1\} \subseteq V(p)$ and so $w1 \Vdash \Box p$. However, the largest neighborhood $\{w1, w2, w3\}$ includes $w3$, where p is false, so not all neighborhoods validate p. Since the semantics requires the existence of at least one neighborhood where p holds universally, $\Box p$ remains true at $w1$.

At $w2$, the neighborhood $\{w2\} \subseteq V(p)$, so $\Box p$ is also true at $w2$. In contrast, at $w3$ all neighborhoods intersect with worlds where p is true, but none are fully included in $\{w3\} \subseteq W \setminus V(p)$, making $\Box p$ false at $w3$. This illustrates the asymmetry of modal truth induced by the neighborhood system's structure and highlights how local availability of information (as captured by neighborhoods) governs necessity in PNML.

We now present a practical example of an application of PNML. Consider a simplified delivery network with three stations: $w1$, $w2$ and $w3$. A package is confirmed to have arrived at $w1$ and $w2$, but not at $w3$. Each station has limited visibility defined by its neighborhood filter. Station $w1$ can verify its own state, its state plus $w2$, or the entire network. Since it has access to a neighborhood fully contained within the set of stations where the package has arrived, it concludes with certainty that delivery occurred. Similarly, $w2$ sees neighborhoods confirming delivery. Station w3w_3w3, the final user waiting for the package, cannot access any neighborhood entirely confined to non-delivery states, as all of its accessible views include stations where the package has arrived. Consequently, it lacks sufficient local support to conclude necessity.

This example illustrates how PNML captures local, constrained reasoning under uncertainty from the perspective of limited observers. Rather than relying on global reachability or universally available information, PNML evaluates necessity through locally bounded, upward-closed neighborhood structures that reflect partial observability and context-dependent knowledge. More technically, PNML departs from topological or relational semantics, where the truth of $\Box p$ depends on open-set containment or accessibility across the model. Instead, it requires the existence of a specific neighborhood in which the formula holds entirely. As a result, PNML supports modal distinctions even within symmetric or densely connected models. This highlights the central role of pretopological filters in expressing modalities grounded in local informational structure.

7. Extensions and Logical Variants

The core framework of PNML admits natural extensions that retain its semantic foundation while increasing its expressive range. One extension involves the introduction of additional modal operators reflecting **dual or complementary neighborhood behavior**. For instance, a possibility operator $\Diamond\varphi$ can be defined in terms of non-universal failure: $w \Vdash \Diamond\varphi$ if there exists a neighborhood $N \in N(w)$ and some $v \in N$ such that $v \Vdash \varphi$. While definable via the dual of necessity in classical modal logic, this operator gains independent semantic interest in PNML due to the lack of normality and the non-deterministic structure of the neighborhoods.

Another extension concerns the incorporation of **dynamic modalities**. In contexts where the neighborhood filter evolves in response to actions or information updates, one may define dynamic operators transforming $N(w)$ at each step. This can model the refinement of local information states, as seen in update semantics and dynamic epistemic logic. Formal treatment would require an additional specification of update functions acting on pretopological frames and corresponding modifications to the satisfaction relation. This introduces challenges, as updates may break the structural properties of the filter unless carefully constrained.

A further direction involves the **algebraic semantics** of PNML. Pretopological filters correspond to certain closure operators and monotone functions, suggesting a dual algebraic characterization in terms of neighborhood lattices or modal algebras equipped with weakened intersection properties. Investigation of completeness and representation theorems in this setting may yield a deeper understanding of the underlying logical geometry.

Finally, **multi-agent generalizations** of PNML can be introduced by assigning distinct pretopological filters $Na(w)$ for each agent a. This enables modeling differences in local knowledge or information availability across agents, preserving the logic's sensitivity to structure while supporting interaction between heterogeneous perspectives. These systems may incorporate axioms or interaction rules that govern the relationships between agents' filters, leading to a family of sublogics with varying epistemic strength.

Overall, each of these extensions preserves the core features of PNML, namely the use of structured yet minimally constrained neighborhood systems, while enriching the formalism to accommodate more complex or interactive modalities. The resulting variants may serve as useful tools in the study of reasoning under partial observability, heterogeneous knowledge systems and generalized information dynamics.

8. Conclusions

The development of our Pretopologically-Neighborhoood Modal Logic (PNML) yields a formally structured modal framework grounded in minimal but coherent semantic assumptions. By constructing models where each world is associated with a pretopological neighborhood filter—closed under supersets and containing the point itself, We established a logic that preserves the interpretive flexibility of neighborhood semantics while introducing structural constraints absent from more general frameworks (Kuratowski 1966). The modal operator was defined in terms of localized satisfaction across neighborhood filters and the axiom system PNML0_00 was shown to be both sound and complete with respect to this class of frames. The semantics allows for the existence of neighborhoods supporting a formula without requiring universal or intersection-based closure, enabling PNML to represent diverse local reasoning patterns. The system was also shown to preserve desirable properties such as upward monotonicity and reflexivity, with definability and bisimulation conditions adapted to its non-normal nature (Hansen 2003; Pacuit 2017).

The novelty of our approach lies in embedding modal reasoning directly within pretopological spaces, avoiding both the rigidity of relational models and the arbitrariness of unconstrained neighborhood semantics (Scott 1970; Montague 1970). While topological modal logic imposes strong global closure properties (McKinsey and Tarski, 1944) and Kripke semantics relies on binary reachability, our system formalizes necessity using local information patterns rooted in upward-closed filters. This structural minimalism does not require intersection closure, thus occupying a space between the fully general and the overly restrictive. Unlike classical neighborhood logics, which often lack coherent frame-theoretic foundations (Chellas 1980), PNML ensures that modal inference respects an order-theoretic organization compatible with generalized convergence and local reasoning. In contrast to peritopological approaches that remain relational at the core (Hamal and Terziler, 2015), our framework defines modality through directly neighborhood-based conditions and emphasizes syntactic completeness alongside semantic interpretation. Compared to topological modal logics, our logic omits idempotence and compactness, which may not be valid in dynamic or observational systems. The absence of necessitation reflects a more conservative account of knowledge or belief, applicable to partial or evolving environments.

Nonetheless, several limitations emerge. First, the absence of intersection closure means that conjunctions of necessary formulas do not yield further necessary formulas, limiting deductive strength in classical terms. While this feature is consistent with the intended semantics, it may hinder applications that assume distributive modal behavior. Second, the omission of necessitation, though deliberate, weakens the inferential machinery relative to normal modal systems (Blackburn, et al., 2001). Extensions requiring stronger meta-properties such as interpolation or uniform substitution will require substantial modification. In terms of semantic complexity, the need to define canonical models through filtered neighborhood constructions may present technical challenges in developing decision procedures or model-checking algorithms. Finally, PNML as currently formulated is propositional; its extension to quantified or predicate forms is nontrivial, particularly due to the local nature of neighborhood filters and the absence of global constraints. These limitations suggest that further refinements are needed to scale it toward more expressive logical environments.

PNML has potential applications in fields where local knowledge and partial observability are fundamental. In distributed computing, agents often make decisions based on partial inputs or neighborhood-based communication, conditions well-modeled by pretopological filters. In quantum mechanics, where observation and state determination are contextual (van Fraassen 1991), PNML may provide a way to model knowledge acquisition without global state assumptions. In biology, particularly in cellular signaling or ecological reasoning, local interactions without full system knowledge are prevalent and reasoning about such conditions may benefit from this framework. Future research could explore modal update operations on neighborhood filters, enabling dynamic reasoning under information change (van Benthem 2011). Another avenue involves the development of a proof calculus more closely aligned with neighborhood filter behavior, possibly incorporating labeled or tableau methods tailored to the logic's structure. Empirically testable hypotheses include the ability of PNML to model real-world inference scenarios more accurately than Kripkean systems in domains like belief revision or protocol verification (Fagin et al., 1995). We suggest further exploration of algebraic dualities and canonical extensions as a means of connecting PNML to modal fixpoint calculi and coalgebraic frameworks, potentially revealing deeper structural correspondences and expanding its applicability within abstract semantic models (Venema 2007).

In summary, we asked whether a semantically grounded modal system could be built on pretopological structures while supporting sound inference under minimal constraints. We answered affirmatively by presenting a complete and well-defined logic that interprets modality through local, upward-closed neighborhood filters. Our framework reconstructs modality directly from neighborhood-based locality, without relying on relational or topological assumptions. PNML occupies a middle ground between fully unrestricted neighborhood semantics and highly constrained topological or relational systems. Its structural simplicity make it well suited for reasoning under uncertainty in settings with partial, locally available information.

Ethics approval and consent to participate. This research does not contain any studies with human participants or animals performed by the Author.

Consent for publication. The Author transfers all copyright ownership, in the event the work is published. The undersigned author warrants that the article is original, does not infringe on any copyright or other proprietary right of any third part, is not under consideration by another journal and has not been previously published.

Availability of data and materials. All data and materials generated or analyzed during this study are included in the manuscript. The Author had full access to all the data in the study and took responsibility for the integrity of the data and the accuracy of the data analysis.

Competing interests. The Author does not have any known or potential conflict of interest including any financial, personal or other relationships with other people or organizations within three years of beginning the submitted work that could inappropriately influence or be perceived to influence their work.

Funding. This research did not receive any specific grant from funding agencies in the public, commercial or not-for-profit sectors.

Acknowledgements: none.

Authors' contributions. The Author performed: study concept and design, acquisition of data, analysis and interpretation of data, drafting of the manuscript, critical revision of the manuscript for important intellectual content, statistical analysis, obtained funding, administrative, technical and material support, study supervision.

Declaration of generative AI and AI-assisted technologies in the writing process. During the preparation of this work, the author used ChatGPT 4o to assist with data analysis and manuscript drafting and to improve spelling, grammar and general editing. After using this tool, the author reviewed and edited the content as needed, taking full responsibility for the content of the publication.

References

1. Blackburn, Patrick, Maarten de Rijke, and Yde Venema. 2002. *Modal Logic*. Cambridge Tracts in Theoretical Computer Science. Cambridge: Cambridge University Press.
2. Chellas, Brian F. 1980. *Modal Logic: An Introduction*. Cambridge: Cambridge University Press.
3. Chen, Jinsheng, Hans van Ditmarsch, Giuseppe Greco, and Apostolos Tzimoulis. 2021. "Neighbourhood Semantics for Graded Modal Logic." *Bulletin of the Section of Logic* 50, no. 3. https://doi.org/10.18778/0138-0680.2021.12.
4. Fagin, Ronald, Joseph Y. Halpern, Yoram Moses, and Moshe Y. Vardi. *Reasoning About Knowledge*. Cambridge, MA: MIT Press, 2004.
5. Fan, Jie. 2020. "Notes on Neighborhood Semantics for Logics of Unknown Truths and False Beliefs." *arXiv* preprint. https://arxiv.org/abs/2002.09622.
6. Ferenz, Niklas, and Andrew Tedder. 2023. "Neighbourhood Semantics for Modal Relevant Logics." *Journal of Philosophical Logic* 52: 145–181. https://doi.org/10.1007/s10992-022-09668-2.
7. Hamal, Ahmet and Mehmet Terziler. 2015. "Peritopological Spaces and Bisimulations." *Reports on Mathematical Logic* 50: 67–81.
8. Hansen, Helle Hvid, Clemens Kupke, and Eric Pacuit. 2009. "Neighbourhood Structures: Bisimilarity and Basic Model Theory." *Logical Methods in Computer Science* 5, no. 2. https://doi.org/10.2168/LMCS-5(2:2)2009.
9. Hansen, H. H. 2003. *Monotonic Modal Logic*. PhD diss., University of Amsterdam.
10. Holliday, Wesley H. 2024. "Modal Logic, Fundamentally." *arXiv* preprint. https://arxiv.org/abs/2403.14043.
11. Kuratowski, Kazimierz. 1966. *Topology. Volume I*. New York: Academic Press.
12. McKinsey, J. C. C. and Alfred Tarski. 1944. "The Algebra of Topology." *Annals of Mathematics* 45(1): 141–91.
13. Moniri, Morteza, and Fatemeh Shirmohammadzadeh Maleki. 2015. "Neighborhood Semantics for Basic and Intuitionistic Logic." *Logic and Logical Philosophy* 24, no. 3: 339–355. https://doi.org/10.12775/LLP.2015.015.
14. Montague, R. "Universal Grammar", *Theoria* 36, 373–98, 1970.

 doi:10.20944/preprints202506.1902.v1

15. Pacuit, Eric. 2017. *Neighborhood Semantics for Modal Logic*. Short Textbooks in Logic. Cham: Springer. https://doi.org/10.1007/978-3-319-67149-9.
16. Scott, D. "Advice on modal logic", in *Philosophical Problems in Logic*, ed. Karel Lambert. Reidel, 1970.
17. Tedder, Andrew, and Niklas Ferenz. 2022. "Neighbourhood Semantics for Quantified Relevant Logics." *Journal of Philosophical Logic* 51: 457–484. https://doi.org/10.1007/s10992-021-09637-1.
18. van Benthem, Johan. 2011. *Logical Dynamics of Information and Interaction*. Cambridge: Cambridge University Press.
19. van Fraassen, Bas C. 1991. *Quantum Mechanics: An Empiricist View*. Oxford: Oxford University Press.
20. Venema, Yde. 2007. "Algebras and Coalgebras." In *Handbook of Modal Logic*, edited by Patrick Blackburn, Johan van Benthem and Frank Wolter, 331–426. Amsterdam: Elsevier.
21. Weiss, Yale, and Romina Birman, eds. 2024. *Saul Kripke on Modal Logic*. Outstanding Contributions to Logic, vol. 30. Cham: Springer.