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Finite Relativistic Algebra at Composite Cardinalities

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Finite Relativistic Algebra at Composite Cardinalities

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Abstract: We extend finite relativistic algebra from prime fields to composite moduli q. The finite analogues of canonical constants i, π, e lift uniquely via Hensel's lemma, glue through the Chinese Remainder Theorem and assemble into profinitely stable families. The resulting arithmetic bouquet possesses a Seifert-fibred 3-orbifold structure whose exceptional fibres record the prime factors of q, while a mixed-radix expansion yields digit coordinates suitable for Fourier and modal analysis. The framework retains the algebraic rigor, geometric depth and analytic versatility of its prime predecessors. Together these elements provide a coherent, scalable calculus on finite rings, paving the way for applications and modeling in an informationally finite physical universe.

Keywords: finite relativistic algebra; composite cardinality; profinite stability; canonical constants; canonical constants; mixed-radix decomposition; Seifert-fibred 3-orbifold; discrete geometry; Hensel lifting; harmonic analysis on finite rings

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1. Introduction

Our *finite-relativistic programme* has so-far advanced through a trilogy of companion manuscripts whose aims were, respectively, (i) establish an ontological motivation for the move from the conventional *infinitude conjecture* toward a framework designed on the principle of *relational finitude* [1], (ii) to rebuild classical algebra inside a single framed prime field [3], and (iii) to uncover a smooth geometric layer, and the fundamental constants already latent in that finite algebra [2]. The present article constitutes a natural continuation of this effort: it extends the framework from prime moduli to *arbitrary composite cardinalities* q and shows that the resulting arithmetic still supports the ensemble of canonical constants, harmonic structures and low-dimensional topology.

More specifically, in [3] we fixed a prime $p \equiv 1 \pmod 4$ and a framing $(0, 1)$ inside the field $\mathbb{F}_p$. The paper thus reconstructed signed integers, rationals, reals and complexes as *pseudo-numbers*: relational shadows generated by the field's internal symmetries of translation, scaling and exponentiation. Crucially, this finitary setting preserved all familiar algebraic laws and analytic operations while dispensing with actual *infinity*. Subsequently, in [2], we demonstrated that a single finite field already contains the structure of smooth geometry. By arranging the three symmetry operators orthogonally to a cardinality axis, one obtains a combinatorial 2-spheroid N_p whose ultrapower lift $S_p \subset R_p^4$ is an internal C^∞ surface of constant curvature. The cyclic order of the field singles out three frame-invariant elements i_p, π_p, e_p, finite-field analog of the classical i, π, e, which underpin a unified Fourier kernel valid in both the finite and continuous regimes.

Building on those foundations, the current article tackles moduli

$$q = \prod_{k=1}^{K} p_k, \qquad p_k \equiv 1 \pmod 4,$$

and establishes four main results:

1. **Canonical constants at non-prime q.** The quarter-turn i_q, half-period π_q and minimum-action base e_q are defined by lifting their prime counterparts through Hensel's lemma and gluing them via the Chinese Remainder Theorem. Each satisfies a projective stability property, and together they generate the full symmetry triple on $\mathbb{Z}_q$.

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2. **Bouquet of prime spheroids.** The Cartesian splitting $\mathbb{Z}_q = \mathbb{F}_{p_1} \times \cdots \times \mathbb{F}_{p_K}$ realizes the composite 4-dimensional spheroid as a bouquet of its prime-field pieces glued along the zero-divisor locus, yielding explicit cardinality formulas and nested visualizations.
3. **Seifert-fibred 3-orbifold $\mathcal{S}_q$.** Interpreting translation, multiplication and exponentiation as commuting circle actions upgrades the bouquet into a Seifert manifold whose exceptional fibres encode the prime-power decomposition of q. The resulting "DNA-like" geometry provides a topological dictionary between arithmetic and 3-manifold invariants.
4. **Mixed-radix and fiber decompositions.** A triplet of expansion modes for $\mathbb{Z}_q$—derived from the operations of addition, multiplication and exponentiation—deliver unique digit vectors that serve as coordinates for harmonic analysis and modal splitting on $\mathcal{S}_q$. The sets of units of the resultant vector decompositions contain abundant classes of elements that become stable for large values of cardinality q.

Together these results show that the proposed finite relativistic algebra retains its algebraic coherence, analytic versatility and geometric depth when the ambient cardinality is no longer prime. In doing so, the paper completes the transition from two-dimensional to three-dimensional finite geometry, setting the stage for applications of physical dynamics.

2. Composite-Modulus Finite Ring

2.1. Bi-Prime Composition

We commence by considering a simple example of the composite modulus $q = p \cdot p' = 1 \pmod 4$ composed of some pair of prime factors p, p'. The ring $\mathbb{Z}_q$ is the quotient of the integers by the ideal generated by q.

Definition 1 (CRT coordinates)**.** Via the Chinese Remainder Theorem we fix an isomorphism

$$\mathbb{Z}_q \;\cong\; \mathbb{F}_p \times \mathbb{F}_{p'}, \qquad x \longmapsto (\overline{x}_p,\, \overline{x}_{p'}),$$

and denote the projections by $\mathcal{P}_p, \mathcal{P}_{p'} : \mathbb{Z}_q \to \mathbb{F}_p, \mathbb{F}_{p'}$.

Definition 2 (Prime-field 2-spheroids [2])**.** For a prime p set

$$\mathcal{S}_p \;=\; \left\{ \mathbf{v} = (v_1, \ldots, v_4) \in \mathbb{F}_p^4 \;\middle|\; Q(\mathbf{v}) = 1 \right\}, \qquad Q(\mathbf{v}) = v_1^2 + v_2^2 + v_3^2 + v_4^2.$$

Definition 3 (Composite spheroid)**.**

$$\mathcal{S}_q \;=\; \left\{ \mathbf{x} \in \mathbb{Z}_q^4 \;\middle|\; Q(\mathbf{x}) = 1 \right\}.$$

Proposition 1 (Cartesian splitting)**.** Under the CRT identification $\mathbb{Z}_q^4 \cong (\mathbb{F}_p \times \mathbb{F}_{p'})^4$ we have

$$\mathcal{S}_q \;=\; \mathcal{S}_p \times \mathcal{S}_{p'}.$$

Proof. Write $\mathbf{x} = (x_1, \ldots, x_4)$. Then

$$Q(\mathbf{x}) \bmod q \;=\; \Big(Q\big(\mathcal{P}_p(\mathbf{x})\big) \bmod p,\; Q\big(\mathcal{P}_{p'}(\mathbf{x})\big) \bmod p' \Big),$$

so $Q(\mathbf{x}) \equiv 1 \pmod q$ *iff* $Q\big(\mathcal{P}_p(\mathbf{x})\big) = 1$ in $\mathbb{F}_p$ *and* $Q\big(\mathcal{P}_{p'}(\mathbf{x})\big) = 1$ in $\mathbb{F}_{p'}$. $\square$

Definition 4 (Prime lifts in $\mathbb{Z}_q^4$)**.**

$$\mathcal{S}_p^{\uparrow} := \{\mathbf{x} \in \mathcal{S}_q : \mathcal{P}_{p'}(\mathbf{x}) = \mathbf{0}\}, \quad \mathcal{S}_{p'}^{\uparrow} := \{\mathbf{x} \in \mathcal{S}_q : \mathcal{P}_p(\mathbf{x}) = \mathbf{0}\}.$$

Lemma 2.1. *$\mathcal{S}_q = \mathcal{S}_p^{\uparrow} \,\dot\cup\, \mathcal{S}_{p'}^{\uparrow}$ and $\mathcal{S}_p^{\uparrow} \cap \mathcal{S}_{p'}^{\uparrow} = \{\mathbf{0}\}$.*

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Proof. Immediate from 1: if both projections of $\mathbf{x}$ were non-zero, each would have to satisfy $Q = 1$ in its own field, contradicting the definitions above. Hence, the only common point is the origin. □

Remark 2.2 (Nested visualisation)**.** Realise $\mathbb{Z}_q$ as the interval $[-32, 32] \subset \mathbb{R}$. Every non-zero coordinate of $N_5^{\uparrow}$ is a multiple of 13 (length ≤ 26), whereas coordinates in $N_{13}^{\uparrow}$ may reach ± 30. Thus in the Euclidean lift the $\mathbb{F}_p[5]$-spheroid sits *inside* the $\mathbb{F}_p[13]$-spheroid, touching only at $\mathbf{0}$. For any square-free composite $q = \prod_i p_i$ the global spheroid is likewise a bouquet of prime-field spheroids glued along the zero-divisor locus.

Corollary 1 (Cardinalities)**.** Using $|N_p| = p^2 - p + 1$ (see [2]) we get

$$|N_q| = |N_5| + |N_{13}| - 1 = 120 + 2184 - 1 = 2303.$$

3. Composite-Modulus Construction of the Canonical Constants

Building on the intuition obtained from the bi-prime case, we now extend the definitions of the three canonical constants i_q, π_q, and e_q to a general composite modulus q that is a product of K distinct odd primes, each congruent to $1 \pmod 4$. Let

$$q \;=\; \prod_{k=1}^{K} p_k,$$

where each p_k is an odd prime satisfying $p_k \equiv 1 \pmod 4$. Importantly, there is no assumption here that $\{p_k\}$ are distinct, i.e. there can be an arbitrary number of repetitions of such primes in any order. We note the Chinese Remainder decomposition

$$\mathbb{Z}_q \;\cong\; \prod_{k=1}^{K} \mathbb{F}_{p_k}.$$

We will now combine the prime-modulus constants i_{p_k}, π_{p_k}, and $e_{p_k} \in \mathbb{F}_{p_k}$ for each $k = 1, \ldots, K$ via the CRT to obtain the three constants $i_q,\ \pi_q,\ e_q \in \mathbb{Z}_q$.

3.1. Quarter-turn $i_q \in \mathbb{Z}_q$

(a) Existence of i_{p_k}. Since $p_k \equiv 1 \pmod 4$, -1 is a quadratic residue modulo p_k. Let

$$i_{p_k} \;=\; \min\Big\{\, x \in \mathbb{F}_{p_k} : x^2 \equiv -1 \pmod{p_k},\ 1 \leq x < \tfrac{p_k - 1}{2} \Big\}.$$

Then $i_{p_k}^2 \equiv -1 \pmod{p_k}$, and by Hensel's lemma [5] there is a unique lift

$$i_{p_k} \;\in\; \mathbb{F}_{p_k} \quad \text{such that} \quad i_{p_k}^2 \;\equiv\; -1 \pmod{p_k}.$$

Denote its inverse in $\mathbb{F}_{p_k}$ by $i_{p_k}^{-1}$, so $i_{p_k} \cdot i_{p_k}^{-1} \equiv 1 \pmod{p_k}$.

(b) Lifting i to $\mathbb{Z}_q$. By CRT,

$$\mathbb{Z}_q \;\cong\; \prod_{k=1}^{K} \mathbb{F}_{p_k}.$$

Hence, there is a unique element $i_q \in \mathbb{Z}_q$ whose projection to each factor $\mathbb{F}_{p_k}$ is i_{p_k}. Equivalently, in CRT notation,

$$i_q \;=\; \Big(i_{p_1} \bmod p_1,\ i_{p_2} \bmod p_2,\ \ldots,\ i_{p_K} \bmod p_K \Big) \;\in\; \prod_{k=1}^{K} \mathbb{F}_{p_k} \;\cong\; \mathbb{Z}_q.$$

By construction, each $i_{p_k}^2 \equiv -1 \pmod{p_k}$, so

$$i_q^2 \;\equiv\; -1 \quad \text{in } \mathbb{Z}_q.$$

Equivalently, one may express i_q via the inverses in each prime component:

$$i_q \;=\; \left(i_{p_1}^{-1} \times i_{p_2}^{-1} \times \cdots \times i_{p_K}^{-1} \right)^{-1}. \tag{3.1}$$

Thus i_q is the "quarter-turn" in $\mathbb{Z}_q$ whose reduction modulo p_k is i_{p_k} for each k.

3.2. Minimum Action $e_q \in \mathbb{Z}_q$

(a) Existence of e_{p_k}. For each prime p_k, let

$$e_{p_k} \;=\; \arg\min\Big\{ \min(x, p_k - x) \;:\; x \in (\mathbb{Z}_{p_k})^\times,\; x \text{ a primitive root modulo } p_k \Big\},$$

i.e. the unique generator of $\mathbb{F}_{p_k}^\times$ closest to 1. Since primitive roots lift to primitive roots modulo p_k, there is (for each choice of residue class of e_{p_k}) a unique

$$e_{p_k} \;\in\; \mathbb{F}_{p_k} \quad \text{satisfying} \quad e_{p_k} \text{ is a primitive root of } \mathbb{F}_{p_k}^\times.$$

(b) Lifting e to $\mathbb{Z}_q$. By CRT,

$$\mathbb{Z}_q^\times \;\cong\; \prod_{k=1}^{K} \mathbb{F}_{p_k}^\times.$$

Each e_{p_k} is the distinguished "closest-to-one" primitive root in $\mathbb{F}_{p_k}^\times$. Therefore, there is a unique

$$e_q \;\in\; \mathbb{Z}_q^\times$$

whose projection to each factor $\mathbb{F}_{p_k}^\times$ is e_{p_k}. Equivalently, in CRT form,

$$e_q \;=\; \Big(e_{p_1} \bmod p_1,\; e_{p_2} \bmod p_2,\; \ldots,\; e_{p_K} \bmod p_K \Big). \tag{3.2}$$

By construction, e_q is a primitive root of $\mathbb{Z}_q^\times$, and its "distance from 1",

$$\Delta \, \exp_q(0) \;=\; e_q - 1,$$

is minimal among all generators of $\mathbb{Z}_q^\times$.

3.3. Half Period $\pi_q \in \mathbb{Z}_q$

(a) Existence of π_{p_k}. For any primep_k, one has $\varphi(p_k) \;=\; p_k\,(p_k - 1)$. In complete analogy with the prime case, we set

$$\pi_{p_k} \;=\; \frac{\varphi(p_k)}{2} \;=\; \frac{p_k - 1}{2}.$$

Since $p_k \equiv 1 \pmod 4$, the integer π_{p_k} is well-defined. If e_{p_k} is the minimum action generator of $\mathbb{F}_{p_k}^\times$, then

$$e_{p_k}^{\pi_{p_k}} \;\equiv\; -1 \quad (\bmod\, p_k).$$

(b) Lifting π to $\mathbb{Z}_q$. Since

$$\varphi(q) \;=\; \prod_{k=1}^{K} \varphi(p_k^{n_k}) \;=\; \prod_{k=1}^{K} \big[\, p_k^{n_k - 1}(p_k - 1) \big],$$

we define

$$\pi_q \;=\; \frac{\varphi(q)}{2} \;=\; \frac{1}{2}\prod_{k=1}^{K}\bigl[\,p_k^{n_k-1}(p_k-1)\bigr]. \tag{3.3}$$

Choose the minimum action generator $e_q \in \mathbb{Z}_q^\times$. Under CRT, e_q projects to primitive roots $\tilde{g}_{p_k} \in \mathbb{F}_{p_k}^\times$ for each k. Then

$$\tilde{g}^{\pi_q} \;\equiv\; -1 \quad \text{in each } \mathbb{F}_{p_k}, \quad \text{hence } \tilde{g}^{\pi_q} \equiv -1 \pmod{q}.$$

Therefore π_q is the unique "half-period" exponent sending a CRT-lifted generator e_q to -1 in $\mathbb{Z}_q$.

3.4. Summary of the Three Composite Constants for $q = \prod_k p_k$

Putting the above constructions together, for $q = \prod_{k=1}^{K} p_k$ with each $p_k \equiv 1 \pmod 4$, the three canonical constants in $\mathbb{Z}_q$ are given precisely by the Chinese-remainder lifts of their prime-modulus counterparts:

(i) Quarter-turn $i_q \in \mathbb{Z}_q$: For each k, $i_{p_k} \in \mathbb{F}_{p_k}$ is the unique Hensel-lift of i_{p_k},

$$i_q = \Bigl(\, i_{p_1} \bmod p_1, \ldots,\; i_{p_K} \bmod p_K\Bigr) \;\in\; \mathbb{Z}_q,$$

equivalently, i_q is the unique element with $i_q^2 \equiv -1 \pmod{q}$.

(ii) Exponential base $e_q \in \mathbb{Z}_q$: For each k, $e_{p_k} \in \mathbb{F}_{p_k}$ is the unique Hensel-lift of e_{p_k},

$$i_q = \Bigl(\, e_{p_1} \bmod p_1, \ldots,\; e_{p_K} \bmod p_K\Bigr) \;\in\; \mathbb{Z}_q,$$

so that e_q is a primitive root of $\mathbb{Z}_q^\times$ minimizing $e_q - 1$.

(iii) Half-period $\pi_q \in \mathbb{Z}_q$: For each k, $\pi_{p_k} = \frac{p_k-1}{2}$, and

$$\pi_q = \frac{1}{2}\prod_{k=1}^{K} p_k(p_k-1) \;=\; \frac{\varphi(q)}{2} \;\in\; \mathbb{Z}_q,$$

and for any CRT-lifted generator $\tilde{g}$, $\tilde{g}^{\,\pi_q} \equiv -1 \pmod{q}$.

In each case, the element in $\mathbb{Z}_q$ is uniquely determined by its reductions modulo each prime factor p_k and is obtained via the Chinese Remainder Theorem. This completes the generalization of the three canonical constants $i,\ \pi,\ e$ from prime moduli to an arbitrary composite modulus $q = \prod_k p_k$.

3.5. Profinite Stability of the Canonical Constants

The construction of canonical constants $i_q,\ \pi_q,\ e_q$ offers a finite, relational foundation for the analogs of i, π, e in $\mathbb{Z}_q$. However, for this framework to represent a truly coherent and scalable system, these constants must exhibit a form of consistency as q evolves. Proving their profinite stability demonstrates precisely this: it shows that these definitions are not ad-hoc, but rather are specific realizations of underlying, universal p-adic entities that project consistently across the entire family of admissible moduli. This stability ensures that as we consider increasingly complex (larger q) arithmetic systems, the "finite versions" of i, π and e behave in a compatible and nested way, reflecting a deep structural integrity and pointing towards a unified description that transcends any single choice of modulus [5].

Write every modulus as an *unordered product of primes*

$$q = \prod_{k=1}^{K} p_k, \qquad p_k \equiv 1 \pmod{4}.$$

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The multiset $\{p_k\}$ allows repetitions, so if the same prime appears r times the factor p^r is present in q; we simply suppress the exponent in the notation.

For each admissible q the finite-relativistic framework assigns three canonical constants

$$i_q, \pi_q, e_q \in \mathbb{Z}_q^{\times},$$

obtained by Chinese remaindering the prime-power values $i_{p^r}, \pi_{p^r}, e_{p^r}$, where r is the multiplicity of p in q.

Lemma 3.1 (Hensel-compatible lifts). *Fix a prime $p \equiv 1 \pmod 4$ and choose $c \in \{i, \pi, e\}$. There exists a unique sequence $(c_{p^r})_{r\geq 1}$ satisfying*

1. $c_{p^1} = c_p$;
2. $c_{p^{r+1}} \equiv c_{p^r} \pmod{p^r}$ *for all* $r \geq 1$;
3. c_{p^r} *realises the same "nearest" minimization modulo* p^r *that defines* c_p *modulo* p *[5].*

Hence (c_{p^r}) is Cauchy in $\mathbb{F}_p$ and converges to a unit $C_p \in \mathbb{F}_p^{\times}$ [5].

Sketch. The defining polynomials x^2+1, $x^{p-1}+1$ and the cyclotomic polynomial for primitive roots have derivatives coprime to p at c_p; Hensel's lemma therefore yields a *unique* lift to $c_{p^{r+1}}$ that agrees with c_{p^r} modulo p^r and stays nearest to its reference point [5]. The congruence makes the sequence Cauchy. □

Theorem 3.2 (Projective stability of i_q, π_q, e_q). *For each $c \in \{i, \pi, e\}$ there exists a unit*

$$C = \prod_{p\equiv 1(4)} C_p \in \widehat{\mathbb{Z}}^{\times}, \qquad C_p = \lim_{r\to\infty} c_{p^r},$$

such that for every modulus $q = \prod_k p_k$ (with multiplicities understood) one has

$$\pi_q(C) = c_q \quad (\textit{equality in } \mathbb{Z}_{\mathbf{q}}).$$

Consequently, whenever $m \mid q$ the congruence $c_q \equiv c_m \pmod m$ holds. Thus the families $(i_q)_q$, $(\pi_q)_q$, $(e_q)_q$ stabilise projectively *across the entire directed system of moduli written in prime-product form.*

Proof. Fix c. Lemma 3.1 provides a compatible tower $(c_{p^r})_{r\geq 1}$ for each prime $p \equiv 1 \pmod 4$. Define $C := \prod_{p\equiv 1(4)} C_p \in \widehat{\mathbb{Z}}^{\times}$. For $q = \prod_k p_k$ let r_p denote the multiplicity of p in $\{p_k\}$. The projection $\mathcal{P}_q : \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}_q$ acts coordinate-wise, sending the p-adic component C_p to $c_{p^{r_p}}$ and ignoring all other primes. The Chinese Remainder Theorem re-assembles these images into c_q. If $m \mid q$ then $r_p(m) \leq r_p(q)$ for every prime, so the same argument gives $\pi_m(C) = c_m$ and hence $c_q \equiv c_m \pmod m$. □

4. Detailed Description of the Established Morphology

This section presents a comprehensive account of the geometric and topological structure arising from the three independent arithmetic motions on $\mathbb{Z}_q$ when

$$q = \prod_{k=1}^{K} p_k$$

is composite. We describe the total space as a Seifert-fibred 3-orbifold [8], explain how its 2-dimensional quotient recovers a pinched torus [9], and analyze the local fiber behavior along each zero-divisor seam.

4.1. Three Independent Circle Actions on $\mathbb{Z}_q$

On the set of residues $\mathbb{Z}_q$, one identifies three natural S^1-type symmetries:

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1. *Translation (add 1)*:
$$x \longmapsto x+1 \pmod{q}.$$
Denote this action by $T : x \mapsto x+1$. It provides one global "longitude" circle.
2. *Multiplication by a unit (stretch/shrink)*:
$$x \longmapsto u \cdot x, \quad u \in \mathbb{Z}_q^\times.$$
Because $\mathbb{Z}_q^\times$ is generally *not* cyclic when q is composite [6], varying u sweeps out a second circle.
3. *Exponentiation by a generator (twist)*:
$$x \longmapsto x^k, \quad k \in \langle g \rangle \subset \mathbb{Z}_q^\times$$
for a chosen "phase generator" g. When $\mathbb{Z}_q^\times$ is not cyclic, "multiply by u" and "raise to a power" become two *distinct* circle directions [7].

Because all three actions commute, their combined orbits form a 3-dimensional CW-complex. Topologically, the *full* space of orbits is a Seifert-fibred 3-orbifold
$$S_q = (\mathbb{Z}_q) \times_{(T,\times,\exp)} S^1,$$
endowed with a circle fiber over each base point in a 2-dimensional quotient.

4.2. The Seifert-Fibred 3-Orbifold Structure

Base 2-Orbifold.

When we *quotient out* the "twist" (exponentiation) circle, the remaining orbits of translation and multiplication form a 2-dimensional orbifold:
$$B_q = S_q / \langle \text{exponentiation} \rangle,$$
whose underlying topological surface is a *genus-1 torus* [4]. However, along each set of zero–divisor residues (those divisible by some prime $p_i \mid q$), the "multiply" and "twist" fibers collapse to a point, creating *cone points* (pinch loci) on the torus.

- **Generic points:** For any invertible residue $x \in (\mathbb{Z}_q)^\times$, none of the three actions collapses. The projection of its orbit (under translation $\times$ multiplication) down to B_q is a smooth point on the torus, carrying a full circle fiber upstairs.
- **Zero-divisor loops:** Fix a prime divisor $p_i \mid q$. The set
$$\mathbb{Z}_{p_k} = \{ x \in \mathbb{Z}_q : p_k \mid x \}$$
forms a closed loop (an embedded S^1) in the translation-multiplication quotient. Along Z_{p_k}, multiplication by p_k (hence by any power of p_k) is no longer invertible, so one of the two "latitude" circles collapses. Consequently, each point of Z_{p_k} becomes a cone point of order m_k in the base orbifold B_q [8].

Hence B_q is precisely a torus with r cone points, one of order m_k for each prime $p_i \mid q$. Equivalently,
$$B_q \cong T^2(p_1, p_2, \ldots, p_r)$$
as a 2-orbifold with cone points of multiplicities
$$m_k = \frac{q}{p_k}, \qquad 1 \le k \le r.$$

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Seifert Fibers and Exceptional Fibers.

Over each smooth point of B_q, the "twist" circle remains intact, forming a *regular Seifert fiber* in S_q. Over each cone point (zero–divisor loop) $Z_{p_i} \subset B_q$, the circle action collapses so that:

$$\text{fiber over } \mathbb{Z}_{p_i} \;\cong\; S^1/\big(\text{rotation by angle } 2\pi/m_i\big),$$

which is a singular (exceptional) fiber of type (m_i). Concretely, a small neighborhood of a cone point in B_q is modeled on $D^2/\big(e^{2\pi i/m_i}\big)$, and the corresponding upstairs neighborhood is

$$\big(D^2/(e^{2\pi i/m_i})\big) \times \big(S^1/\mathrm{Id}\big) \;\cong\; \frac{D^2 \times S^1}{(e^{2\pi i/m_i}, \mathrm{Id})},$$

which is the standard local model of a Seifert fibered neighborhood with exceptional fiber of order m_i [8].

5. Modal Decomposition

Lemma 5.1. *For any natural number q and the corresponding set of prime divisors of $\prod_n p_n = q-1$, p_n have multiplicative inverses in $\mathbb{Z}_q$ for all n.*

Proof. It is sufficient to show that $\gcd(q, p_n) = 1$ for all n, where $\gcd()$ stands for Greatest Common Divisor. p_n are primes so they don't have any divisors greater than 1. Suppose that p_n divides q, so that $q = m \cdot p_n$ for some integer m. Since p_n divides q and also divides $Q-1$ it has to divide the difference

$$(m \cdot p_n) - (m \cdot p_n - 1) = 1 \tag{5.1}$$

But a prime number p_n cannot divide 1 since primes are greater than 1. □

5.1. Mixed Radix Decomposition of $\mathbb{Z}_q$

Theorem 5.2 (Mixed-radix decomposition in $\mathbb{Z}_q$)**.** *Let an integer $q > 2$ be written as an* ordered *product of (not necessarily distinct) primes*

$$q \;=\; p_1\, p_2 \cdots p_r, \qquad p_i \geq 2.$$

Define mixed-radix weights

$$u_1 \;:=\; 1, \qquad u_i \;:=\; p_1 p_2 \cdots p_{i-1} \quad (2 \leq i \leq r), \qquad \Longrightarrow \qquad u_{i+1} = p_i\, u_i.$$

(a) *Set $\mathcal{K} := [0, p_1-1] \times \cdots \times [0, p_r-1]$. The map*

$$\Phi : \mathcal{K} \longrightarrow \mathbb{Z}_q, \qquad \Phi(k_1, \ldots, k_r) \;=\; \sum_{i=1}^{r} k_i\, u_i \pmod q$$

is a ***bijection****. Consequently, every residue $k \in \mathbb{Z}_q$ has a* unique ordered digit vector $(k_1, \ldots, k_r) = \Phi^{-1}(k)$ *with $0 \leq k_i < p_i$.*

(b) *Let $\mathcal{P}$ be any permutation of $\{1, \ldots, r\}$. If one simultaneously permutes the primes and the weights,*

$$p_i^{(\mathcal{P})} := p_{\mathcal{P}(i)}, \qquad u_i^{(\mathcal{P})} := \prod_{\mathcal{P}(j) < \mathcal{P}(i)} p_{\mathcal{P}(j)},$$

then

$$k \;=\; \sum_{i=1}^{r} k_i\, u_i \pmod q \iff k \;=\; \sum_{i=1}^{r} k_{\mathcal{P}(i)}\, u_i^{(\mathcal{P})} \pmod q.$$

Thus, re-labelling the independent subsystems merely permutes the ordered list $(k_1, \ldots, k_r)$; the unordered multiset $\{k_1, \ldots, k_r\}$ *is invariant.*

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Proof. *Injectivity.* Assume $\sum_i k_i u_i \equiv \sum_i k'_i u_i \pmod q$. Reducing modulo p_1 annihilates every term except $k_1 - k'_1$, hence $k_1 = k'_1$. Dividing the congruence by p_1 inside $\mathbb{Z}_q$ and repeating the argument inductively yields $k_i = k'_i$ for all i.

Surjectivity. Given $k \in [0, q-1]$, perform successive divisions:

$$k_1 := k \bmod p_1,\ k^{(1)} := \lfloor k/p_1 \rfloor; \quad k_2 := k^{(1)} \bmod p_2,\ k^{(2)} := \lfloor k^{(1)}/p_2 \rfloor; \ \ldots$$

After r steps $k^{(r)} = 0$ and $k = \sum_{i=1}^{r} k_i u_i$, so k lies in the image of Φ. Parts (a) and (b) follow immediately, completing the proof. □

The resultant ordered digit vector $(k_1, \ldots, k_r)$ provides a *mixed radix decomposition* of elements $k \in \mathbb{Z}_q$ as follows:

$$k = \sum_{i=1}^{r} k_i u_i \pmod q,$$

5.2. Logarithmic Mixed Radix Decomposition

We already established that for any non-zero element $k \in \mathbb{Z}_q$, it can be represented as power of a ring generator $k = e^m$, where $m \in \{0, \ldots, q-2\}$, and we denote the ring generator as e for brevity. Applying the Theorem 5.2 to the element m of the reduced ring $\mathbb{Z}_q^\times$ we obtain a unique ordered digit vector representation of m in terms of the prime factors of $q-1$

$$m = \sum_{n=1}^{s} m_n v_n \pmod{q-1}$$

and can now decompose k into a so-called *logarithmic mixed-radix decomposition* as follows:

$$k = e^m = \exp\left(\sum_{n=1}^{s} m_n v_n \pmod{q-1}\right) = \prod_{n=1}^{s} e^{m_n v_n}.$$

5.3. Primary Component Decomposition of $\mathbb{Z}_q^\times$

Theorem 5.3 (Fibre decomposition of the unit group $\mathbb{Z}_q^\times$)**.** *Let $q > 1$ have prime-power factorization*

$$q = \prod_{i=1}^{s} p_i^{e_i}, \qquad p_i \text{ prime},\ e_i \geq 1.$$

For each i put

$$M_i := \frac{q}{p_i^{e_i}}, \qquad c_i \equiv M_i^{-1} \pmod{p_i^{e_i}}, \qquad e_i^* := c_i M_i \pmod q,$$

so that $e_i^* \equiv 1 \pmod{p_i^{e_i}}$, $e_i^* \equiv 0 \pmod{p_j^{e_j}}$ $(j \neq i)$. Choose a generator g_i of the cyclic unit group $(\mathbf{Z}_{p_i^{e_i}})^\times$ $(:= \langle g_i \rangle)$; for $p_i = 2$ and $e_i \geq 3$ one may take $g_i = 5$. Define the fibre generators

$$G_i := 1 - e_i^* + g_i e_i^* \ \in \mathbb{Z}_q, \qquad i = 1, \ldots, s.$$

(i) *$G_i \equiv g_i \pmod{p_i^{e_i}}$ and $G_i \equiv 1 \pmod{p_j^{e_j}}$ for $j \neq i$; hence the G_i commute and each has multiplicative order $\varphi(p_i^{e_i}) = p_i^{e_i - 1}(p_i - 1)$.*

(ii) *Every unit $u \in (\mathbb{Z}_q)^\times$ can be written*

$$u = \prod_{i=1}^{s} G_i^{\ell_i}, \qquad 0 \leq \ell_i < \varphi(p_i^{e_i}),$$

and the exponent tuple $(\ell_1, \ldots, \ell_s)$ is unique.

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(iii) *The map*

$$(\ell_1, \dots, \ell_s) \longmapsto \prod_{i=1}^{s} G_i^{\ell_i} \pmod q$$

is a group isomorphism

$$\prod_{i=1}^{s} \mathbb{Z}_{\varphi(p_i^{e_i})} \cong \mathbb{Z}_q^{\times}.$$

Sketch. (i) By construction e_i^* are orthogonal idempotents ($e_i^* e_j^* = 0$ for $i \neq j$ and $\sum_i e_i^* = 1$), so G_i reduces to g_i modulo its own prime power and to 1 modulo all others; hence G_i has the claimed orders and commutes with G_j ($i \neq j$).

(ii) Given $u \in \mathbb{Z}_q^{\times}$, let $\ell_i := \log_{g_i}(u \bmod p_i^{e_i})$ in $\mathbf{Z}_{\varphi(p_i^{e_i})}$. Then $u \equiv G_i^{\ell_i} \pmod{p_i^{e_i}}$ and $u \equiv 1 \pmod{p_j^{e_j}}$ for $j \neq i$. Multiplying the $G_i^{\ell_i}$ reproduces u modulo every prime power, hence modulo q. Uniqueness of the tuple follows by comparing residues mod p_i.

(iii) Parts (i)-(ii) show the map is bijective and respects multiplication.

□

Consequently every non-zero-divisor of $\mathbb{Z}_q$ (unit) factors *uniquely* as a product of the fibre generators $\{G_i\}$; the exponents ℓ_i serve as "multiplicative digits" in the *fibre decomposition* of $\mathbb{Z}_q^{\times}$. Zero-divisors cannot be so decomposed, as at least one of their coordinates modulo p_i vanishes.

6. Conclusions

The present manuscript further develops our *finite-relativistic programme* by extending every construction hitherto restricted to a single prime modulus to *all* composite cardinalities

$$q = \prod_{k=1}^{K} p_k, \qquad p_k \equiv 1 \pmod 4.$$

Our results demonstrate that the algebraic, analytic and geometric machinery developed in the prime setting survives—and in fact acquires new richness—once Chinese–remainder structure and mixed radices are admitted.

1. **Universality of the canonical constants.** The quarter-turn i_q, half-period π_q and minimum-action base e_q are obtained by a prime-wise Hensel lift followed by a CRT amalgamation. They form a projectively compatible family, so each is the reduction of a single profinite unit $i, \pi, e \in \mathbb{Z}_q$, ensuring consistency across all moduli. This settles the logical question of whether "finite versions" of i, π, e can coexist in a single framework—*they can and do.*
2. **From bouquets to Seifert 3-orbifolds.** The discrete 2-spheroid N_q factors as a bouquet of its prime-field companions; equipping it with the commuting circle actions of translation, multiplication and exponentiation lifts the bouquet to a Seifert-fibred 3-orbifold S_q. Each prime divisor of q produces an exceptional fibre whose multiplicity records its power in q, furnishing a direct correspondence between arithmetic data and low-dimensional topology.
3. **Digitised arithmetic, harmonic and modal analysis.** A mixed-radix decomposition yields unique additive digit vectors, while multiplicative and "twist" fibres give logarithmic and primary-component coordinate systems on $\mathbb{Z}_q^{\times}$. These digit spaces form natural arenas for finite Fourier analysis, providing basis functions that already respect the Seifert fibration and the zero-divisor seams.
4. **Stability and profinite coherence.** Not only do the constants i_q, π_q, e_q stabilise; whole *classes* of mixed-radix vector units are shown to be prefix-stable. Fixed "low" digits therefore behave as conserved infrared data when the modulus is enlarged, a property expected to be crucial for multiscale simulations of finite-precision physics.:contentReference[oaicite:7]index=7

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Outlook: With the composite-cardinality foundations now in place, the programme is poised to move from construction to application. The profinite stability of the canonical constants, the explicit Seifert-fibred geometry of S_q, and the mixed-radix harmonic toolkit together constitute a self-contained calculus on finite, relational spacetime. We are therefore ready to *deploy this framework to concrete physical settings*—from finite-precision lattice gauge theories and discrete quantum systems to number-theoretic models of cosmology—and to test how faithfully it can capture, unify and predict observable phenomena within a single, fully finite universe.

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