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Article

Depicting Falsifiability in Algebraic Modelling

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Abstract: One key distinction between science and pseudoscience lies in whether a theory can, at least in principle, be falsified by empirical measurements. In physics, in particular, the prevailing view is that reality is defined by what can be measured. All minimal theories that are consistent with observed data are regarded as equally valid. However, the possibility that two objects could differ despite sharing all currently measurable properties is often overlooked. As a result, many theories may be considered non-minimal when viewed from a micrological perspective. In this initial attempt, we introduce an algebraic framework to represent minimalist modeling assumptions. From a physical perspective, we explore and analyze weakened forms of the associative and commutative laws. From a mathematical standpoint, we demonstrate that assuming the existence of a weak neutral element leads to the emergence of several transversal algebraic laws. Each of these laws is individually weaker than the combination of associativity and commutativity, but any two of them together are equivalent this combination.

Keywords: magma, hemi-associativity, hemi-commutativity, falsifiability, measurement

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1. Introduction

Let x and y be two objects that can be combined via a dyadic operation, denoted by $+$. There are two ways to apply this operation to x and y:

$$x + y, \quad \text{and} \quad y + x.$$

The objects x and y are said to commute if $x + y = y + x$.

We now shift from a mathematical to a physical perspective. From this standpoint, the expressions $x + y$ and $y + x$ are accessible only through measurements. Let M denote such a (possibly multivariate) measurement. If x and y commute, then we have:

$$M(x + y) = M(y + x). \tag{1}$$

The converse holds only if the measurement function M is sufficiently expressive.

Notable contributions in the philosophy of science—such as Quine's notion of underdetermination [1] and the Copenhagen interpretation of quantum mechanics—suggest that measurement functions are generally not expressive enough to establish the identity. So, even if the best possible M is chosen, algebra may add auxiliary hypotheses [2], namely that $x + y$ is identical to $y + x$, although we only observe commutativity by all of the measurements. In such cases, the algebraic representation becomes unfalsifiable and, according to Popper [3], should be replaced by a testable formulation such as Equation (1).

A physical law is considered sound if its validity extends across a range of environments. We propose translating this notion into algebra by defining a weaker form of commutativity. Specifically, the objects x and y are called *hemi-commuting* if

$$M(a + (x + y) + b) = M(a + (y + x) + b)$$

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for all relevant objects a and b. Clearly, commuting objects are also hemi-commuting.

Our approach relies on an additional assumption: the existence of a weak form of a neutral element. Since the theoretical presence or absence of such an element is rarely critical in physics, this assumption is considered reasonable.

Beyond introducing this new framework in Section 2, we demonstrate that under these weakened assumptions, associativity and commutativity can be expressed in over twenty equivalent formulations (see Theorem 1 and Proposition 4). Furthermore, we shows that certain algebraic identities hold under weaker assumptions than hitherto used. For instance, Proposition 2 and Corollary 1 together show that in case of hemi-associativity all parentheses can be removed.

Proofs and additional results are deferred to Section 3. Section 4 connects our findings to relevant philosophical, physical, and algebraic literature, including a brief presentation of the origins of the technical terms used. All definitions employed herein are internally consistent.

2. Main Result

In the sequel we always understand $x + y + z$ as $(x + y) + z$. We use the prefix "hemi" to indicate that we use a kind of weak form of a certain property. We need several definitions to formulate our main result (Theorem 1).

Definition 1. *Let G and S be non-empty sets. Let $+$ be a dyadic operation on G and $M : G \to S$ a map. Denote $M(x) = M(y)$ by $x \boxminus y$ and let*

$$G_s = \{\varepsilon \in G : x + \varepsilon \boxminus x \quad \forall x \in G\}. \tag{2}$$

If G_s is not empty and closed, i.e., $\varepsilon, \tilde{\varepsilon} \in G_s$ implies $\varepsilon + \tilde{\varepsilon} \in G_s$, then the tuple $(G, +, M)$ is called a hemi-unital magma.

Definition 2. *Let $(G, +, M)$ be a hemi-unital magma. Assume further that*

$$\varepsilon + x \quad \boxminus \quad x, \qquad \forall x \in G, \varepsilon \in G_s. \tag{3}$$

Then, the operation $+$ is called hemi-associative, *if*

$$a + ((x + y) + z) + b \quad \boxminus \quad a + (x + (y + z)) + b, \qquad \forall a, b, x, y, z \in G, \tag{4}$$

hemi-commutative, *if*

$$a + (y + x) + b \quad \boxminus \quad a + (x + y) + b \qquad \forall a, b, x, y \in G, \tag{5}$$

hemi-right-cyclic *if*

$$a + ((x + y) + z) + b \quad \boxminus \quad a + ((y + z) + x) + b, \quad \forall a, b, x, y, z \in G, \tag{6}$$

hemi-left-cyclic *if*

$$a + (x + (y + z)) + b \quad \boxminus \quad a + (y + (z + x)) + b, \quad \forall a, b, x, y, z \in G,$$

hemi-right-modular, *if*

$$a + ((x + y) + z) + b \quad \boxminus \quad a + ((z + y) + x) + b \quad \forall a, b, x, y, z \in G,$$

and hemi-left-modular, *if*

$$a + (x + (y + z)) + b \quad \boxminus \quad a + (z + (y + x)) + b \quad \forall a, b, x, y, z \in G.$$

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It can easily be checked that genuine associativity implies Equation (4) and genuine commutativity implies Equation (5).

Definition 3. *Let $(G, +, M)$ be a hemi-unital magma. The dyadic operation $+$ is called* wide-left-modular*, if, for all $a, b, c, x, y, z \in G$, we have*

$$\begin{aligned} x + y + z + b + c &\boxminus y + (x + z + b + c), \\ a + (x + y) + z + b &\boxminus a + x + (z + y) + b. \end{aligned} \tag{7}$$

Example 1. *Let G be the set of finite sequences of the symbols $\square$ and $\Diamond$ including the empty set. For elements $g = g_1 g_2 \dots g_n, h = h_1 h_2 \dots h_m \in G$ we define*

$$\begin{aligned} M(g) &:= |\{i \in \{1, \dots, n\} \mid g_i = \square\}|, \\ g \circ h &:= g_1 \dots g_n h_1 \dots h_m, \\ \varphi(g, h) &:= \begin{cases} g_1 h_1 g_2 h_2 \dots g_m h_m g_{m+1} \dots g_n, & n > m \\ g_1 h_1 g_2 h_2 \dots g_n h_n h_{n+1} \dots h_m, & n \leqslant m \end{cases}. \end{aligned}$$

We define

$$\begin{aligned} \psi(g) &:= \bigcirc_{i=1}^{n/2} \begin{cases} \Diamond, & g_{2i-1} g_{2i} = \Diamond\Diamond \\ g_{2i-1} g_{2i}, & \text{else} \end{cases}, \quad n \text{ even}, \\ \psi(g) &:= \psi(g_1 \dots g_{n-1}) \circ g_n, \quad n \text{ odd}, \\ \psi(\varnothing) &:= \varnothing. \end{aligned}$$

Finally, we establish $g + h := \psi(\varphi(g, h))$. For instance,

$$\square\Diamond\Diamond + \Diamond\square\Diamond\square = \psi(\square\Diamond\Diamond\square\Diamond\Diamond\square) = \square\Diamond\Diamond\square\Diamond\square.$$

One can see that $G_s = \{\Diamond^i \mid i \in \mathbb{N}_0\}$. Further, $+$ is not commutative nor associative but it is hemi-commutative and hemi-associative as well as wide-left-modular. Notice that we have infinite elements behaving like a neutral element with respect to M while only one is a genuine neutral element.

The following proposition shows the strong interlacing of the rather different concepts above:

Theorem 1. *Let $(G, +, M)$ be a hemi-unital magma such that Equation (3) holds true. With the exception of the two pairs "hemi-left-modular and wide-left-modular" and "hemi-left-modular and hemi-right-modular", any two pairs of the following seven properties*

- *$+$ is hemi-commutative;*
- *$+$ is hemi-associative;*
- *$+$ is hemi-right-cyclic;*
- *$+$ is hemi-left-cyclic;*
- *$+$ is hemi-right-modular;*
- *$+$ is hemi-left-modular;*
- *$+$ is wide-left-modular*

are equivalent.

Remark 1. *Considering hemi-associativity hemi-commutativity as standards, Theorem 1 also states that these two properties together can be weakened by one of the above properties, but not by two of them, since this would fall back to the standard. Since the set S and the map M are not specified, we may choose $S = G$ and M the identity, so that Theorem 1 and its proof below imply general assertions on standard algebra.*

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3. Proof and further results

We first show some properties that facilitate the later proofs. Some of the Lemmata have their value on their own. We slowly accumulate all the necessary implications for Theorem 1 over the course of this chapter. We use the symbol ε for a fixed element in G_s, whereas 0 stands symbolically for some element in G_s, so that the meaning of 0 may change even within the same mathematical term.

Proposition 1. *Let $(G, +, M)$ be a wide-left-modular, hemi-unital magma. Then, for all $a, b, x, y, z \in G$ and $\varepsilon, \tilde{\varepsilon} \in G_s$,*

$$(x+y)+z+b \;\; \boxminus \;\; y+(x+z)+(\varepsilon+b), \tag{8}$$
$$x+y \;\; \boxminus \;\; y+x, \tag{9}$$
$$\varepsilon+x \;\; \boxminus \;\; x, \tag{10}$$
$$a+(\varepsilon+(\tilde{\varepsilon}+x))+b \;\; \boxminus \;\; a+x+b.$$

Furthermore, the operation + is hemi-left-modular.

Proof. Equations (8) follows from

$$\begin{aligned}(x+y)+z+b &\boxminus (x+y)+z+b+0\\ &\boxminus y+(x+z+b+0)\\ &\boxminus y+((x+z)+b)+(0+0)\\ &\boxminus y+(x+z)+(0+b).\end{aligned}$$

Equations (2), (8) and (7) yield

$$\begin{aligned}x+y &\boxminus (x+y)+0+0 \boxminus y+(x+0)+0+0\\ &\boxminus y+x+(0+0)+0 \boxminus y+x.\end{aligned}$$

Equation (10) follows from (2) with (9). By Equations (2), (8), (7), (10) and (9), we attain

$$\begin{aligned}a+(\varepsilon+(\tilde{\varepsilon}+x))+b &\boxminus (a+(\varepsilon+(\tilde{\varepsilon}+x)))+b+0\\ &\boxminus \varepsilon+(\tilde{\varepsilon}+x)+(a+b)+0\\ &\boxminus (\varepsilon+\tilde{\varepsilon})+((a+b)+x)+0\\ &\boxminus (a+b)+x+0\\ &\boxminus b+(a+x)\\ &\boxminus a+x+b.\end{aligned}$$

Finally, Equations (8) and (7) yield

$$\begin{aligned}a+(x+(y+z))+b &\boxminus (x+(y+z))+(a+b)\\ &\boxminus (y+x)+z+(0+(a+b))\\ &\boxminus (y+x)+(z+(a+b))\\ &\boxminus (z+(y+x))+(a+b)\\ &\boxminus a+(z+(y+x))+b.\end{aligned}$$

□

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Proposition 2. *Let $(G, +, M)$ be a hemi-unital magma. The operation $+$ shall fulfill Equation (3) and have the property that*

$$a + (x + y) + z + b \mathrel{\vdash\!\!\dashv} a + x + (y + z) + b, \qquad \forall a, b, x, y, z \in G. \tag{11}$$

Then, all parentheses can be removed in an algebraic expression. In particular,

$$x_0 + (x_1 + \ldots + x_i) + y_1 + \ldots + y_j \mathrel{\vdash\!\!\dashv} x_0 + x_1 + \ldots + x_i + y_1 + \ldots + y_j \tag{12}$$

for all $i, j \in \mathbb{N}_0, x_0, \ldots, x_i, y_1, \ldots, y_j \in G$.

Proof. The proof is separated into two steps, in the first one we see that for all $x \in G$

$$x + y_1 + \ldots + y_j \mathrel{\vdash\!\!\dashv} x + (y_1 + (\ldots + (y_{j-2} + (y_{j-1} + y_j)) \ldots)). \tag{13}$$

In the second step, we show that for all $y \in G$

$$x_0 + (x_1 + \ldots + x_i) + y \mathrel{\vdash\!\!\dashv} x_0 + x_1 + \ldots + x_i + y.$$

Equation (12) follows after applying step 1, step 2 and then step 1 again. If $j = 0$, then we use Equation (2). Both steps are shown by means of induction. For the first step, the cases $j = 0, 1$ are trivial. By induction, assume (13) holds for some $j \in \mathbb{N}$.

$$\begin{aligned}
x + y_1 + \ldots + y_{j+1} &\mathrel{\vdash\!\!\dashv} 0 + ((x + y_1 + \ldots + y_j) + y_{j+1}) + 0 + 0 \\
&\mathrel{\vdash\!\!\dashv} (0 + (x + y_1 + \ldots + y_{j-1} + y_j)) + (y_{j+1} + 0) + 0 + 0 \\
&\mathrel{\vdash\!\!\dashv} 0 + ((x + y_1 + \ldots + y_{j-1}) + y_j) + y_{j+1} + (0 + 0) \\
&\mathrel{\vdash\!\!\dashv} 0 + (x + y_1 + \ldots + y_{j-1}) + (y_j + y_{j+1}) + (0 + 0) \\
&\mathrel{\vdash\!\!\dashv} (0 + (x + y_1 + \ldots + y_{j-1})) + (y_j + y_{j+1}) + (0 + 0) + 0 \\
&\mathrel{\vdash\!\!\dashv} 0 + (x + y_1 + \ldots + y_{j-1}) + ((y_j + y_{j+1}) + 0) + 0 \\
&\mathrel{\vdash\!\!\dashv} 0 + ((x + y_1 + \ldots + y_{j-1}) + (y_j + y_{j+1})) + 0 + 0 \\
&\mathrel{\vdash\!\!\dashv} x + y_1 + \ldots + y_{j-1} + (y_j + y_{j+1})
\end{aligned}$$

This shows the first step. In the second step, the cases $i = 0, 1$ are trivial. By induction, assume that (12) holds for some $i \in \mathbb{N}$. We use the induction hypothesis in the second step.

$$\begin{aligned}
x_0 + (x_1 + \ldots + x_{i+1}) + y &\mathrel{\vdash\!\!\dashv} x_0 + (x_1 + \ldots + x_i) + (x_{i+1} + y) + 0 \\
&\mathrel{\vdash\!\!\dashv} x_0 + x_1 + \ldots + x_i + (x_{i+1} + y) + 0 + 0 \\
&\mathrel{\vdash\!\!\dashv} (x_0 + x_1 + \ldots + x_i) + x_{i+1} + (y + 0) + 0 \\
&\mathrel{\vdash\!\!\dashv} (x_0 + x_1 + \ldots + x_i + x_{i+1}) + y + (0 + 0) \\
&\mathrel{\vdash\!\!\dashv} x_0 + x_1 + \ldots + x_i + x_{i+1} + y.
\end{aligned}$$

□

Corollary 1. *Let $(G, +, M)$ be a hemi-unital magma, such that Equation (3) holds. Then, Equations (11) and (4) are equivalent.*

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Proof. We only have to show that Equation (4) implies (11):

$$\begin{aligned} a+(x+y)+z+b &\boxminus 0+(((a+(x+y))+z)+b) \\ &\boxminus 0+(a+((x+y)+(z+b))) \\ &\boxminus a+(x+(y+(z+b))) \\ &\boxminus 0+((a+x)+(y+(z+b)))) \\ &\boxminus (a+x)+((y+z)+b) \\ &\boxminus 0+(((a+x)+(y+z))+b) \\ &\boxminus a+x+(y+z)+b. \end{aligned}$$

□

In the following, whenever $+$ is hemi-associative, we are going to use Proposition 2 together with Corollary 1.

Corollary 2. *Let $(G,+,M)$ be a hemi-commutative, hemi-associative, hemi-unital magma. Then, for all $a,b,x,y \in G$, $\varepsilon \in G_S$,*

$$\begin{aligned} a+x+y+b &\boxminus a+y+x+b \\ a+x+b &\boxminus a+(\varepsilon+x)+b. \qquad (14) \\ a+(x+y)+b &\boxminus a+y+(b+x). \qquad (15) \end{aligned}$$

Furthermore, the operation $+$ is hemi-right-cyclic, hemi-left-cyclic, hemi-right-modular, hemi-left-modular and wide-left-modular.

Proof. Proposition 2 and Equation (5) yield

$$a+x+y+b \boxminus a+(x+y)+b \boxminus a+(y+x)+b \boxminus a+y+x+b.$$

Equations (2) and (5) and Proposition 2 yield

$$a+x+b \boxminus a+x+(\varepsilon+b)+0 \boxminus a+(\varepsilon+x)+b.$$

Further, we have

$$\begin{aligned} a+(x+y)+b &\boxminus a+(y+x)+b \\ &\boxminus (a+y)+(x+b)+0 \\ &\boxminus a+y+(b+x). \end{aligned}$$

Hemi-right-cyclicity and hemi-left-cyclicity are trivial. Hemi-right-modularity follows from

$$\begin{aligned} a+((x+y)+z)+b &\boxminus a+x+(y+z)+b \\ &\boxminus a+(x+(z+y))+b \\ &\boxminus a+((z+y)+x)+b. \end{aligned}$$

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Hemi-left-modularity follows from Proposition 2:

$$\begin{aligned} a+(x+(y+z))+b &\mathrel{\vDash\!\!\Dashv} a+((x+y)+z)+b \\ &\mathrel{\vDash\!\!\Dashv} a+(z+(x+y))+b \\ &\mathrel{\vDash\!\!\Dashv} (a+z)+(x+y)+b \\ &\mathrel{\vDash\!\!\Dashv} (a+z)+(y+x)+b \\ &\mathrel{\vDash\!\!\Dashv} a+(z+(y+x))+b. \end{aligned}$$

The operation $+$ is wide-left-modular, since, by Proposition 2 and Equations (2) and (5) we have

$$(x+y)+z+b+c \mathrel{\vDash\!\!\Dashv} 0+(y+x)+(z+b+c) \mathrel{\vDash\!\!\Dashv} y+(x+z+b+c).$$

Proposition 2 and Equation (5) yield Equation (7),

$$a+(x+y)+z+b \mathrel{\vDash\!\!\Dashv} (a+x)+(y+z)+b \mathrel{\vDash\!\!\Dashv} a+x+(z+y)+b.$$

□

Proposition 3. *Let $(G,+,M)$ be a hemi-unital magma. Then, any two properties of*

- *$+$ is hemi-associative*
- *$+$ is hemi-commutative*
- *$+$ is hemi-left-modular*

are equivalent.

Proof. Hemi-left-modularity and hemi-associativity imply hemi-commutativity, since, by Proposition 2,

$$\begin{aligned} a+(x+y)+b &\mathrel{\vDash\!\!\Dashv} (a+(x+y))+(b+(0+0))+0 \\ &\mathrel{\vDash\!\!\Dashv} (a+(x+y))+(0+(0+b))+0 \\ &\mathrel{\vDash\!\!\Dashv} a+(x+(y+(0+0)))+b \\ &\mathrel{\vDash\!\!\Dashv} a+((0+0)+(y+x)+b \\ &\mathrel{\vDash\!\!\Dashv} a+(0+(0+(y+x))+b \\ &\mathrel{\vDash\!\!\Dashv} a+(y+x)+0+0+b \\ &\mathrel{\vDash\!\!\Dashv} a+(y+x)+b. \end{aligned}$$

Hemi-left-modularity and hemi-commutativity imply hemi-associativity, as

$$\begin{aligned} a+((x+y)+z)+b &\mathrel{\vDash\!\!\Dashv} a+(z+(x+y))+b \\ &\mathrel{\vDash\!\!\Dashv} 0+(a+(z+(x+y))+b)+0 \\ &\mathrel{\vDash\!\!\Dashv} 0+(b+(a+(z+(x+y))))+0 \\ &\mathrel{\vDash\!\!\Dashv} 0+((z+(x+y))+(a+b))+0 \\ &\mathrel{\vDash\!\!\Dashv} z+(x+y)+(a+b) \\ &\mathrel{\vDash\!\!\Dashv} z+(y+x)+(a+b) \end{aligned}$$

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By permuting x and y this way, we obtain

$$\begin{aligned} a+((x+y)+z)+b &\vDash a+(z+(x+y))+b \\ &\vDash a+(z+(y+x))+b \\ &\vDash a+(x+(y+z))+b. \end{aligned}$$

The remaining implication follows from Corollary 2. □

Proposition 4. *Let $(G,+,M)$ be a hemi-unital magma. Any two of the following four properties imply the other two:*

- $+$ *is hemi-associative;*
- $+$ *is hemi-commutative;*
- $+$ *is wide-left-modular;*
- *Equation (15) holds true.*

Proof. The equivalence of all the pairs among the first three properties follow from Corollary 2, Proposition 3 and Proposition 1. The combinations with the last one remain to show. If $+$ is wide-left-modular, we have, by Equation (15) and (7),

$$a+(x+y)+b \vDash a+y+(b+x) \vDash a+(y+x)+b$$

and $+$ is hemi-commutative. Now, assume $+$ is hemi-associative. This results in

$$\begin{aligned} a+(x+y)+b &\vDash a+y+(b+x) \\ &\vDash (a+y)+(b+x)+0 \\ &\vDash (a+y)+x+(0+b)+0 \\ &\vDash (a+y)+x+b+(0+0) \\ &\vDash a+(y+x)+b. \end{aligned}$$

Lastly, assume $+$ is hemi-commutative. Then,

$$\begin{aligned} a+((x+y)+z)+b &\vDash 0+((a+((x+y)+z))+b)+0 \\ &\vDash 0+b+(0+(a+((x+y)+z))) \\ &\vDash 0+b+(a+((x+y)+z))+(0+0) \\ &\vDash 0+b+(((x+y)+z)+a)+0 \\ &\vDash 0+(a+b)+((x+y)+z) \qquad (16) \\ &\vDash 0+(a+b)+z+(0+(x+y))+0 \\ &\vDash 0+(a+b)+z+(x+y)+(0+0) \\ &\vDash 0+(a+b)+(y+z)+(0+x) \\ &\vDash 0+(a+b)+(x+(y+z)). \end{aligned}$$

We successfully reordered x, y and z into the desired form. Going backwards with the last equation from step (16) yields the desired result. □

Proposition 5. *Let $(G,+,M)$ be a hemi-unital magma. Any two of the three properties*

- $+$ *is hemi-commutative;*
- $+$ *is hemi-associative;*
- $+$ *is hemi-right-cyclic*

are equivalent.

Proof. We have to show that (4) and (6) imply (5). We use Proposition 2 to see

$$
\begin{aligned}
a + (x + y) + b &\boxminus a + (x + y) + b + 0 + 0 + 0 \\
&\boxminus (a + x + y) + ((b + 0) + 0) + 0 \\
&\boxminus (a + x + y) + ((0 + 0) + b) + 0 \\
&\boxminus a + ((x + y) + (0 + 0)) + b \\
&\boxminus a + ((y + (0 + 0)) + x) + b \\
&\boxminus (a + y) + ((0 + 0) + x) + b \\
&\boxminus (a + y) + ((x + 0) + 0) + b \\
&\boxminus a + (y + x) + b.
\end{aligned}
$$

The remaining implication is trivial. □

Proposition 6. *Let $(G, +, M)$ be a hemi-right-modular, hemi-untial magma. Any of properties*

- *$+$ is hemi-associative;*
- *$+$ is hemi-commutative;*
- *$+$ is hemi-right-cyclic;*

are equivalent. Furthermore, any of these three properties imply that $+$ is hemi-left-modular.

Proof. Let $+$ be hemi-associative. We show that $+$ is then hemi-commutative. Corollary 2 then implies hemi-right-cyclicity and hemi-left-modularity. By Proposition 2, we have

$$
\begin{aligned}
a + (x + y) + b &\boxminus (a + (x + y)) + ((b + 0) + 0) + 0 \\
&\boxminus (a + (x + y)) + 0 + 0 + b \\
&\boxminus a + ((x + y) + (0 + 0)) + b \\
&\boxminus a + (((0 + 0) + y) + x) + b \\
&\boxminus a + ((0 + 0) + y) + (x + b) \\
&\boxminus (a + y) + ((0 + 0) + x) + b \\
&\boxminus (a + y + x) + ((0 + 0) + b) + 0 \\
&\boxminus a + (y + x) + b.
\end{aligned}
$$

Let $+$ be hemi-commutative. We have

$$
\begin{aligned}
a + ((x + y) + z) + b &\boxminus a + ((z + y) + x) + b \\
&\boxminus 0 + ((a + ((z + y) + x) + b) + 0) + 0 \\
&\boxminus (0 + b) + (a + ((z + y) + x)) \\
&\boxminus (0 + b) + (((z + y) + x) + a) \\
&\boxminus (0 + b) + ((a + x) + (z + y)) \\
&\boxminus 0 + ((0 + b) + ((a + x) + (z + y))) + 0 \\
&\boxminus (a + x) + (z + y) + (0 + b) \\
&\boxminus (a + x) + (y + z) + (0 + b).
\end{aligned}
$$

From here, we walk the equations backwards and obtain

$$
a + ((x + y) + z) + b \boxminus a + ((y + z) + x) \boxminus a + (x + (y + z)) + b.
$$

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Assume $+$ is hemi-right-cyclic. It is enough to verify that $+$ is being hemi-commutative.

$$\begin{aligned}
a+(x+y)+b &\mathrel{|{=}|} 0+((a+(x+y))+b)+0 \\
&\mathrel{|{=}|} 0+((b+a)+(x+y))+0 \\
&\mathrel{|{=}|} ((x+y)+a)+b+0 \\
&\mathrel{|{=}|} (0+b)+((x+y)+a) \\
&\mathrel{|{=}|} (0+b)+((a+y)+x) \\
&\mathrel{|{=}|} (0+b)+((y+x)+a).
\end{aligned}$$

We were able to swap x and y and, hence, have hemi-commutativity. □

Proposition 7. *Let $(G,+,M)$ be a hemi-left-cyclic, hemi-untial magma. Any of*

- $+$ *is hemi-associative*
- $+$ *is hemi-commutative*
- $+$ *is hemi-right-cyclic*
- $+$ *is hemi-right-modular*
- $+$ *is hemi-left-modular*

are equivalent.

Proof. It can easily been seen, that if $+$ is hemi-commutative, $+$ is also hemi-associative. Therefore, in all other cases, by Corollary 2, it is enough to see that $+$ is hemi-commutative. Assume $+$ is hemi-associative. We use Proposition 2.

$$\begin{aligned}
a+(x+y)+b &\mathrel{|{=}|} (a+(x+y))+(b+(0+0))+0 \\
&\mathrel{|{=}|} a+(x+(y+0))+0+b \\
&\mathrel{|{=}|} a+y+(0+(x+0))+b \\
&\mathrel{|{=}|} a+(y+x)+(0+(0+b))+0 \\
&\mathrel{|{=}|} a+(y+x)+b.
\end{aligned}$$

For the next step, assume $+$ is hemi-right-cyclic.

$$\begin{aligned}
a+(x+y)+b &\mathrel{|{=}|} 0+((a+(x+y))+b)+0 \\
&\mathrel{|{=}|} (b+a)+(x+y)+0 \\
&\mathrel{|{=}|} ((x+y)+0)+(b+a)+0 \\
&\mathrel{|{=}|} 0+((x+y)+0)+(b+a) \\
&\mathrel{|{=}|} (y+0)+x+(b+a) \\
&\mathrel{|{=}|} (x+(b+a))+(y+0) \\
&\mathrel{|{=}|} ((y+0)+0)+(x+(b+a))+0 \\
&\mathrel{|{=}|} 0+((y+0)+0)+(x+(b+a)) \\
&\mathrel{|{=}|} 0+((0+0)+y)+(x+(b+a)) \\
&\mathrel{|{=}|} y+(x+(b+a)).
\end{aligned}$$

In the next step is the only use of hemi-left-cyclicity at all in these equations.

$$\begin{aligned}
&\mathrel{|{=}|} (b+a)+(y+x) \\
&\mathrel{|{=}|} a+(y+x)+b.
\end{aligned}$$

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Let $+$ be hemi-right-modular. We get

$$\begin{aligned}
a+(x+y)+b &\mathrel{|\!\!=\!\!|} 0+(0+(a+(x+y)+b))+0 \\
&\mathrel{|\!\!=\!\!|} 0+((a+(x+y))+(b+0))+0 \\
&\mathrel{|\!\!=\!\!|} ((b+0)+(x+y))+a+0 \\
&\mathrel{|\!\!=\!\!|} (0+a)+((b+0)+(x+y)) \\
&\mathrel{|\!\!=\!\!|} (0+a)+(x+(y+(b+0))) \\
&\mathrel{|\!\!=\!\!|} x+(y+(b+0))+a \\
&\mathrel{|\!\!=\!\!|} x+(0+(y+b))+a \\
&\mathrel{|\!\!=\!\!|} (0+a)+(x+(0+(y+b))) \\
&\mathrel{|\!\!=\!\!|} (0+a)+(0+((y+b)+x)) \\
&\mathrel{|\!\!=\!\!|} ((y+b)+x)+(0+a) \\
&\mathrel{|\!\!=\!\!|} a+((y+b)+x).
\end{aligned}$$

Similar to the proof of Proposition 6 we now apply hemi-right-modularity and flip x and y. Then, we walk the equations backwards. This yields hemi-commutativity. Assume now, that $+$ is hemi-left-modular. We have

$$\begin{aligned}
a+(x+y)+b &\mathrel{|\!\!=\!\!|} 0+(0+(a+(x+y)+b))+0 \\
&\mathrel{|\!\!=\!\!|} 0+((a+(x+y))+(b+0))+0 \\
&\mathrel{|\!\!=\!\!|} b+(a+(x+y)) \\
&\mathrel{|\!\!=\!\!|} b+(x+(y+a))+0 \\
&\mathrel{|\!\!=\!\!|} b+(a+(y+x))+0.
\end{aligned}$$

We swapped x and y and now we can walk from the third step backwards to procure hemi-commutativity. □

Proposition 8. *Let $(G,+,M)$ be a hemi-untial magma. If $+$ is hemi-left-modular and hemi-right-cyclic, then $+$ is also hemi-commutative.*

Proof. We have

$$\begin{aligned}
a+(x+y)+b &\mathrel{|\!\!=\!\!|} 0+((a+(x+y))+b)+0 \\
&\mathrel{|\!\!=\!\!|} 0+((b+a)+(x+y))+0 \\
&\mathrel{|\!\!=\!\!|} y+(x+(b+a)) \\
&\mathrel{|\!\!=\!\!|} y+(a+(b+x))+0 \\
&\mathrel{|\!\!=\!\!|} (0+y)+(a+(b+x)) \\
&\mathrel{|\!\!=\!\!|} (b+x)+(a+(0+y)) \\
&\mathrel{|\!\!=\!\!|} (b+x)+(y+(0+a)) \\
&\mathrel{|\!\!=\!\!|} (0+a)+(y+(b+x)) \\
&\mathrel{|\!\!=\!\!|} a+(y+(b+x)) \\
&\mathrel{|\!\!=\!\!|} a+(x+(b+y)).
\end{aligned}$$

Since we were able to permute x and y, we can do the same in the original expression by walking the equations back. □

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4. Discussion

Indistinguishability [4] is a common concept in physics, i.e., an object does not change it properties by certain algebraic operations. Here, symmetry considerations play an important role. Our approach might be seen as a generalization of this concept, leaving open whether an objects and its transformation are identical or have only undistinguishable properties. A related approach is physical equivalence [5].

4.1. Philosophical aspects

We touch here indeterminacy of the model choice, which might be addressed by an adapted algebraic approach, in contrast to indeterminacy of prediction, which can be at least partially addressed by a stochastic approach. We emphasize that theories might be heavier semantically theory-laden [6] than believed at first glance. We do not claim at all, that our suggested approach is a must for modelling. It might be a path, if subsequential investigations show its practical applicability.

4.2. The eponymous concepts

A set G together with operation $+$ is called a magma or groupoid. A unital magma [7] possesses an element ε such that

$$x = \varepsilon + x = x + \varepsilon \qquad \forall x \in G.$$

An operation is called left modular [8], if

$$x + (y + z) = z + (y + x) \qquad \forall x, y, z \in G.$$

Finally, in theoretical computer science, cyclically-invariant functions $f : G^d \to G$ are of interest, i.e., functions with the property that $f(x_1, \ldots, x_d) = f(x_2, \ldots, x_d, x_1)$ for all $x_1, \ldots, x_d \in G$ [9].

4.3. Related mathematical approaches

Weakening fundamental laws in algebra is not new, and the names we use in the Definitions 2–3 are borrowed from these generalizations. The main difference to all approaches we have seen in literature is, that generalizations in standard algebra restrict the applicability of a law to certain situations and keep the assumptions that the resulting objects are identical. For instance, a magma is called alternative, if for all elements x and y, we have $(xx)y = x(xy)$ and $y(xx) = (yx)x$; a magma is called flexible, if for all elements x and y we have $x(yx) = (xy)x$. In this paper, we keep the general applicability of the laws, but release the strong assumption that we get identical objects. Among the papers we have found, [10] is the closest to ours, showing that entropicity and the existence of a neutral element implies in the binary case associativity and commutativity.

5. Conclusions

This trial shows that primitive algebraic laws can be weakened, so that they better reflect the actual practical situation. Two novel elements, a stylized environment and a weak neutral element allow for executing algebraic transformations and drawing non-trivial conclusions. Revealing hidden assumptions, the paper addresses also philosophical questions such as falsifiability and theory-ladenness. Although fulfilling the five virtues of [11], this case study leaves its practical applicability open.

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