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Article

Binomial Convolution of Sequences

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Abstract: Given any two sequences of complex numbers, we establish simple relations between their binomial convolution and the binomial convolution of their individual binomial transforms. We employ these relations to derive new identities involving Fibonacci numbers, Bernoulli numbers, harmonic numbers, odd harmonic numbers and binomial coefficients.

Keywords: binomial transform; binomial coefficient; combinatorial identity; sequence; Bernoulli numbers; harmonic numbers

MSC: Primary 05A19; Secondary 05A10, 11B68, 11B39

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1. Introduction

Let n be a non-negative integer. Let (s_k) and (σ_k), $k = 0, 1, 2, \ldots$, be sequences of complex numbers. It is known that

$$s_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma_k \iff \sigma_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} s_k. \tag{1}$$

Such sequences (s_k) and (σ_k), as given in (1), will be called a binomial-transform pair of the first kind.

Consider two binomial-transform pairs of the first kind, namely, $\{(s_k), (\sigma_k)\}$ and $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \ldots$. We will establish the following convolution identity:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s_k t_{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma_k \tau_{n-k}.$$

Let $(\bar{s}_k)$ and $(\bar{\sigma}_k)$, $k = 0, 1, 2, \ldots$, be sequences of complex numbers. It is also known that

$$\bar{s}_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \bar{\sigma}_k \iff \bar{\sigma}_n = \sum_{k=0}^{n} \binom{n}{k} \bar{s}_k. \tag{2}$$

Such sequences $(\bar{s}_k)$ and $(\bar{\sigma}_k)$, as given in (2), will be called a binomial-transform pair of the second kind. In this case we also call $(\bar{\sigma}_k)$ the binomial transform of $(\bar{s}_k)$.

Given two binomial-transform pairs of the second kind, say $\{(\bar{s}_k), (\bar{\sigma}_k)\}$ and $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \ldots$, we will derive the following convolution identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \bar{s}_k \bar{t}_{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \bar{\sigma}_k \bar{\tau}_{n-k}.$$

Throughout, we will distinguish binomial-transform pairs of the second kind with an overbar on the letter representing each sequence. We will use a Greek alphabet to denote each binomial transform and the corresponding Latin alphabet to represent the original sequence.

We will also derive the following convolution of one sequence and the binomial transform of the other:

$$\sum_{k=0}^{n} \binom{n}{k} \bar{s}_k \bar{\tau}_{n-k} = \sum_{k=0}^{n} \binom{n}{k} \bar{\sigma}_k \bar{t}_{n-k}.$$

Given a binomial-transform pair of the first kind, $\{(s_k), (\sigma_k)\}$ and a binomial-transform pair of the second kind, $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \ldots$, we will derive the following convolution:

$$\sum_{k=0}^{n} \binom{n}{k} s_k \bar{t}_{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma_k \bar{\tau}_{n-k}.$$

The convolution identities developed in this paper facilitate the discovery of an avalanche of new combinatorial identities. To close this section, we present a couple of identities, selected from our results, to whet the reader's appetite for reading on.

We derived various general identities involving binomial transform pairs, such as

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{y-k}{x} t_{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{y-k}{y-x} \tau_{n-k},$$

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{m\, H_{k+m}}{k+m}\, t_{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} (H_{k+m} - H_k) \tau_{n-k},$$

and

$$\sum_{k=0}^{n} \binom{n}{k} H_{k+m} \bar{t}_{n-k} = H_m \bar{\tau}_n - \sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} \frac{1}{k}\, \bar{\tau}_{n-k},$$

where H_k is a harmonic number and m is a complex number that is not a negative integer.

We discovered the following convolution of harmonic and odd harmonic numbers:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_k O_{n-k} = \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} \frac{2^{2(n-k)-1}}{k(n-k)} \binom{2(n-k)}{n-k}^{-1}.$$

We also obtained some identities involving Bernoulli numbers and Bernoulli polynomials, including the following:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{y}{w}\right)^k B_{n-k}(y) B_k(w) = \frac{ny}{2w} B_{n-1} \left(1 - \left(\frac{y}{w}\right)^{n-2}\right)$$

and

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{x}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{x+k}{k} B_{n-k},$$

where x, y and w are complex numbers.

We found the following biomial convolution of Fibonacci and Bernoulli numbers, valid for n an even integer:

$$\sum_{k=0}^{n} \binom{n}{k} F_k B_{n-k} = 0.$$

We obtained the following polynomial identities involving, respectively, harmonic numbers and odd harmonic numbers:

$$\sum_{k=0}^{n} \binom{n}{k} H_k t^k = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{1}{k}\, t^k (1+t)^{n-k}$$

and

$$\sum_{k=0}^{n} \binom{n}{k} O_k t^k = \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} 2^{2k-1} \binom{2k}{k}^{-1} \frac{1}{k}\, t^k (1+t)^{n-k},$$

with the special values

$$\sum_{k=0}^{n} \binom{n}{k} H_k = \sum_{k=1}^{n} (-1)^{k-1} 2^{n-k} \binom{n}{k} \frac{1}{k}$$

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and
$$\sum_{k=0}^{n}\binom{n}{k}O_k=\sum_{k=1}^{n}(-1)^{k-1}2^{n+k-1}\binom{n}{k}\binom{2k}{k}^{-1}\frac{1}{k}.$$

2. Required Identities

In this section we will state some binomial transform identities which we will use for illustrative purposes in the sequel. The identities are drawn mostly from Boyadzhiev's book [2].

2.1. Identities Involving Binomial Coefficients

Binomial coefficients are defined, for non-negative integers i and j, by
$$\binom{i}{j}=\begin{cases}\dfrac{i!}{j!(i-j)!}, & i\geq j;\\ 0, & i<j;\end{cases}$$
the number of distinct sets of j objects that can be chosen from i distinct objects.

Generalized binomial coefficients are defined for complex numbers r and s by
$$\binom{r}{s}=\frac{\Gamma(r+1)}{\Gamma(s+1)\Gamma(r-s+1)},$$
where the Gamma function, $\Gamma(z)$, is defined for $\Re(z)>0$ by
$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt=\int_0^\infty(\log(1/t))^{z-1}dt,$$
and is extended to the rest of the complex plane, excluding the non-positive integers, by analytic continuation.

In the identities below, n is a non-negative integer and x, y and z are complex numbers.
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{y}{k}^{-1}=\binom{y-x}{n}\binom{y}{n}^{-1},\quad y\notin\mathbb{N}_0, \tag{3}$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\binom{y-k}{x}=\binom{y-n}{y-x}, \tag{4}$$
$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{k+y}{x}=\binom{y}{x-n}, \tag{5}$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}=\binom{n+x}{n}, \tag{6}$$
and
$$\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k+z}=\binom{n+x}{n+z}. \tag{7}$$

2.2. Identities Involving Harmonic Numbers

Harmonic numbers, H_n, and odd harmonic numbers, O_n, are defined for non-negative integers n by
$$H_n=\sum_{k=1}^{n}\frac{1}{k},\quad O_n=\sum_{k=1}^{n}\frac{1}{2k-1},\quad H_0=0=O_0.$$

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In the following identities, m and n are non-negative integers.

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{H_k}{k+1}=-\frac{H_n}{n+1}, \tag{8}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}H_{k+m}=-\frac{1}{n}\binom{n+m}{m}^{-1},\quad n\neq 0, \tag{9}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{H_{k+m}}{k+m}=\frac{H_{n+m}-H_n}{m}\binom{n+m}{n}^{-1},\quad m\neq 0, \tag{10}$$

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}H_k=\binom{n+m}{m}(H_m+H_n-H_{m+n}), \tag{11}$$

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{k+m}{m}(H_{k+m}-H_m)=\binom{m}{n}(H_m-H_{m-n}), \tag{12}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}O_k=-\binom{2n}{n}^{-1}\frac{2^{2n-1}}{n}. \tag{13}$$

2.3. Identities Involving Bernoulli Numbers

The Bernoulli numbers, B_k, are defined by the generating function

$$\frac{z}{e^z-1}=\sum_{k=0}^{\infty}B_k\frac{z^k}{k!},\quad z<2\pi\,,$$

and the Bernoulli polynomials by the generating function

$$\frac{ze^{xz}}{e^z-1}=\sum_{k=0}^{\infty}B_k(x)\frac{z^k}{k!},\quad |z|<2\pi\,.$$

Clearly, $B_k=B_k(0)$.

The first few Bernoulli numbers are

$$B_0=1,\ B_1=-\frac{1}{2},\ B_2=\frac{1}{6},\ B_3=0,\ B_4=-\frac{1}{30},\ B_5=0,\ B_6=\frac{1}{42},\ B_7=0,\ \ldots\,,$$

while the first few Bernoulli polynomials are

$$B_0(x)=1,\quad B_1(x)=x-\frac{1}{2},\quad B_2(x)=x^2-x+\frac{1}{6},\quad B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x\,,$$
$$B_4(x)=x^4-2x^3+x^2-\frac{1}{30},\quad B_5(x)=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x\,,$$
$$B_6(x)=x^6-3x^5+\frac{5}{2}x^4-\frac{1}{2}x^2+\frac{1}{42}\,.$$

An explicit formula for the Bernoulli polynomials is

$$\frac{B_n(x)}{x^n}=\sum_{k=0}^{n}\binom{n}{k}\frac{B_k}{x^k}\,,$$

while a recurrence formula for them is

$$B_n(x+1)=\sum_{k=0}^{n}\binom{n}{k}B_k(x), \tag{14}$$

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or more generally,

$$\frac{B_n(x+y)}{y^n} = \sum_{k=0}^{n} \binom{n}{k} \frac{B_k(x)}{y^k}. \tag{15}$$

From (14), we have

$$\sum_{k=0}^{n} \binom{n}{k} B_k = (-1)^n B_n. \tag{16}$$

2.4. An Identity Involving Fibonacci Numbers

The Fibonacci numbers, F_n, and the Lucas numbers, L_n, are defined, for $n \in \mathbb{Z}$, through the recurrence relations

$$F_n = F_{n-1} + F_{n-2}, (n \geq 2), \quad F_0 = 0, F_1 = 1;$$

and

$$L_n = L_{n-1} + L_{n-2}, (n \geq 2), \quad L_0 = 2, L_1 = 1;$$

with

$$F_{-n} = (-1)^{n-1} F_n\,, \quad L_{-n} = (-1)^n L_n\,. \tag{17}$$

Explicit formulas (Binet formulas) for the Fibonacci and Lucas numbers are

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad L_n = \alpha^n + \beta^n, \quad n \in \mathbb{Z},$$

where $\alpha = (1+\sqrt{5})/2$ is the golden ratio and $\beta = (1-\sqrt{5})/2 = -1/\alpha$.

The generalized Fibonacci sequence, (G_n), $n = 0, 1, 2, \ldots$, the so-called gibonacci sequence, is a generalization of F_n and L_n; having the same recurrence relation as the Fibonacci sequence but with arbitrary initial values. Thus

$$G_n = G_{n-1} + G_{n-2}, \quad (n \geq 2);$$

with

$$G_{-n} = G_{-n+2} - G_{-n+1}.$$

where G_0 and G_1 arbitrary numbers (usually integers) not both zero.

In the identity listed below, n is a non-negative integer, r and t are integers.

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{G_{tk+r}}{L_t^k} = \frac{(-1)^r}{L_t^n} (G_0 L_{tn-r} - G_{tn-r}). \tag{18}$$

3. Identities Involving Binomial-Transform Pairs of the First Kind

Our first main result is stated in Theorem 19; we require the result stated in the next lemma.

Lemma 1. *Let $\{(a_k), (\alpha_k)\}$, $k = 0, 1, 2, \ldots$, be a binomial-transform pair of the first kind. Let $\mathcal{L}_x$ be a linear operator defined by $\mathcal{L}_x(x^j) = a_j$ for every complex number x and every non-negative integer j. Then $\mathcal{L}_x((1-x)^j) = \alpha_j$.*

Proof. We have

$$\begin{aligned}
\mathcal{L}_x\left((1-x)^j\right) = \mathcal{L}_x\left(\sum_{k=0}^{j} (-1)^k \binom{j}{k} x^k\right) &= \sum_{k=0}^{j} (-1)^k \binom{j}{k} \mathcal{L}_x\left(x^k\right) \\
&= \sum_{k=0}^{j} (-1)^k \binom{j}{k} a_k \\
&= \alpha_j.
\end{aligned}$$

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□

Theorem 1. *Let n be a non-negative integer. If $\{(s_k),(\sigma_k)\}$ and $\{(t_k),(\tau_k)\}$, $k=0,1,2,\ldots$, are binomial-transform pairs of the first kind, then*

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}s_k t_{n-k}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\sigma_k\tau_{n-k}. \tag{19}$$

Proof. Let n be a non-negative integer. Let x and y be complex numbers. Consider the following identity whose veracity is readily established by the binomial theorem:

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}y^{n-k}x^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1-y)^{n-k}(1-x)^k. \tag{20}$$

Let (t_j), $j=0,1,2,\ldots$, be a sequence of complex numbers. Let $\mathcal{L}_x(x^j)=t_j$.

Operate on both sides of (20) with $\mathcal{L}_x$ to obtain

$$\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}y^{n-k}t_k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(1-y)^{n-k}\tau_k,$$

where

$$\tau_k=\mathcal{L}_x((1-x)^k)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}t_i.$$

Thus,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}y^{k}t_{n-k}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}(1-y)^{k}\tau_{n-k}. \tag{21}$$

Let (s_j), $j=0,1,2,\ldots$, be a sequence of complex numbers. Let $\mathcal{L}_y(y^j)=s_j$. The action of $\mathcal{L}_y$ on (21) produces

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}s_k t_{n-k}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\sigma_k\tau_{n-k},$$

where

$$\sigma_k=\mathcal{L}_y((1-y)^k)=\sum_{i=0}^{k}(-1)^i\binom{k}{i}s_i,$$

and the proof is complete.

□

Example 1. *From (4), we can choose*

$$s_k=\binom{y-k}{x},\quad \sigma_k=\binom{y-k}{y-x},$$

and use these in (19) to obtain the following identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y-k}{x}t_{n-k}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{y-k}{y-x}\tau_{n-k}, \tag{22}$$

which holds for every binomial-transform pair of the first kind $\{(t_k),(\tau_k)\}$.

As an immediate consequence of (22), we have the following polynomial identity in the complex variable t:

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y-k}{x}t^{n-k}=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\binom{y-k}{y-x}(1-t)^{n-k},$$

which can also be written as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{y+k}{x}t^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y+k}{y+n-x}(1-t)^k, \tag{23}$$

which is valid for all complex numbers x, y and t.

Example 2. *If we identify*

$$s_k=\frac{H_{k+m}}{k+m},\quad \sigma_k=\frac{H_{k+m}-H_k}{m}\binom{k+m}{k}^{-1},$$

from (10), then, from (19), we obtain the following identity

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{m\,H_{k+m}}{k+m}t_{n-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+m}{m}^{-1}(H_{k+m}-H_k)\tau_{n-k}, \tag{24}$$

which holds for any binomial-transform pair of the first kind, $\{(t_k),(\tau_k)\}$, for every complex number m that is not a non-positive number and every non-negative integer n.

In particular, using

$$t_k=\frac{G_{tk+r}}{L_t^k},\quad \tau_k=\frac{(-1)^r}{L_t^k}(G_0L_{tk-r}-G_{tk-r}),$$

obtained from (18), in (24) yields the following general Fibonacci-harmonic number identity:

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{m\,H_{k+m}}{k+m}L_t^kG_{t(n-k)+r},\\&\quad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+m}{m}^{-1}(H_{k+m}-H_k)L_t^k\left(G_0L_{t(n-k)-r}-G_{t(n-k)-r}\right);\end{aligned} \tag{25}$$

and, in particular,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{m}{k+m}H_{k+m}F_{n-k}=\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}\binom{k+m}{m}^{-1}(H_{k+m}-H_k)F_{n-k}, \tag{26}$$

with the special value

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{H_{k+1}F_{n-k}}{k+1}=\sum_{k=0}^{n}(-1)^{k-1}\binom{n}{k}\frac{F_{n-k}}{(k+1)^2}. \tag{27}$$

Example 3. *From (3) and (4), we can choose*

$$s_k=\binom{y-k}{x},\quad \sigma_k=\binom{y-k}{y-x},$$

and

$$t_k=\binom{u}{k}\binom{v}{k}^{-1},\quad \tau_k=\binom{v-u}{k}\binom{v}{k}^{-1}.$$

Using these in (19), we obtain the following identity

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{y-k}{x}\binom{u}{n-k}\binom{v}{n-k}^{-1}\\&\quad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{y-k}{y-x}\binom{v-u}{n-k}\binom{v}{n-k}^{-1},\end{aligned} \tag{28}$$

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which is valid for complex numbers x, y, u and v such that v is not a non-negative number that is less than n.

In particular, at $v = n$, we get

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{y-k}{x} \binom{u}{n-k} = \sum_{k=0}^{n} (-1)^{k} \binom{y-k}{y-x} \binom{n-u}{n-k}, \tag{29}$$

with the special value

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \binom{y-k}{y-x} = \binom{y-n}{x}, \tag{30}$$

which is the binomial transform of the first kind of (4).

Example 4. *From (9) and (13), we can identify*

$$s_k = H_{k+m}, \quad \sigma_k = \frac{\delta_{k0}(1+H_m)-1}{k+\delta_{k0}} \binom{k+m}{m}^{-1},$$

and

$$t_k = O_k, \quad \tau_k = -\frac{1-\delta_{k0}}{k+\delta_{k0}} \binom{2k}{k}^{-1} 2^{2k-1}.$$

Here and throughout this paper, δ_{ij} is Kronecker's delta having the value 1 when $i = j$ and is zero otherwise.

Using these in (19) leads to the following harmonic number identity:

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_{k+m} O_{n-k} \\ = -\frac{H_m}{n} \binom{2n}{n}^{-1} 2^{2n-1} + \sum_{k=1}^{n-1} (-1)^{k} \binom{n}{k} \frac{2^{2(n-k)-1}}{k(n-k)} \binom{k+m}{m}^{-1} \binom{2(n-k)}{n-k}^{-1},\end{aligned} \tag{31}$$

which is valid for every positive integer n and every complex number m that is not a negative integer.

In particular, we have the following alternating binomial convolution of harmonic number and odd harmonic number:

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} H_k O_{n-k} = \sum_{k=1}^{n-1} (-1)^{k} \binom{n}{k} \frac{2^{2(n-k)-1}}{k(n-k)} \binom{2(n-k)}{n-k}^{-1}. \tag{32}$$

Corollary 1. *Let n be a non-negative integer. If $\{(s_k),(\sigma_k)\}$ and $\{(t_k),(\tau_k)\}$, $k = 0,1,2,\ldots$, are binomial-transform pairs, then*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \sigma_k t_{n-k} = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} s_k \tau_{n-k}. \tag{33}$$

Corollary 2. *Let n be a non-negative integer. If $\{(s_k),(\sigma_k)\}$, $k = 0,1,2,\ldots$, is a binomial-transform pair, then*

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} s_k s_{n-k} = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \sigma_k \sigma_{n-k}. \tag{34}$$

Remark 1. *We deduce from (34) that if n is a non-negative integer, then*

$$\sum_{k=0}^{n} (-1)^{k} \binom{n}{k} s_k s_{n-k} = \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} \sigma_k \sigma_{n-k}, \tag{35}$$

with each sum being 0 for odd n.

Example 5. *Plugging s_k and σ_k from Example 2 into (35) yields, for n a non-negative integer and m a complex number that is not a negative integer,*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{m^2H_{k+m}H_{n-k+m}}{(k+m)(n-k+m)}$$
$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+m}{k}^{-1}\binom{n-k+m}{n-k}^{-1}(H_{k+m}-H_k)(H_{n-k+m}-H_{n-k}), \tag{36}$$

and, in particular,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{H_{k+1}H_{n-k+1}}{(k+1)(n-k+1)}=\sum_{k=0}^{n}\frac{(-1)^k}{(k+1)^2(n-k+1)^2}\binom{n}{k}. \tag{37}$$

Example 6. *From (3), with*

$$s_k=\binom{x}{k}\binom{y}{k}^{-1},\quad \sigma_k=\binom{y-x}{k}\binom{y}{k}^{-1},$$

identity (35) gives

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{x}{n-k}\binom{y}{k}^{-1}\binom{y}{n-k}^{-1}$$
$$=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{y-x}{k}\binom{y-x}{n-k}\binom{y}{k}^{-1}\binom{y}{n-k}^{-1}, \tag{38}$$

which, upon setting $x=n$ and $y=n$, in turn, yields

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3\binom{y}{k}^{-1}\binom{y}{n-k}^{-1}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{y-n}{k}\binom{y-n}{n-k}\binom{y}{k}^{-1}\binom{y}{n-k}^{-1} \tag{39}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{x}{k}\binom{x}{n-k}\binom{n}{k}^{-1}=\sum_{k=0}^{n}(-1)^k\binom{n-x}{k}\binom{n-x}{n-k}\binom{n}{k}^{-1}. \tag{40}$$

Setting $y=-1$ in (39) gives, after some simplification,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}\binom{2n-k}{n-k},$$

the left hand side of which is known to be

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\begin{cases}0, & \text{if } n \text{ is odd;}\\ (-1)^{n/2}(3n/2)!((n/2)!)^{-3}, & \text{if } n \text{ is even.}\end{cases}$$

We therefore obtain

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}\binom{2n-k}{n-k}=\begin{cases}0, & \text{if } n \text{ is odd;}\\ (-1)^{n/2}(3n/2)!((n/2)!)^{-3}, & \text{if } n \text{ is even.}\end{cases} \tag{41}$$

Similarly, setting $x=-1$ in (40) gives

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}^{-1}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(n+1)^2}{(n-k+1)(k+1)},$$

from which we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{(n-k+1)(k+1)}=\frac{1+(-1)^n}{(n+1)(n+2)},\tag{42}$$

upon using [7]

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^{-1}=(1+(-1)^n)\frac{n+1}{n+2}.$$

A sequence (s_k), $k=0,1,2,\ldots$, for which

$$s_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}s_k,$$

for every non-negative integer n will be called an invariant sequence.

A sequence (s_k), $k=0,1,2,\ldots$, for which

$$-s_n=\sum_{k=0}^{n}(-1)^k\binom{n}{k}s_k,$$

for every non-negative integer n will be called an inverse invariant sequence.

Many invariant sequences are known (see e.g. Sun [5], Boyadzhiev [2]). Examples include (L_k), (kF_{k-1}), $(\binom{x/2}{k}/\binom{x}{k})$, $0\neq x\notin\mathbb{Z}^+$, $(\binom{2k}{k}/2^{2k})$. Inverse invariant sequences include (F_k) and $(H_k/(k+1))$. Here L_k is the k^{th} Lucas number, F_k is the k^{th} Fibonacci number and H_k is the k^{th} harmonic number.

Corollary 3. *Let (s_k) and (t_k), $k=0,1,2,\ldots$, be two sequences of complex numbers. Let n be a non-negative integer. If both sequences are invariant or both are inverse invariant, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}s_kt_{n-k}=0,\ \textit{if } n \textit{ is odd;}\tag{43}$$

while if one of the sequences is invariant and the other is inverse invariant, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}s_kt_{n-k}=0,\ \textit{if } n \textit{ is even.}\tag{44}$$

Remark 2. *The results stated in Corollary 3 were also obtained by Wang [8].*

Example 7. *If n is a non-negative odd integer and x is a non-zero complex number that is not a positive integer that is less than n, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x/2}{k}\binom{x}{k}^{-1}L_{n-k}=0.\tag{45}$$

4. Identities Involving Binomial-Transform Pairs of the Second Kind

Lemma 2. *Let $\{(\bar{a}_k),(\bar{\alpha}_k)\}$, $k=0,1,2,\ldots$, be a binomial-transform pair of the second kind. Let $\mathcal{M}_x$ be a linear operator defined by $\mathcal{M}_x(x^j)=\bar{a}_j$ for every complex number x and every non-negative integer j. Then $\mathcal{M}_x((1+x)^j)=\bar{\alpha}_j$.*

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Proof. We have

$$\mathcal{M}_x\Big((1+x)^j\Big) = \mathcal{M}_x\left(\sum_{k=0}^{j}\binom{j}{k}x^k\right) = \sum_{k=0}^{j}\binom{j}{k}\mathcal{M}_x\Big(x^k\Big) = \sum_{k=0}^{j}\binom{j}{k}\bar{a}_k = \bar{\alpha}_j.$$

□

Theorem 2. *Let n be a non-negative integer. If $\{(\bar{s}_k),(\bar{\sigma}_k)\}$ and $\{(\bar{t}_k),(\bar{\tau}_k)\}$, $k=0,1,2,\ldots$, are binomial-transform pairs of the second kind, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{s}_k\bar{t}_{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{\sigma}_k\bar{\tau}_{n-k}. \tag{46}$$

Proof. Write (20) as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}y^{n-k}x^k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+y)^{n-k}(1+x)^k. \tag{47}$$

Let $(\bar{t}_j)$, $j=0,1,2,\ldots$, be a sequence of complex numbers. Let $\mathcal{M}_x(x^j)=\bar{t}_j$.

Operate on both sides of (47) with $\mathcal{M}_x$ to obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}y^{n-k}\bar{t}_k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+y)^{n-k}\bar{\tau}_k,$$

where

$$\bar{\tau}_k = \mathcal{M}_x((1+x)^k) = \sum_{i=0}^{k}\binom{k}{i}\bar{t}_i.$$

Thus,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}y^{k}\bar{t}_{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(1+y)^{k}\bar{\tau}_{n-k}. \tag{48}$$

Let $(\bar{s}_j)$, $j=0,1,2,\ldots$, be a sequence of complex numbers. Let $\mathcal{M}_y(y^j)=\bar{s}_j$. The action of $\mathcal{M}_y$ on (48) produces

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{s}_k\bar{t}_{n-k} = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{\sigma}_k\bar{\tau}_{n-k},$$

where

$$\bar{\sigma}_k = \mathcal{M}_y((1+y)^k) = \sum_{i=0}^{k}\binom{k}{i}\bar{s}_i,$$

and the proof is complete.

□

Example 8. *Considering (15), if we choose*

$$\bar{s}_k = \frac{B_k(x)}{y^k}, \quad \bar{\sigma}_k = \frac{B_k(x+y)}{y^k},$$

in (46), we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}y^kB_{n-k}(x)\bar{t}_k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}y^kB_{n-k}(x+y)\bar{\tau}_k, \tag{49}$$

which is valid for complex variables x and y and any binomial transform pair $\{(\bar{s}_k),(\bar{\sigma}_k)\}$ of the second kind.

Many results can be derived from (49). To begin with, we have the following polynomial identity:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}y^kB_{n-k}(x)t^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}y^kB_{n-k}(x+y)(1+t)^k, \tag{50}$$

which holds for complex numbers x, y and t.

Choosing

$$\bar{t}_k=\frac{B_k(z)}{w^k},\quad \bar{\tau}_k=\frac{B_k(z+w)}{w^k},$$

in (49) leads to

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}(y/w)^kB_{n-k}(x)B_k(z)\\&\quad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(y/w)^kB_{n-k}(x+y)B_k(z+w),\end{aligned} \tag{51}$$

giving, in particular,

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}B_{n-k}(x)B_k(z)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}B_{n-k}(x+w)B_k(z+w), \tag{52}$$

and

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(y/w)^kB_{n-k}(y)B_k(w)=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(y/w)^kB_{n-k}B_k. \tag{53}$$

Proposition 1. *If n is a positive odd integer and y and w are complex numbers, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\left(\frac{y}{w}\right)^kB_{n-k}(y)B_k(w)=\frac{ny}{2w}B_{n-1}\left(1-\left(\frac{y}{w}\right)^{n-2}\right). \tag{54}$$

Proof. A consequence of the fact that the right-hand side of (53) can be written as

$$\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}(y/w)^{2k}B_{n-2k}B_{2k}-\sum_{k=1}^{\lceil n/2\rceil}\binom{n}{2k-1}(y/w)^{2k-1}B_{n-2k+1}B_{2k-1},$$

and the fact that B_j is zero for odd j greater than unity. □

Example 9. *Using the binomial-transform pairs*

$$\bar{s}_k=\binom{x}{k},\quad \bar{\sigma}_k=\binom{x+k}{k},\ \text{From (6)}, \tag{55}$$

and

$$\bar{t}_k=B_k,\quad \bar{\tau}_k=(-1)^kB_k,\ \text{From (16)}, \tag{56}$$

in (46) gives

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{x}{k}B_{n-k}=\sum_{k=0}^{n}\binom{n}{k}\binom{x+k}{k}B_{n-k}. \tag{57}$$

Corollary 4. *Let n be a non-negative integer. If $\{(\bar{s}_k),(\bar{\sigma}_k)\}$ is a binomial-transform pair of the second kind, then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{s}_k\bar{s}_{n-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{\sigma}_k\bar{\sigma}_{n-k}. \tag{58}$$

Example 10. *Using, in (58), the $\bar{s}_k$ and $\bar{\sigma}_k$ given in (55), we have*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x}{k}\binom{x}{n-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{x+k}{k}\binom{x+n-k}{n-k}; \tag{59}$$

which holds for every non-negative integer n and every complex number x.

Example 11. *From (11) we identify the following binomial-transform pair of the second kind:*

$$\bar{s}_k=\binom{m}{k}H_k,\quad \bar{\sigma}_k=\binom{k+m}{m}(H_m+H_k-H_{k+m}), \tag{60}$$

which when used in (58) gives

$$\begin{aligned}&\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{m}{k}\binom{m}{n-k}H_kH_{n-k}\\&\quad=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+m}{m}\binom{n-k+m}{m}(H_m+H_k-H_{k+m})(H_m+H_{n-k}-H_{n-k+m}),\end{aligned} \tag{61}$$

which is valid for every non-negative integer n and every complex number m that is not a negative integer.

Theorem 3. *Let n be a non-negative integer. If $\{(\bar{s}_k),(\bar{\sigma}_k)\}$ and $\{(\bar{t}_k),(\bar{\tau}_k)\}$, $k=0,1,2,\ldots$, are binomial-transform pairs of the second kind, then*

$$\sum_{k=0}^{n}\binom{n}{k}\bar{s}_k\bar{\tau}_{n-k}=\sum_{k=0}^{n}\binom{n}{k}\bar{\sigma}_k\bar{t}_{n-k}. \tag{62}$$

Proof. Consider the following variation on (47):

$$\sum_{k=0}^{n}\binom{n}{k}y^{n-k}(1+x)^k=\sum_{k=0}^{n}\binom{n}{k}(1+y)^{n-k}x^k, \tag{63}$$

and proceed as in the proof of Theorem 2. □

Example 12. *Using the binomial-transform pairs of the second kind*

$$\bar{s}_k=\binom{m}{k}H_k,\quad \bar{\sigma}_k=\binom{k+m}{m}(H_m+H_k-H_{k+m}),\ \textit{Equation (60)},$$

and

$$\bar{t}_k=\binom{x}{k},\quad \bar{\tau}_k=\binom{x+k}{k},\ \textit{Equation (55) relabelled},$$

in (62) yields

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}\binom{x+n-k}{n-k}H_k=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{n-k}\binom{k+m}{m}(H_m+H_k-H_{k+m}). \tag{64}$$

By considering the binomial-transform pair of the second kind $\{(1),(2^k)\}$, $k=0,1,2,\ldots$, we obtain a corollary to Theorem 3.

Corollary 5. *Let n be a non-negative integer. If $\{(\bar{s}_k),(\bar{\sigma}_k)\}$ is a binomial-transform pair of the second kind, then*

$$\sum_{k=0}^{n}\binom{n}{k}\bar{\sigma}_k=\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\bar{s}_k. \tag{65}$$

Remark 3. *Identity (65) corresponds to Example 4 of Boyadzhiev [1].*

Example 13. *With (7) in mind, choosing*

$$\bar{s}_k = \binom{x}{k+z}, \quad \bar{\sigma}_k = \binom{k+x}{k+z}, \tag{66}$$

in (65) yields

$$\sum_{k=0}^{n} \binom{n}{k}\binom{k+x}{k+z} = \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k+z} 2^{n-k}. \tag{67}$$

for n a non-negative integer and x and z complex numbers.

5. Identities Involving Mixed Binomial Transform Pairs

Theorem 4. *Let n be a non-negative integer. If $\{(s_k),(\sigma_k)\}$ is a binomial transform pair of the first kind and $\{(\bar{t}_k),(\bar{\tau}_k)\}$, $k = 0,1,2,\ldots$, is a binomial-transform pair of the second kind, then*

$$\sum_{k=0}^{n} \binom{n}{k} s_k \bar{t}_{n-k} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sigma_k \bar{\tau}_{n-k}. \tag{68}$$

Proof. Let $M_x(x^k) = \bar{t}_k$ and $\mathcal{L}_y(y^k) = s_k$.

Act on the identity

$$\sum_{k=0}^{n} \binom{n}{k} y^{n-k} x^k = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (1-y)^{n-k}(1+x)^k,$$

with $\mathcal{M}_x$ and $\mathcal{L}_y$, in succession. □

Example 14. *Use of the binomial-transform pair of the first kind $\{(F_k),(-F_k)\}$, and the binomial-transform pair of the second kind $\{(B_k),((-1)^k B_k)\}$, $k = 0,1,2,\ldots$, in (68) gives*

$$\sum_{k=0}^{n} \binom{n}{k} F_k B_{n-k} = 0, \tag{69}$$

for n an even integer.

Similarly $\{(L_k),(-L_k)\}$ and $\{(B_k),((-1)^k B_k)\}$, $k = 0,1,2,\ldots$, in (68) yields

$$\sum_{k=0}^{n} \binom{n}{k} L_k B_{n-k} = 0, \tag{70}$$

for n an odd integer.

Example 15. *Use of the following binomial-transform pair (see Example 4):*

$$s_k = H_{k+m}, \quad \sigma_k = \frac{\delta_{k0}(1+H_m)-1}{k+\delta_{k0}} \binom{k+m}{m}^{-1},$$

in (68) leads to the following identity

$$\sum_{k=0}^{n} \binom{n}{k} H_{k+m} \bar{t}_{n-k} = H_m \bar{\tau}_n - \sum_{k=1}^{n} (-1)^k \binom{n}{k} \binom{k+m}{m}^{-1} \frac{1}{k} \bar{\tau}_{n-k}, \tag{71}$$

which is valid for every binomial-transform pair of the second kind $\{(\bar{t}_k),(\bar{\tau}_k)\}$, $k = 0,1,2,\ldots$ and every complex number m that is not a negative integer.

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In particular, the following polynomial identity holds:

$$\sum_{k=0}^{n}\binom{n}{k}H_{k+m}t^{n-k}=(1+t)^nH_m-\sum_{k=1}^{n}(-1)^k\binom{n}{k}\binom{k+m}{m}^{-1}\frac{(1+t)^{n-k}}{k},$$

and can be recast as stated in proposition 2 by writing $1/t$ for t.

Proposition 2. *If n is a non-negative integer, m is a complex number that is not a negative integer and t is a complex variable, then*

$$\sum_{k=0}^{n}\binom{n}{k}H_{k+m}t^{k}=(1+t)^nH_m+\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\binom{k+m}{m}^{-1}\frac{1}{k}t^k(1+t)^{n-k}. \tag{72}$$

In particular, evaluation at $m=0$ and $m=-1/2$, respectively, gives

$$\sum_{k=0}^{n}\binom{n}{k}H_kt^k=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}t^k(1+t)^{n-k} \tag{73}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}O_kt^k=\sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}2^{2k-1}\binom{2k}{k}^{-1}\frac{1}{k}t^k(1+t)^{n-k}, \tag{74}$$

with the special values

$$\sum_{k=0}^{n}\binom{n}{k}H_k=\sum_{k=1}^{n}(-1)^{k-1}2^{n-k}\binom{n}{k}\frac{1}{k} \tag{75}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}O_k=\sum_{k=1}^{n}(-1)^{k-1}2^{n+k-1}\binom{n}{k}\binom{2k}{k}^{-1}\frac{1}{k}. \tag{76}$$

In deriving (74), we used

$$H_{k-1/2}=2O_k-2\log 2$$

and

$$\binom{k-1/2}{k}=2^{-2k}\binom{2k}{k}.$$

Corollary 6. *Let n be a non-negative integer. If $\{(s_k),(\sigma_k)\}$ is a binomial transform pair of the first kind and $\{(\bar{t}_k),(\bar{\tau}_k)\}$, $k=0,1,2,\ldots$, is a binomial-transform pair of the second kind, then*

$$\sum_{k=0}^{n}\binom{n}{k}\sigma_k\bar{t}_{n-k}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}s_k\bar{\tau}_{n-k}. \tag{77}$$

6. Polynomial Identities Involving Binomial Transform Pairs

We have already encountered the identities stated in the next theorem; they are variations on (21) and (48).

Theorem 5. *If n is a non-negative integer and y is a complex variable, then*

$$\sum_{k=0}^{n}\binom{n}{k}s_{n-k}y^k=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sigma_{n-k}(1+y)^k, \tag{78}$$

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{s}_{n-k}y^k=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\bar{\sigma}_{n-k}(1+y)^k. \tag{79}$$

The implication of Theorem 5 is that every binomial-transform pair has associated with it a polynomial identity.

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Remark 4. *Identity (79) is equivalent to Boyadzhiev [1, Corollary 1].*

7. Binomial Transform Identities Associated with Polynomial Identities of a Certain Type

Polynomial identities of the following form abound in the literature:

$$\sum_{k=0}^{n} f(k)t^{p(k)} = \sum_{k=0}^{m} g(k)(1-t)^{q(k)}; \tag{80}$$

where m and n are non-negative integers, t is a complex variable, $p(k)$ and $q(k)$ are sequences of integers, and $f(k)$ and $g(k)$ are sequences of complex numbers.

The binomial identities stated in Theorems 6 and 7 are readily derived using the operators $\mathcal{L}_y$ and $\mathcal{M}_y$.

Theorem 6. *Let a polynomial identity have the form stated in (80). If $\{(s_k),(\sigma_k)\}$ is a binomial-transform pair of the first kind, then*

$$\sum_{k=0}^{n} f(k)s_{p(k)} = \sum_{k=0}^{m} g(k)\sigma_{q(k)} \tag{81}$$

and

$$\sum_{k=0}^{n} f(k)\sigma_{q(k)} = \sum_{k=0}^{m} g(k)s_{p(k)}. \tag{82}$$

Theorem 7. *Let a polynomial identity have the form stated in (80). If $\{(\bar{s}_k),(\bar{\sigma}_k)\}$ is a binomial-transform pair of the second kind, then*

$$\sum_{k=0}^{n} (-1)^{p(k)} f(k)\bar{s}_{p(k)} = \sum_{k=0}^{m} g(k)\bar{\sigma}_{q(k)} \tag{83}$$

and

$$\sum_{k=0}^{n} f(k)\bar{\sigma}_{p(k)} = \sum_{k=0}^{n} (-1)^{q(k)} g(k)\bar{s}_k. \tag{84}$$

Lemma 3. *Sun [6, Lemma 3.1] If m, n and r are non-negative integers and t is a complex variable, then*

$$\sum_{k=0}^{n} (-1)^{k-r}\binom{n}{k}\binom{k+m}{r}t^{k+m-r} = \sum_{k=0}^{m} (-1)^{k}\binom{m}{k}\binom{k+n}{r}(1-t)^{n+k-r}. \tag{85}$$

Example 16. *In (85) we can identify*

$$f(k) = (-1)^{k-r}\binom{n}{k}\binom{k+m}{r}, \quad g(k) = \binom{m}{k}\binom{k+n}{r}, \tag{86}$$

and

$$p(k) = k+m-r \text{ and } q(k) = n+k-r. \tag{87}$$

Using these in Theorem 6 gives

$$\sum_{k=0}^{n} (-1)^{k-r}\binom{n}{k}\binom{k+m}{r}s_{k+m-r} = \sum_{k=0}^{m} (-1)^{k}\binom{m}{k}\binom{k+n}{r}\sigma_{n+k-r}, \tag{88}$$

and

$$\sum_{k=0}^{n} (-1)^{k-r}\binom{n}{k}\binom{k+m}{r}\sigma_{k+m-r} = \sum_{k=0}^{m} (-1)^{k}\binom{m}{k}\binom{k+n}{r}s_{n+k-r}; \tag{89}$$

for every binomial-transform pair $\{(s_k),(\sigma_k)\}$ of the first kind.

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8. Conversions Between Binomial Transform Identities

If a binomial transform identity involves a binomial-transform pair of the first kind, it is straightforward to convert it to an identity involving a binomial-transform pair of the second kind; and vice versa. An identity involving $\{s(k), \sigma(k)\}$, $k = 0, 1, 2, \ldots$, can be converted to that involving $\{\bar{s}(k), \bar{\sigma}(k)\}$ by choosing $s(k) = x^k$ and $\sigma(k) = (1-x)^k$, replacing x with $-x$ and then operating on the resulting identity with $\mathcal{M}_x$. Similarly, an identity involving $\{\bar{s}(k), \bar{\sigma}(k)\}$ can be converted to that involving $\{s(k), \sigma(k)\}$ by choosing $\bar{s}(k) = x^k$ and $\bar{\sigma}(k) = (1+x)^k$, replacing x with $-x$ and then operating on the resulting identity with $\mathcal{L}_x$.

Example 17. *Let $\{(\bar{t}_k), (\bar{\tau}_k)\}$, $k = 0, 1, 2, \ldots$, be a binomial-transform pair of the second kind. Chen's main result [3, Theorem 3.1] (after Gould and Quaintance's simplification [4, Equation (10)]) for non-negative integers m, n and s is*

$$\begin{aligned}\sum_{k=0}^{m} \binom{m}{k} \binom{n+k+s}{s}^{-1} \bar{t}_{n+k+s} \\ = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{m+k+s}{s}^{-1} \bar{\tau}_{m+k+s} \\ + \sum_{k=0}^{s-1} \frac{(-1)^{n+s-k} s}{m+n+s-k} \binom{s-1}{k} \binom{m+n+s-k-1}{n}^{-1} \bar{\tau}_k.\end{aligned} \tag{90}$$

In order to convert (90) to an identity for the binomial-transform pair of the first kind, $\{(t_k), (\tau_k)\}$, $k = 0, 1, 2, \ldots$, we choose $\bar{t}_k = x^k$ and $\bar{\tau}_k = (1+x)^k$ and replace x with $-x$ to obtain

$$\begin{aligned}\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n+k+s}{s}^{-1} x^{n+k+s} \\ = (-1)^s \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k+s}{s}^{-1} (1-x)^{m+k+s} \\ + \sum_{k=0}^{s-1} \frac{(-1)^k s}{m+n+s-k} \binom{s-1}{k} \binom{m+n+s-k-1}{n}^{-1} (1-x)^k.\end{aligned}$$

Operating on the above equation with the linear operator $\mathcal{L}_x$ now gives

$$\begin{aligned}\sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n+k+s}{s}^{-1} t_{n+k+s} \\ = (-1)^s \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k+s}{s}^{-1} \tau_{m+k+s} \\ + \sum_{k=0}^{s-1} \frac{(-1)^k s}{m+n+s-k} \binom{s-1}{k} \binom{m+n+s-k-1}{n}^{-1} \tau_k.\end{aligned} \tag{91}$$

Example 18. *Another example, Chen [3, Theorem 3.2] is*

$$\sum_{k=0}^{m} \binom{m}{k} \binom{n+k}{s} \bar{t}_{n+k-s} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{m+k}{s} \bar{\tau}_{m+k-s}; \tag{92}$$

whose corresponding version for a binomial-transform pair of the first kind is

$$\sum_{k=0}^{m} (-1)^{s-k} \binom{m}{k} \binom{n+k}{s} t_{n+k-s} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{m+k}{s} \tau_{m+k-s}. \tag{93}$$

Example 19. *Gould and Quaintance [4, Theorem 3] gave the following identity involving a binomial transform pair of the second kind:*

$$\sum_{k=0}^{s} \binom{s}{k} \binom{m+n+s-k}{m}^{-1} \frac{\bar{t}_k}{m+n+s+1-k} = \sum_{k=0}^{s} \binom{s}{k} \binom{m+n+s-k}{n}^{-1} \frac{(-1)^{s-k}\bar{\tau}_k}{m+n+s+1-k}, \tag{94}$$

which is valid for every non-negative integer s and all complex numbers m and n excluding the set of negative integers.

The corresponding result for a binomial-transform pair of the first kind is

$$\sum_{k=0}^{s} (-1)^k \binom{s}{k} \binom{m+n+s-k}{m}^{-1} \frac{t_k}{m+n+s+1-k} = \sum_{k=0}^{s} \binom{s}{k} \binom{m+n+s-k}{n}^{-1} \frac{(-1)^{s-k}\tau_k}{m+n+s+1-k}. \tag{95}$$

Remark 5. *An identity involving a binomial-transform pair of the first kind $\{s_k, \sigma_k\}$, $k = 0, 1, 2, \ldots$, can always be converted to a binomial-transform pair identity of the second kind $\{\bar{s}_k, \bar{\sigma}_k\}$ by doing*

$$s_k \to (-1)^k \bar{s}_k, \quad \sigma_k \to \bar{\sigma}_k.$$

Similarly, conversion of a second kind transform pair identity to a first kind transform pair is achieved through

$$\bar{s}_k \to (-1)^k s_k, \quad \bar{\sigma}_k \to \sigma_k.$$

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