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A Conjecture on Large Prime Gaps

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Abstract: We propose a conjecture on large prime gaps based on Littlewood's oscillatory theorem. While this conjecture is inconsistent with the classical conjecture of Cramér, we will provide evidences to this new conjecture on large prime gaps.

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1. Prime Gaps

Throughout let p denote a prime number and let p_n denote the n-th prime. Let x denote a positive integer. The n-th prime gap is $d_n := p_{n+1} - p_n$. Let $\Omega_\pm$ denote the big omega notation and O the big O notation. For the basic theory on prime numbers see any standard books on prime number theory and analytic number theory.

1.1. Small Prime Gaps

In 2005 Goldston, Pintz and Yıldırım [6] proved that

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0, \tag{1}$$

and improved this bound a bit in [7]. By a refinement of the method of Goldston-Pintz-Yıldırım, Zhang [16] proved that

$$\liminf_{n\to\infty} (p_{n+1} - p_n) < 7 \cdot 10^7, \tag{2}$$

which is the first result of bounded gaps between primes. Later on Maynard [10], Tao and the Polymath Project [12] reduced the bound of Zhang. The current bound is the following

$$\liminf_{n\to\infty} (p_{n+1} - p_n) \le 246. \tag{3}$$

These are the recent works towards to the old twin prime conjecture, which says that there are infinitely many twin prime numbers. In general Polignac's conjecture says that for every positive even integer $2k$, there are infinitely many primes p such that $p + 2k$ is also prime.

1.2. Large Prime Gaps

Unlike the case of small prime gaps, which has been made much progress in recent years. The advance to the problem of large prime gaps is slow and it seems that studying the large prime gaps is more difficult than the case of small gaps.

In 1930 Hoheisel [8] showed that there is a constant $\theta < 1$ such that for sufficiently large n,

$$d_n < p_n^\theta.$$

The current best result of this type is due to Baker, Harman and Pintz [1], who proved in 2001 that for sufficiently large n,

$$d_n < p_n^{0.525}.$$

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It should be mentioned that assuming the Riemann hypothesis, H. Cramér proved [3] that $d_n = O(\sqrt{p_n}\log p_n)$, and thus

$$d_n = O(p_n^{1/2+\varepsilon}) \tag{4}$$

where $\varepsilon > 0$. We shall give a new proof of this estimate later.

Another type of results on large prime gaps was started by Westzynthius [15], who showed in 1931 that

$$\limsup_{n\to\infty} \frac{d_n}{\log p_n} = \infty.$$

Later Rankin [14] improved this and proved that there is a $c > 0$ such that

$$d_n > \frac{c\log n \log\log n \log\log\log\log n}{(\log\log\log n)^2}$$

holds infinitely often. The current record of this type is due to Ford, Green, Konyagin, Maynard and Tao [5], who showed that

$$d_n > \frac{c\log n \log\log n \log\log\log\log n}{\log\log\log n}$$

holds infinitely often. This result improves the one of Westzynthius by logarithmic factors.

2. A New Proof of (4)

Recall that the first Chebyshev function [11] is define as

$$\vartheta(x) = \sum_{p\le x} \log p.$$

It is well known [4] that the Riemann hypothesis is equivalent to that for $\varepsilon > 0$,

$$\vartheta(x) = x + O\left(x^{1/2+\varepsilon}\right). \tag{5}$$

In the following we give a new proof of (4) under the Riemann hypothesis.

Theorem 1 (Cramér)**.** *The Riemann hypothesis implies $d_n = O(p_n^{1/2+\varepsilon})$.*

Proof. By (5) the Riemann hypothesis implies that as $x \to \infty$,

$$\vartheta(x) = x + O\left(x^{1/2+\varepsilon}\right).$$

Thus we have as $n \to \infty$,

$$\vartheta(p_n) = p_n + O\left(p_n^{1/2+\varepsilon}\right), \tag{6}$$

$$\vartheta(p_{n+1}-1) = p_{n+1} - 1 + O\left((p_{n+1}-1)^{1/2+\varepsilon}\right). \tag{7}$$

Since $\vartheta(p_n) = \vartheta(p_{n+1}-1)$, thus

$$p_n + O\left(p_n^{1/2+\varepsilon}\right) = p_{n+1} - 1 + O\left((p_{n+1}-1)^{1/2+\varepsilon}\right). \tag{8}$$

Therefore

$$d_n = O\left(p_n^{1/2+\varepsilon}\right) - O\left((p_{n+1}-1)^{1/2+\varepsilon}\right). \tag{9}$$

Notice that $(p_{n+1}-1)^{1/2+\varepsilon} = O\left(p_n^{1/2+\varepsilon}\right)$ since $p_{n+1} < 2p_n$, we conclude

$$d_n = O(p_n^{1/2+\varepsilon}). \tag{10}$$

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3. Our Conjecture on Large Prime Gaps

Note that we let $\Omega_{\pm}$ denote the big omega notation. The following is Littlewood's oscillatory theorem.

Theorem 2 ([9]). *As $x \to \infty$,*

$$\vartheta(x) - x = \Omega_{\pm}\left(x^{1/2} \log\log\log x\right).$$

By [11, p481, Exercise 15.2.1-5], the implicit constant in Littlewood's theorem can be taken to be 1/2. There is a similar result for prime arguments and the implicit constant in this version can be taken to be 1/4, followed from the proof of [2, Lemma 9.14].

Lemma 1 ([2, Lemma 9.14]). *There are infinitely many primes p such that*

$$\vartheta(p) < p - \frac{1}{4}\sqrt{p} \log\log\log p,$$

and also infinitely many primes p such that

$$\vartheta(p) > p + \frac{1}{4}\sqrt{p} \log\log\log p.$$

For the large prime gaps Cramér conjectured [3] that

$$\limsup_{n\to\infty} \frac{d_n}{\log^2 p_n} = 1 \tag{11}$$

and it fits well for small prime numbers. Nevertheless in the theory of prime numbers, the oscillatory phenomena often happens for large numbers. Examples include Skewes's number and Littlewood's oscillatory theorem.

By Lemma 1 there are infinitely many primes p such that

$$\vartheta(p) > p + \frac{1}{4}\sqrt{p} \log\log\log p,$$

and infinitely many primes p such that

$$\vartheta(p) < p - \frac{1}{4}\sqrt{p} \log\log\log p.$$

Thus it is highly likely that there exists n such that

$$p_n < \vartheta(p_n),$$

and

$$p_{n+1} > \vartheta(p_{n+1}) + \frac{1}{4}\sqrt{p_{n+1}} \log\log\log p_{n+1} = \vartheta(p_n) + \log p_{n+1} + \frac{1}{4}\sqrt{p_{n+1}} \log\log\log p_{n+1}.$$

If so, then $d_n > \log p_{n+1} + \frac{1}{4}\sqrt{p_{n+1}} \log\log\log p_{n+1}$ which suggests the falsity of Cramér's conjecture. Based on the oscillatory property of $\vartheta(p) - p$ we propose the following conjecture.

Conjecture 3. *There are infinitely many n such that*

$$d_n > \log p_{n+1} + \frac{1}{4}\sqrt{p_{n+1}} \log\log\log p_{n+1} > \frac{1}{4}\sqrt{p_n} \log\log\log p_n. \tag{12}$$

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In particular,

$$\limsup_{n\to\infty} \frac{d_n}{\sqrt{p_n}} = \infty, \quad \limsup_{n\to\infty} \frac{d_n}{\sqrt{p_n}\log\log\log p_n} > 0. \tag{13}$$

If this conjecture is true then Cramér's theorem 1 indicates that the Riemann hypothesis implies essentially the near best possible estimate of large prime gaps. In the next section we give two evidences to our conjecture.

4. Evidences to Conjecture 3

4.1. First Evidence

Let

$$M(n) = \max_{0<i<n} p_{n-i}p_{n+i}.$$

Pomerance [13] proved that

$$\limsup_{n\to\infty} \frac{p_n^2 - M(n)}{\log^2 n} \geq 1. \tag{14}$$

See [13, Theorem 3.1] and the remark that follows. He then made the following conjecture.

Conjecture 4 ([13], p.405, (5.4))**.**

$$\limsup_{n\to\infty} \frac{p_n^2 - M(n)}{p_n} > 0. \tag{15}$$

Pomerance noticed that this conjecture would be true from the proof of [13, Theorem 3.1], together with an additional condition (see [13, p.405]). For simplicity we denote this additional condition by X.

Now let us assume that Conjecture 4 is true, then since $p_{n+1}p_{n-1} \leq M(n)$, we have

$$\limsup_{n\to\infty} \frac{p_n^2 - p_{n+1}p_{n-1}}{p_n} \geq \limsup_{n\to\infty} \frac{p_n^2 - M(n)}{p_n} > 0. \tag{16}$$

The left limit is

$$\limsup_{n\to\infty} \frac{p_n^2 - p_{n+1}p_{n-1}}{p_n} = \limsup_{n\to\infty} \frac{p_n^2 - (p_n + d_n)(p_n - d_{n-1})}{p_n}. \tag{17}$$

Suppose there is an infinite subsequence of n for condition X that also satisfies

$$d_n \geq d_{n-1}. \tag{18}$$

Then

$$\limsup_{n\to\infty} \frac{p_n^2 - (p_n + d_n)(p_n - d_n)}{p_n} \geq \limsup_{n\to\infty} \frac{p_n^2 - (p_n + d_n)(p_n - d_{n-1})}{p_n} > 0, \tag{19}$$

and thus

$$\limsup_{n\to\infty} \frac{d_n^2}{p_n} > 0. \tag{20}$$

To sum up under the condition X and the condition there is an infinite subsequence of n for condition X that also satisfies (18) (both of which are likely to be true) , then there are infinitely many n such that

$$d_n \gg \sqrt{p_n}. \tag{21}$$

4.2. Second Evidence

By (1) we have

$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}}} = 0. \tag{22}$$

If this fraction has a limit, that is if

$$\lim_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}}} = 0, \tag{23}$$

then let us see what will probably happen.

We recall the well known Stolz-Cesàro theorem.

Theorem 5 (Stolz-Cesàro)**.** *Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers. Assume that $(b_n)_{n\geq 1}$ is a strictly monotone and divergent sequence and the following limit exists:*

$$\lim_{n\to\infty} \frac{a_n - a_{n-1}}{b_n - b_{n-1}} = k.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = k.$$

Thus

$$\lim_{n\to\infty} \frac{p_{n+1} - p_n - (p_n - p_{n-1})}{\sqrt{p_{n+1}} - \sqrt{p_n}} = 0 \tag{24}$$

implies that

$$\lim_{n\to\infty} \frac{p_{n+1} - p_n}{\sqrt{p_{n+1}}} = 0. \tag{25}$$

We have

$$\frac{p_{n+1} - p_n - (p_n - p_{n-1})}{\sqrt{p_{n+1}} - \sqrt{p_n}} = \frac{d_n - d_{n-1}}{\sqrt{p_n + d_n} - \sqrt{p_n}} > \frac{d_n - d_{n-1}}{\sqrt{d_n}}. \tag{26}$$

Suppose there is a sequence n_ℓ such that $d_{n_\ell - 1} = o(d_{n_\ell})$ (which is likely true), then

$$\limsup_{n\to\infty} \frac{p_{n+1} - p_n - (p_n - p_{n-1})}{\sqrt{p_{n+1}} - \sqrt{p_n}} \approx \sqrt{d_{n_\ell}} \to \infty, \tag{27}$$

and thus (24) would be false. All the arguments provide evidence to Conjecture 3.

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