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Planck Length and Metric Geometry

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Abstract: In this article, the problem of Planck length is considered in the language of metric geometry. To do this, we explicitly construct a geodesic in the Gromov-Hausdorff space of isometry classes of compact metric spaces connecting a non-Archimedean space with an Archimedean one.

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1. The Planck Length Problem

The Planck length is the following combination of fundamental constants, having the dimension of length:

$$\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}}.$$

Numerical value of the Planck length is $\ell_{Pl} \approx 1.61 \cdot 10^{-33} cm$.

The physical meaning of the Planck length is as follows. This is a scale on which it is fundamentally impossible to consider the theory of gravity without taking into account the quantum effects [1], since it is on the Planck scale that the values with the dimension of length inherent for gravity theory (the Schwarzschild radius of a spherically symmetric black hole) coincide with those for quantum theory (the Compton wavelength). Really, the Compton wavelength is given by the expression

$$\lambda_C = \frac{\hbar}{mc},$$

and the Schwarzschild radius is

$$r_g = \frac{2Gm}{c^2}.$$

It is easy to see that the equality takes place:

$$\ell_{Pl}^2 = \frac{\lambda_C r_g}{2}.$$

The appearance of a black hole on Planck scales does not allow us to obtain information about the structure of space on scales smaller than the Planck length.

In [2], it was conjectured that this kind of effect is associated with a fundamental change in the geometry of space on the Planck scale. Namely, the existence of unmeasurable regions of space is the result of a violation of Archimedes' axiom (the axiom of measurability) in Euclidean geometry. A conjecture about the non-Archimedean nature of space on Planck scales was formulated. This gave rise to the development of a new field in mathematical physics [3,4].

However, the question of the mechanism of changing the metric from Archimedean to non-Archimedean remains open. In this paper, an attempt is made to construct a model of metric change using the apparatus of metric geometry. Namely, a geodesic in the Gromov-Hausdorff space connecting ultrametric and ordinary metric spaces will be explicitly constructed. As a model example of an ultrametric space, we will consider the set $\mathbb{Z}_p$ of p-adic integers with a metric generated by the standard p-adic norm; as a model example of an ordinary metric space, we will choose the unit segment $[0,1] \subset \mathbb{R}$ with a standard metric generated by the absolute value.

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2. Metric Geometry. Basic Notions

A metric space is a pair $X = (X, d_X)$, where X is a set, d_X is a metric on X, that is, a mapping $d_X\colon X \times X \to [0, \infty)$ satisfying the conditions:

- $d_X(x, x') = 0 \iff x = x'$;
- $d_X(x, x') = d_X(x', x)$;
- $d_X(x, x'') \leq d_X(x, x') + d_X(x', x'')$.

If d_X satisfies the condition $d_X(x, x'') \leq \max\{d_X(x, x'), d_X(x', x'')\}$ then this is ultrametric, the space (X, d_X) is ultrametric (or non-Archimedean).

Important examples for the future are the following.

- $\mathbb{I} = [0, 1], d_{\mathbb{I}}(x, x') = |x - x'|$ – Archimedean space;
- $\mathbb{Z}_p, d_{\mathbb{Z}_p}(x, x') = |x - x'|_p$ – non-Archimedean space;
- $\Delta_m = \{x_1, \ldots x_m\}, d_{\Delta_m}(x_i, x_j) = 1, i \neq j, i, j = 1, 2, \ldots, m$ – simplex.

We define two operations on metric spaces: direct product and dilation.

Direct product $(X \times Y, d_{X\times Y})$ of the metric spaces X and Y is the Cartesian product of $X \times Y$ with the metric given by the expression

$$d_{X\times Y}\big((x, y), (x', y')\big) = \max\big\{d_X(x, x'), d_Y(y, y')\big\}.$$

Let $\lambda \in \mathbb{R}_+$ be a positive real number. The space λX obtained from the space (X, d_X) by dilation the metric has the form:

$$\lambda X = (X, \lambda d_X).$$

Let (X, d) be a metric space and $H = H(X)$ be a set of compact subsets of X. We define the metric (Hausdorff metric) d_H on H.

Let $A,\ B \in H(X)$,

$$d_H(A, B) = \inf\{\epsilon > 0 :\ B \subset U_\epsilon(A) \text{ and } A \subset U_\epsilon(B)\},$$

where $U_\epsilon(A) = \{x \in X :\ d(x, A) \leq \epsilon\}$.

$(H(X), d_H)$ is a metric space, and it is true that $H(X)$ is compact if and only if X is compact.

By means of GH, we denote the set of isometric classes of compact metric spaces. We introduce the metric on the set GH as follows [5,6].

The realization of the pair $X,\ Y$ of compact metric spaces is called the triple $(Z,\ X',\ Y')$, where Z is a metric space, $X' \subset Z,\ Y' \subset Z,\ X,\ Y$ are isometric to $X',\ Y'$, respectively, and $d_Z|_{X'} = d_X, d_Z|_{Y'} = d_Y$.

$$d_{GH}(X, Y) = \inf_{\text{realizations of X,Y}} d_H(X', Y').$$

(GH, d_{GH}) is a complete separable metric space.

3. Calculation of Distances

The following Theorems are valid.

Theorem 1.

$$d_{GH}(\mathbb{I}, \mathbb{Z}_p) = \frac{1}{2}.$$

Theorem 2. *Let X be a connected compact metric space,* $\operatorname{diam} X = 1$*. Then we have:*

$$d_{GH}(X, \mathbb{Z}_p) = \frac{1}{2}.$$

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Theorem 3. *Let k be a positive integer such that the inequalities $p^k < q < p^{k+1}$ are satisfied. Then equality is valid:*

$$2d_{GH}(\mathbb{Z}_p, \mathbb{Z}_q) = 1 - \frac{1}{p^k}.$$

A subset of $R(X,Y) \subset X \times Y$ of the direct product of the sets X and Y is called a correspondence if the projections of this subset onto the components of the product are surjective: $\mathrm{pr}_X R(X,Y) = X$, $\mathrm{pr}_Y R(X,Y) = Y$.

The distortion $\mathrm{dist}R(X,Y)$ of a corresponence $R(X,Y)$ is the following number:

$$\mathrm{dist}R(X,Y) = \sup_{(x,y),(x',y') \in R(X,Y)} |d_X(x,x') - d_Y(y,y')|$$

.

The following statement [6] is true:

$$d_{GH}(X,Y) = \frac{1}{2} \inf_{\text{correspondences } R(X,Y)} \mathrm{dist}R(X,Y).$$

This statement provides a convenient way to calculate distances in the Gromov-Hausdorff space. Here are some simple examples.

Example 1. *Let $R(X,Y) = X \times Y$, then*

$$\mathrm{dist}R(X,Y) = \max\{\mathrm{diam}X, \mathrm{diam}Y\}.$$

Therefore,

$$2d_{GH}(X,Y) \leq \max\{\mathrm{diam}X, \mathrm{diam}Y\}.$$

Example 2. $2d_{GH}(X, \Delta_1) = \mathrm{diam}X$. *Using the triangle inequality*

$$d_{GH}(X, \Delta_1) \leq d_{GH}(X,Y) + d_{GH}(Y, \Delta_1),$$

we get:

$$2d_{GH}(X,Y) \geq |\mathrm{diam}X - \mathrm{diam}Y|.$$

Example 3. *Let $f\colon X \to Y$ be surjective. Then the graph $\{(x, f(x)), x \in X\}$ is a correspondence.*

There are two important points:

- there is (not unique) optimal correspondence

$$R_{opt}(X,Y)\colon 2d_{GH} = \mathrm{dist}R_{opt}(X,Y);$$

- to calculate distances in the Gromov-Hausdorff space, it is enough to consider only closed correspondences.

Proof of Theorem 1. Let $R(\mathbb{Z}_p, \mathbb{I})$ be an arbitrary closed correspondence. Let's consider $\mathbb{Z}_p$ as a disjoint union of p balls of radius $1/p$, $\mathbb{Z}_p = \sqcup_{i=1,\ldots,p} B^i_{1/p}$. The family of subsets $\mathbb{I}$ of the form $\{\mathrm{pr}_{\mathbb{I}} R(B^i_{1/p}, \mathbb{I}), i = 1, \ldots, p\}$ forms a covering of the segment $\mathbb{I}$ by closed subsets. Since $\mathbb{I}$ is connected, at least two sets of our coverage have a common point. The projections on $\mathbb{Z}_p$ of the preimages of this common point lie in different balls B_i and B_j. Therefore, the distance in $\mathbb{Z}_p$ between the projections of the preimages is equal to one. Thus, $\mathrm{dist}R(\mathbb{Z}_p, \mathbb{I}) \geq 1$. Since this is true for any correspondence, choosing the optimal one yields $2d_{GH}(\mathbb{Z}_p, \mathbb{I}) \geq 1$. On the other hand, $2d_{GH}(\mathbb{Z}_p, \mathbb{I}) \leq \max\{\mathrm{diam}\,\mathbb{Z}_p, \mathrm{diam}\mathbb{I}\} = 1$.

Since we used only the connectivity of the space $\mathbb{I}$, the same proof works in the case of Theorem 2.

Proof of Theorem 3.

Let N be a positive integer. Let's consider $\mathbb{Z}_p$ as a disjoint union of p^N balls of radius $\epsilon = p^{-N}$. We will choose one point in each ball of the constructed partition. The set $X_N^{(p)}$ obtained in this way, consisting of p^N points, is provided with the metric d_N, induced by the metric on $\mathbb{Z}_p$. As a result, we get the metric space $(X_N^{(p)}, d_N)$. It is useful to note that the space $X_1^{(p)}$ is nothing but a simplex of Δ_p. Note that the Hausdorff distance between $\mathbb{Z}_p$ and $X_N^{(p)}$ is equal to ϵ. This immediately implies the validity of the evaluation of $d_{GH}(\mathbb{Z}_p, X_N^{(p)}) \le \epsilon$ (it suffices to consider the realization of the pair $(\mathbb{Z}_p, X_N^{(p)})$ of the form $Z = \mathbb{Z}_p = Y', X' = X_N^{(p)}$).

The following simple Lemma follows from the triangle inequality.

Lemma 1. *For any metric compact X, the inequality holds:*

$$\left| d_{GH}(X, \mathbb{Z}_p) - d_{GH}(X, X_N^{(p)}) \right| \le p^{-N}.$$

Let $MST(X_N^{(p)})$ be the minimum spanning tree of a finite metric space $X_N^{(p)}$. By means of $\sigma(X_N^{(p)})$, we denote the mst-spectrum of the space $X_N^{(p)}$, that is, the sequence of edge lengths of the minimum spanning tree in decreasing order. The following Lemma is valid.

Lemma 2.

$$\sigma(X_N^{(p)}) = \left\{ \underbrace{1, \ldots, 1}_{p-1} \underbrace{\frac{1}{p}, \ldots, \frac{1}{p}}_{p(p-1)}, \underbrace{\frac{1}{p^2}, \ldots, \frac{1}{p^2}}_{p^2(p-1)} \ldots, \underbrace{\frac{1}{p^{N-1}}, \ldots, \frac{1}{p^{N-1}}}_{p^{N-1}(p-1)} \right\}.$$

Let's decompose $\mathbb{Z}_p$ into a disjoint union of p balls of radius $1/p$: $\mathbb{Z}_p = \sqcup_i^p B_{1/p}^i$. In each of the partition balls, we will choose one element from the set $X_N^{(p)}$. The pairwise distances between the various elements of this set are equal to one, that is, it is a simplex Δ_p. It follows directly from this that $MST(X_N^{(p)})$ has exactly $p-1$ edge of length 1. Now each of the balls $B_{1/p}^i, i = 1, 2, \ldots, p$ of our partition let's decompose into disjoint union of p balls of radius $1/p^2$ (in total, we get p^2 balls of radius $1/p^2$) and let's do a similar reasoning for each of these balls. Continuing these arguments N times, we obtain the statement of the Lemma.

To further prove of Theorem 3, we first prove the estimate from below:

$$1 - \frac{1}{p^k} \le 2d_{GH}(\mathbb{Z}_p, \mathbb{Z}_q).$$

From the triangle inequality for the spaces $\mathbb{Z}_p, \mathbb{Z}_q, \Delta_{p^k}$ we obtain:

$$d_{GH}(\mathbb{Z}_p \mathbb{Z}_q) \ge d_{GH}(\Delta_{p^k}, \mathbb{Z}_q) - d_{GH}(\Delta_{p^k}, \mathbb{Z}_p).$$

Next, we will use the results of [7] (Theorem 3.3). The above theorem states, in particular, the following:

$$2d_{GH}(\Delta_m, X) = \max\{\sigma_1 - 1, \sigma_m, 1 - \sigma_{m-1}\},$$

where X is a finite ultrametric space consisting of n points, and $1 < m < n$.

Let's choose a positive integer $N, q < p^N$. Then the equality

$$2d_{GH}(\Delta_{p^k}, X_N^{(q)}) = 1$$

is valid, because $\sigma_1 = \sigma_{p-1} = \sigma_{p^k} = 1$. In addition, the following equality is true

$$2d_{GH}(\Delta_{p^k}, X_N^{(p)}) = 1/p^k,$$

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because in this case $\sigma_1 = \sigma_{p-1} = 1$ and $\sigma_{p^k} = 1/p^k$. Taking into account the last equalities for sufficiently large N, we obtain the required estimate from below.

To obtain an estimate from above, we construct the correspondence $R(\mathbb{Z}_p, \mathbb{Z}_q)$ explicitly and calculate its distortion.

Let's represent the number q as the sum of positive integers of the following form:

$$q = q_1 + q_2 + \cdots + q_{p^k},\ 1 \le q_i \le p,\ i = 1, 2, \ldots, p^k.$$

Note that in this representation, at least one of the terms is not equal to 1 (since $p^k < q$).

Let's decompose $\mathbb{Z}_p$ into a disjoint union of p^k balls of radius p^{-k}:

$$\mathbb{Z}_p = \sqcup_{i=1}^{p^k} B^i_{p^{-k}}.$$

Let's represent $\mathbb{Z}_q$ as a disjoint union of balls of radius q^{-1} in accordance with the above decomposition of the number q:

$$\mathbb{Z}_q = \sqcup_{i_1=1}^{q_1} B^{i_1}_{q^{-1}} \sqcup_{i_2=1}^{q_2} B^{i_2}_{q^{-1}} \cdots \sqcup_{i_1=1}^{q_{p^k}} B^{i_{p^k}}_{q^{-1}}.$$

Since any compact totally disconnected spaces are homeomorphic, there exists a homeomorphism $\phi\colon \mathbb{Z}_q \to \mathbb{Z}_p$ such that for all $j = 1, 2, \ldots p^k$ the conditions

$$\phi\left(\sqcup_{i_j=1}^{q_j} B^{i_j}_{q^{-1}}\right) = B^j_{p^{-k}}$$

are fulfilled.

As the desired correspondence, $R(\mathbb{Z}_q, \mathbb{Z}_p)$ let's take the graph of the map ϕ.

We'll show that the distortion of this correspondence is $1 - \frac{1}{p^k}$.

Let $x, x' \in \mathbb{Z}_q$: $|x - x'|_q \le \frac{1}{q}$, then the inequality $|\phi(x) - \phi(x')|_p \le \frac{1}{p^k}$ is fulfilled by the definition of the map ϕ. Indeed, the inequality $|x - x'|_q \le \frac{1}{q}$ means that x and x' lie inside a ball of radius $1/q$, and the image of each such ball lies inside a ball of radius $1/p^k$ in $\mathbb{Z}_p$. Therefore, for all such x and x', the inequality $\left||x - x'|_q - |\phi(x) - \phi(x')|_p\right| \le p^{-k}$ holds.

Now let x and x' lie in different balls of radius $1/q$ in $\mathbb{Z}_q$ (in this case, $|x - x'|_q = 1$). There are two possible cases here. The first is when x and x' lie in different groups of balls, and the second is when they lie in the same group of balls.

In the first case, we have $|\phi(x) - \phi(x')|_p \ge p^{-k+1}$, since $\phi(x)$ and $\phi(x')$ lie in different balls of radius p^{-k} in $\mathbb{Z}_p$. Therefore, the inequality

$$\left||x - x'|_q - |\phi(x) - \phi(x')||_p\right| \le 1 - p^{-k+1}$$

holds.

In the second case, $|\phi(x) - \phi(x')|_p \le p^{-k}$, since $\phi(x)$ and $\phi(x')$ lie in the same ball of radius p^{-k} in $\mathbb{Z}_p$.

Now we will impose an additional condition on the map ϕ. As noted earlier, in our partition of a set consisting of q balls of radius $1/q$ in $\mathbb{Z}_q$ into p^k groups of balls, there are groups (at least one) consisting of q_k balls such that the inequalities $2 \le q_k \le p$ are satisfied. The image of each such group under the map ϕ, is a ball of radius p^{-k} in $\mathbb{Z}_p$ (each group has its own). Let's decompose this ball into a disjoint union of p balls of radius p^{-k-1}, and divide this set into q_k groups (recall that $q_k \le p$). We will construct the map ϕ in such a way that each of the q_k balls is mapped into its own group. In this case, if x and x' lie in different balls from the group of q_k balls, then their images lie in different balls of radius p^{-k-1} inside a ball of radius p^{-k} and, thus, $|\phi(x) - \phi(x')|_p = p^{-k}$.

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Thus, we have obtained the following properties of the map ϕ:

$$\big||x-x'|_q - |\phi(x)-\phi(x')|_p\big| \le \frac{1}{p^k}, \text{ if } |x-x'|_q \le \frac{1}{q},$$

and in the case of $|x-x'|_q = 1$:

$$\big||x-x'|_q - |\phi(x)-\phi(x')|_p\big| \le 1 - \frac{1}{p^{k-1}}$$

or

$$\big||x-x'|_q - |\phi(x)-\phi(x')|_p\big| = 1 - \frac{1}{p^k}.$$

It follows directly from the last formulas that the graph of the constructed map ϕ has a distortion equal to $1-\frac{1}{p^k}$. Therefore, the inequality is valid $2d_{GH}(\mathbb{Z}_p,\mathbb{Z}_q) \le 1-\frac{1}{p^k}$. The theorem has been proved.

4. Geodesics

Note that the correspondence constructed during the proof of theorem 3 (the graph of the map $\phi\colon \mathbb{Z}_q \to \mathbb{Z}_p$) is optimal.

It is not difficult to construct an optimal correspondence $R(\mathbb{Z}_p,\mathbb{I})$ between $\mathbb{Z}_p$ and the unit interval $\mathbb{I}$.

As such a correspondence, consider the graph of the Monna map.

Let $\mathbb{Z}_p \ni x = x_0 + x_1 p + \dots x_k p^k + \dots$. The Monna map $\mu\colon \mathbb{Z}_p \to \mathbb{I}$ is given by the expression

$$\mu(x) = \frac{1}{p}(x_0 + x_1 p^{-1} + \dots + x_k p^{-k} \dots).$$

Calculate the distortion of the Monna map's graph. Let $x, x' \in \mathbb{Z}_p\colon |x-x'|_p = p^{-n}$. This means that $x_0 = x'_0, x_1 = x'_1, \dots, x_{n-1} = x'_{n-1}, x_n \ne x'_n$. Then the inequality is valid

$$|\mu(x) - \mu(x')| \le p^{-n}.$$

Therefore, for all $x, x'\colon |x-x'| < 1$, the estimate

$$\big||x-x'|_p - |\mu(x)-\mu(x')|\big| \le 1/p$$

is valid.

Now let $|x-x'|_p = 1$, that is, $x_0 \ne x'_0$. Let $x_0 > x'_0$ be for certainty, then $(x-x')_0 = x_0 - x'_0$ and the inequalities are valid

$$\frac{x_0 - x'_0}{p} \le |\mu(x)-\mu(x')| \le \frac{x_0 - x'_0 + 1}{p}.$$

It immediately follows that the distortion of the Monna map's graph is $1-\frac{1}{p}$. Taking into account theorem 1, it can be concluded that the Monna map's graph defines the optimal correspondence between $\mathbb{Z}_p$ and $\mathbb{I}$.

Our task is to construct a geodesic connecting $\mathbb{Z}_p$ and $\mathbb{I}$ in the Gromov-Hausdorff space. To do this, we will use the following result from the paper [8]:

Proposition 1. *Let $(X,d_X),(Y,d_Y)$ be compact metric spaces, then for any optimal correspondence $R_{opt}(X,Y)$ there is a family of compact metric spaces R_t such that $R_0 = X$, $R_1 = Y$ and for $t \in (0,1)$ $R_t = (R_{opt}(X,Y), d_t)$, where*

$$d_t((x,y),(x',y')) = (1-t)d_X(x,x') + t d_Y(y,y')$$

defines the shortest curve in GH connecting the spaces X and Y.

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Thus, the following statement is true.

Theorem 4. *The family of spaces* $(\mathbb{Z}_p, d_0 = |\cdot|_p)$, (Γ_μ, d_t), $(\mathbb{I}, d_1 = |\cdot|)$, *where* $\Gamma_\mu \subset \mathbb{Z}_p \times \mathbb{I}$ *denotes the graph of the Monna map,*

$$d_t((x, \mu(x)), (y, \mu(y))) = (1-t)|x-y|_p + t|\mu(x) - \mu(y)|,\ t \in (0,1),$$

defines the shortest curve connecting $\mathbb{Z}_p$ *and* $\mathbb{I}$ *in the Gromov-Hausdorff space.*

A geodesic connecting $\mathbb{Z}_p$ and $\mathbb{Z}_q$ is constructed in a similar way. To do this, instead of the graph of the Monna map, we need to take the graph of the map ϕ from the proof of theorem 3.

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