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*Article*

# On Entropy-Driven Hemi-Inner Products

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**Abstract:** It is well known that the inner product defines the norm in a Hilbert space, and conversely, the norm determines the inner product via the polarization identity. Since both norms and entropy functions are non-negative, one may formally define a weak form of an entropy-induced inner product by means of the polarization identity. We present several identities traditionally associated with genuine inner products that continue to hold in a more general setting. In conclusion, an entropy function can endow a simple algebraic structure with properties reminiscent of Hilbert spaces. We consider various examples and show that an entropy can be reconstructed from a weak inner product in case of cancellative commutative semigroups.

**Keywords:** magma; hemi-associativity; entropy; hemi-inner product; Hilbert space

## 1. Introduction

Inner products play an essential role in physics, as they serve to represent measurements. For instance, special relativity postulates that two four-dimensional worlds—the observer's and the observed—are related by an angle determined by their relative velocity, as formalized through the Wick rotation. The observed times and positions are obtained via projections, i.e., projective measurements. Since in physics, measurements determine reality, relativistic time dilation is considered a physical reality. From a mathematical perspective, a projection implies actuality. Although in quantum mechanics a measurement is more generally described by a positive semi-definite operator, its interpretation essentially reduces to that of a projection.

Orthogonal projections are inseparable from Hilbert spaces and their inner products; the inner product defines the size of the shadow cast by a projection [1]. Shadows can also be interpreted linguistically in entirely different frameworks. For instance, the shadow of a set $A$ onto another set $B$ can be understood as $A \cap B$, and the size of the shadow is given by $\mu(A \cap B)$ for some measure $\mu$. [2,3] draw upon these analogies and integrate Hilbert space theory with the intuitive notion of a shadow in set theory in an algebraic approach to entropy, which also encompasses Shannon's entropy. The work proposes a generalized, entropy-driven, weak form of an inner product:

$$\langle x, y \rangle \quad = \quad c([\![x \dotplus y]\!] - [\![x]\!] - [\![y]\!]), \tag{1}$$

where $[\![.]\!]$ denotes a generalized entropy measure and some constant $c \in \mathbb{R}\backslash\{0\}$. For example, in Hilbert space theory, $x$ and $y$ are vectors, $\dotplus$ denotes standard vector addition, $[\![.]\!]$ is the squared norm and $c = 1/2$. In set theory, $x$ and $y$ are sets, $\dotplus$ denotes the union, $[\![.]\!] = \mu$ and $c = -1$. [4] introduces, in a mathematically rigorous manner, a weak associativity law, called hemi-associativity, along with further weak laws; see Definition 5 below. For instance, under the assumption of hemi-associativity, the following identity holds:

$$\langle x \dotplus y, z \rangle \quad = \quad \langle x, y \dotplus z \rangle - \langle x, y \rangle + \langle y, z \rangle, \qquad \forall x, y, z \in G. \tag{2}$$

In quantum mechanics (specifically, in representation theory), a map that satisfies this property up to exponentiation is known as a 2-cocycle [5, p. 113].

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In this paper, we demonstrate that, under such weak assumptions, general identities for weak inner products, as well as connections between inner products and their defining entropy, can be derived. In particular, under more restrictive conditions, the entropy can be reconstructed from a given inner product, although not uniquely.

For the reader's convenience, we recall the necessary definitions from [2–4] in Section 2. The main results are presented in Section 3; see [2] also for preliminary findings. Proofs are provided in Section 4. The paper concludes with some final remarks in Section 5.

## 2. Definitions

Here, we follow closely the wording in [2–4].

**Definition 1.** *Let $G$ and $S$ be non-empty sets. Let $\dotplus$ be a dyadic operation on $G$ and $[\![.]\!] : G \to S$ a map. Abbreviate $[\![x]\!] = [\![y]\!]$ by $x \Join y$ and let*

$$G_s = \{\varepsilon \in G : x \dotplus \varepsilon \Join x \quad \forall x \in G\}. \tag{3}$$

*If $G_s$ is not empty and closed, i.e., $\varepsilon, \tilde{\varepsilon} \in G_s$ implies $\varepsilon \dotplus \tilde{\varepsilon} \in G_s$, then the tuple $(G, \dotplus, [\![.]\!])$ is called a* hemi-unital magma.

**Definition 2.** *Let $G$ be a Hausdorff space and $[\![.]\!] : G \to [0, \infty]$ a map. Let the set of zero elements $G_0 = \{\varepsilon \in G : [\![\varepsilon]\!] = 0\}$ be a Borel set, neither empty nor the whole space $G$. If the map $[\![.]\!]$ is continuous on $G \backslash G_0$, then $[\![.]\!]$ is called an* entropy measure *for $G$ (or an* entropy *on $G$).*

Continuity of the entropy is both physically motivated and mathematically useful, since important properties of the entropy can be shown only on a dense subset, see Theorem 1 below. A discontinuity at $\partial G_0$ appears in natural choices of the entropy, such as the geostatistical variogram [6,7] or the Kullback-Leibler-divergence [8], when the latter is considered as an entropy, cf. [9, Remark 2].

For formal reasons, the set $G$ should be embedded in a larger frame. Speaking in stochastics terms, we are here interested essentially only in a single random variable $z$ living on $G$. The operator $\circ$ in the next definition deals with connecting an independent random variable $z$ to several independent random variables $x$, so that $G$ is formally imbedded into a product space, denoted by $G^*$.

**Definition 3.** *Let $G^*$ be a Hausdorff space and $[\![.]\!] : G^* \to [0, \infty]$ a map. Let $G \subset G^*$ be a measurable subset and $\circ : G^* \times G \to G^*$ a measurable map. If $[\![.]\!]$ restricted to $G$ is an entropy and*

$$[\![x \circ z]\!] = [\![x]\!] + [\![z]\!] \quad \forall x \in G^*, z \in G,$$

*then we call $(G, \circ, [\![.]\!])$ an* entropy-driven magma.

A kernel is a function on $G \times G$.

**Definition 4.** *Let $(G, \circ, [\![.]\!])$ be an entropy-driven magma with $[\![x]\!] < \infty$ for all $x \in G$. Assume, a dyadic operation $\dotplus$ on $G$ exists, such that*

$$[\![x \dotplus \varepsilon]\!] \quad = \quad [\![x]\!], \qquad \forall x \in G, \varepsilon \in G_s \tag{4}$$

*for some non-empty subset $G_s$ of $G_0$. If the function*

$$x \mapsto [\![x \dotplus x]\!] - 2[\![x]\!]$$

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*does not change sign, then the tuple* $(G, \circ, \dotplus, [\![.]\!])$ *is said to have* comparable elements. *The sign is denoted by* sign $[\![\cdot]\!]$ *with* sign $[\![\cdot]\!] \in \{-1, +1\}$. *The kernel*

$$\langle .,. \rangle : G \times G \to \mathbb{R}, \quad (x, y) \mapsto \text{sign}\,[\![\cdot]\!]([\![x \dotplus y]\!] - [\![x \circ y]\!])$$

*is called* hemi-inner product.

In this paper we neglect the fact, that variants of an inner product can be obtained by multiplying with a positive constant, as it is done in Equation (1).

**Remark 1.** *Equation (4) guarantees that $G_s$ is closed with respect to the operation $\dotplus$, hence $(G, \dotplus, [\![.]\!])$ is a hemi-unital magma. There can be a subtle difference between $G_s$ and $G_0$, e.g., when contrasting a deterministic value with an almost surely constant random variable.*

**Definition 5.** *Let $(G, \dotplus, [\![.]\!])$ be a hemi-unital magma. Assume further that*

$$\varepsilon \dotplus x \quad \boxminus \quad x, \qquad \forall x \in G, \varepsilon \in G_s. \tag{5}$$

*Then, the operation $\dotplus$ is called* hemi-associative, *if*

$$a \dotplus ((x \dotplus y) \dotplus z) \dotplus b \quad \boxminus \quad a \dotplus (x \dotplus (y \dotplus z)) \dotplus b, \qquad \forall a, b, x, y, z \in G,$$

*and* hemi-commutative, *if*

$$a \dotplus (y \dotplus x) \dotplus b \quad \boxminus \quad a \dotplus (x \dotplus y) \dotplus b \qquad \forall a, b, x, y \in G.$$

**Definition 6.** *Let $(G, \dotplus, [\![.]\!])$ be a hemi-unital magma. The dyadic operation $\dotplus$ is called* wide-left-modular, *if, for all $a, b, x, y, z \in G$, we have*

$$a \dotplus x \dotplus y \dotplus z \dotplus b \quad \boxminus \quad x \dotplus (a \dotplus y \dotplus z \dotplus b), \tag{6}$$

$$a \dotplus (x \dotplus y) \dotplus z \dotplus b \quad \boxminus \quad a \dotplus x \dotplus (z \dotplus y) \dotplus b. \tag{7}$$

## 3. Main Results

We write $nx = x \dotplus \ldots \dotplus x$ ($n$ times) for $n \in \mathbb{N}$ and $x \in G$, and let $0x \in G_s$. We understand $a \dotplus nx \dotplus b$ as $a \dotplus (nx) \dotplus b$. We write $n * x$ for reverse parantheses, i.e.,

$$n * x \quad = \quad x \dotplus (x \dotplus (\ldots (x \dotplus (x \dotplus x)) \ldots)). \tag{8}$$

Again, we let $0 * x \in G_s$. We write $\star$, if an equation is true when all $\star$ are replaced by either the $*$ or the $\cdot$ operation. We understand $\sum_{i=m}^{m-1}(.)$ as $0 \in \mathbb{R}$.

**Proposition 1.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be an entropy-driven magma with comparable elements. If $\dotplus$ is hemi-associative, then*

$$\langle x \dotplus y, z \rangle \quad = \quad \langle x, y \dotplus z \rangle - \langle x, y \rangle + \langle y, z \rangle \qquad \forall x, y, z \in G. \tag{9}$$

*If $\dotplus$ is wide-left-modular, then*

$$\langle x \dotplus y, z \rangle \quad = \quad \langle y, x \dotplus z \rangle - \langle x, y \rangle + \langle x, z \rangle \qquad \forall x, y, z \in G. \tag{10}$$

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**Proposition 2.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be a entropy-driven magma with comparable elements. If the operation $\dotplus$ is hemi-commutative or wide-left-modular, then, for all $x, y \in G$,*

$$\langle x, y\rangle \quad = \quad \langle y, x\rangle.$$

**Proposition 3.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be an entropy-driven magma with comparable elements. If $\dotplus$ is hemi-associative, hemi-commutative or wide-left-modular, then, for all $x \in G$ and any choice of $\star$,*

$$\langle i \star x, j \star x\rangle \quad = \quad \langle j \star x, i \star x\rangle, \quad i, j \in \mathbb{N}_0. \tag{11}$$

**Proposition 4.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be an entropy-driven magma with comparable elements. Fix some choice of $\star$. If*

$$\langle x, i \star x\rangle = \langle i \star x, x\rangle \qquad \forall i \in \mathbb{N}_0, x \in G, \tag{12}$$

*then, for all $x \in G$,*

$$\operatorname{sign}[\![\cdot]\!] \sum_{k=1}^{j} \langle (i+k-1) \star x, x\rangle \quad = \quad [\![(i+j) \star x]\!] - [\![i \star x]\!] - j[\![x]\!], \quad i, j \in \mathbb{N}_0, \tag{13}$$

$$\operatorname{sign}[\![\cdot]\!] \sum_{k=1}^{j} \langle k \star x, x\rangle \quad = \quad [\![(j+1) \star x]\!] - (j+1)[\![x]\!], \quad j \in \mathbb{N}_0. \tag{14}$$

**Proposition 5.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be a entropy-driven magma with comparable elements. If one the following conditions*

1. *$\dotplus$ is hemi-associative,*
2. *$\dotplus$ is wide-left-modular and $\star$ equals $*$,*

*is satisfied, then, for all $x \in G$ and $i, j, m \in \mathbb{N}$, we have*

$$\langle i \star x, j \star x\rangle \quad = \quad \sum_{k=1}^{i+j-1} \langle k \star x, x\rangle - \sum_{k=1}^{j-1} \langle k \star x, x\rangle - \sum_{k=1}^{i-1} \langle k \star x, x\rangle, \tag{15}$$

$$\sum_{k=1}^{m-1} \langle i \star x, (ki) \star x\rangle \quad = \quad \sum_{k=1}^{mi-1} \langle k \star x, x\rangle - m \sum_{k=1}^{i-1} \langle k \star x, x\rangle. \tag{16}$$

**Proposition 6.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be a wide-left-modularentropy-driven magma with comparable elements. If, additionally, one of the following conditions is satisfied,*

1. *$\dotplus$ is hemi-associative,*
2. *$\star$ equals $*$,*

*then, for all $x, y \in G$ and $r, s \in \mathbb{N}$, we have*

$$\begin{aligned}
\langle r \star x \dotplus r \star y, x \dotplus y\rangle \quad &= \quad \langle r \star (x \dotplus y), x \dotplus y\rangle, \\
\sum_{k=1}^{rs-1} \langle k \star (x \dotplus y), x \dotplus y\rangle \quad &= \quad \langle r \star (s \star x), s \star (r \star y)\rangle - rs\langle x, y\rangle \\
&\quad + \sum_{k=1}^{r-1} \langle k \star (s \star x), s \star x\rangle + \sum_{k=1}^{s-1} \langle k \star (r \star y), r \star y\rangle \\
&\quad + r \sum_{k=1}^{s-1} \langle k \star x, x\rangle + s \sum_{k=1}^{r-1} \langle k \star y, y\rangle.
\end{aligned} \tag{17}$$

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Crucial for the reconstruction of $[\![.]\!]$ from the hemi-inner product is the condition that $M_{x,e} \in (0, \infty)$ for

$$M_{x,e} = \sup_{y \in G, m,n \in \mathbb{N}, m\star y = n \star x} \frac{e}{n}\left(\sum_{k=1}^{m-1} \langle k \star y, y\rangle - \sum_{k=1}^{n-1} \langle k \star x, x\rangle\right) \tag{18}$$

and some $x \in G$ and some $e \in \{-1, +1\}$. Since $M_{x,e}$ is always non-negative, $M_{x,e}$ might be considered as an entropy if $y \mapsto M_{y,e}$ is not identically 0.

**Example 1** (Pre-Hilbert space)**.** *Let $H$ be a pre-Hilbert space with inner product $\langle .,.\rangle_H$. Let $[\![x]\!] = \langle x, x\rangle_H$ and $\dotplus$ be the standard addition in $H$. Then, $\langle .,.\rangle = 2\langle .,.\rangle_H$. Let $y = nx/m$. Then,*

$$\begin{aligned} 2\sum_{k=1}^{m-1}\langle ky, y\rangle_H - 2\sum_{k=1}^{n-1}\langle kx, x\rangle_H &= 2\left(\frac{m(m-1)}{2}\cdot\frac{n^2}{m^2} - \frac{n(n-1)}{2}\right)\langle x, x\rangle_H \\ &= n\left(1 - \frac{n}{m}\right)\langle x, x\rangle_H. \end{aligned}$$

*Hence, $M_{x,-1} = \infty$ and $M_{x,1} = \langle x, x\rangle_H = [\![x]\!] = \langle x, x\rangle/2$. Obviously, Eq. (18) is far away from the standard interpretation of what a squared norm is.*

**Example 2** (Periodic semigroups)**.** *A semigroup is called periodic if $\{\ell x : \ell \in \mathbb{N}\}$ is finite for all $x \in G$ [10], i.e., for $x \in G$ minimal natural numbers $r, s \in \mathbb{N}$ exist, such that $rx = (r+s)x$. Let $A = \sum_{i=r}^{r+s-1}\langle ix, x\rangle$. If $A > 0$, then*

$$M_{x,e} \geqslant \frac{e}{r+ks}\left(\sum_{k=1}^{r+\ell s-1}\langle k\star x, x\rangle - \sum_{k=1}^{r+ks-1}\langle k\star x, x\rangle\right) = \frac{e}{r+ks}(\ell - k)A, \qquad k,\ell \in \mathbb{N},$$

*so that $M_{x,\operatorname{sign} A} = \infty$ and*

$$M_{x,-\operatorname{sign} A} \geqslant \frac{|A|}{s}.$$

*In some special cases equality can be shown. If $G$ is idempotent, i.e., $2x = x$ for all $x \in G$, then $M_{x,-1} = \langle x, x\rangle$. An element $x$ of a semigroup $G$ with neutral element 0 is called nilpotent, if $nx = 0$ for some $n \in \mathbb{N}$. We are interested in the case $2x = 0$, but consider the slightly more general case of an arbitrary magma $G$ such that, for all $x \in G$, $k \star x = x$ if $k$ is odd and $\langle k\star x, x\rangle = 0$ if $k$ is even. Then, $M_{x,1} = \infty$ and $M_{x,-1} = \langle x, x\rangle/2$.*

**Example 3** (Variogram)**.** *A symmetric, real-valued kernel $g$ over a set $G$ is called negative definite if*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j g(x_i, x_j) \leqslant 0$$

*for all $n \in \mathbb{N}$, $x_i \in G$ and $a_i \in \mathbb{R}$, with $\sum_{i=1}^{n} a_i = 0$. Let $G = \mathbb{R}^d$. A function $\gamma : G \to [0, \infty)$ is called a variogram if the map $(x, y) \mapsto \gamma(x - y)$ is a negative definite kernel and $\gamma(0) = 0$. The corresponding covariance $C$ satisfies [6]*

$$C(x, y) = \gamma(x) + \gamma(y) - \gamma(x - y).$$

*In most practical cases, $\gamma$ has a jump at the origin. In all practical cases, $\gamma$ is continuous outside the origin [7]. Let $x \dotplus y$ be defined as $x - y$ and $[\![.]\!] = \gamma$. Then, $\dotplus$ is wide-left-modular and the covariance $C$ is the hemi-inner product. Example 2 shows that $\gamma(x)$ can be retrieved by $C$ as $M_{x,-1}$. Note that $C$ is an ordinary positive semi-definite kernel [6].*

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**Example 4** (Extreme values). *Let $G = [0, \infty)$, and $\dotplus$ be the ordinary maximum denoted by $\vee$. Let $[\![x]\!] = x^\alpha$ for some fixed $\alpha > 0$. Then, $\operatorname{sign}[\![\cdot]\!] = -1$ and*

$$\langle x, y\rangle = x^\alpha + y^\alpha - (x \vee y)^\alpha = (x \wedge y)^\alpha, \qquad x, y \in [0, \infty).$$

*Since $m * x = x$ we have $y = x$ and*

$$\sum_{k=1}^{m-1}\langle ky, y\rangle - \sum_{k=1}^{n-1}\langle kx, x\rangle = (m-n)x^\alpha.$$

*Hence, $M_{x,1} = \infty$ and $M_{x,-1} = x^\alpha = [\![x]\!]$. Similar to the hemi-inner product, [2,3] also introduce a canonical hemi-metric, which in this example is also a genuine metric, used in extreme value theory [11,12].*

**Example 5** (Cyclic semigroup). *Let $G$ be a cyclic (monogenic) semigroup, i.e., $G = \{\ell x : \ell \in \mathbb{N}\}$ for some $x \in G$ [10,13]. Additionally, a neutral element may be included. If $|G| = \infty$, the kernel $\langle ., .\rangle$ is symmetric and Equation (16) holds, then*

$$\begin{aligned} M_{\ell x, e} &= \sup_{n,m\in\mathbb{N}, m|n\ell} \frac{e}{n}\left(\sum_{k=1}^{m-1}\langle k\frac{n\ell}{m}x, \frac{n\ell}{m}x\rangle - \sum_{k=1}^{n-1}\langle k\ell x, \ell x\rangle\right) \\ &= \sup_{n,m\in\mathbb{N}, m|n\ell} e\left(\sum_{k=1}^{\ell-1}\langle kx, x\rangle - \frac{m}{n}\sum_{k=1}^{n\ell/m-1}\langle kx, x\rangle\right). \end{aligned}$$

*If $\sum_{k=1}^{n}\langle kx, x\rangle \geqslant 0$ for all $n \in \mathbb{N}$, then $M_{\ell x,1} = \sum_{k=1}^{\ell-1}\langle kx, x\rangle$.*
*For instance, let $G = \mathbb{N}_0$, the operation $\dotplus$ be the standard addition and $[\![\ell]\!] = \ell^2$, $\ell \in \mathbb{N}_0$. Then, $\langle \ell_1, \ell_2\rangle = 2\ell_1\ell_2$ and $M_{\ell,1} = \ell(\ell-1)$. Let $[\![.]\!]_\#$ be another entropy defined as $[\![\ell]\!]_\# = \ell(\ell-1)$, $\ell \in \mathbb{N}_0$. Then, its hemi-inner product $\langle ., .\rangle_\#$ satisfies $\langle ., .\rangle_\# \equiv \langle ., .\rangle$. This example shows, that a hemi-inner product does not uniquely define a corresponding entropy. Furthermore, $\langle \ell, \ell\rangle$ and $[\![\ell]\!]$ need not be proportional.*

**Theorem 1.** *Let $(G, \dotplus)$ be a magma and $\langle ., .\rangle : G \times G \to \mathbb{R}$ be a map, such that Equation (12) holds true. Let $G_s = \{\varepsilon \in G : \langle \varepsilon, .\rangle \equiv 0\}$ be non-empty. We assume further that for any $z, y \in G\backslash G_s$, natural numbers $m, n \in \mathbb{N}$ exist, such that $m \star z = n \star y$. Assume that an element $x \in G$ and an $e \in \{-1, +1\}$ exist such that $M_{x,e} \in (0, \infty)$. Let $E \in \mathbb{R}$, $E \geqslant M_{x,e}$,*

$$\begin{aligned} f_n(z) &= e\sum_{i=1}^{n-1}\langle i \star z, z\rangle, \\ F_{m,n}(z) &= \frac{1}{m}(nE + f_n(x) - f_m(z)) \end{aligned}$$

*and $n(z)$ and $m(z)$ be a pair of positive integers, such that $m(z)z = n(z)x$. If for all $y, z \in G$,*

$$F_{m(y\dotplus z),n(y\dotplus z)}(y \dotplus z) = \langle y, z\rangle + F_{m(y),n(y)}(y) + F_{m(z),n(z)}(z). \tag{19}$$

*and for all $m', n' \in \mathbb{N}$, $z, y \in G\backslash G_s$ the implication*

$$m'y = n'x \quad\Rightarrow\quad F_{m',n'}(y) = F_{m(y),n(y)}(y) \tag{20}$$

*holds true, then a non-negative function $[\![.]\!]$ exists with $\langle ., .\rangle$ as hemi-inner product.*

The requirement in the theorem, that for all $y, z \in G$ some numbers $n, m \in \mathbb{N}$ exist such that $m \star y = n \star z$, is rather strong in the context of commutative semigroups. An element $x \in G$ is called left cancellative, if $x \dotplus y = x \dotplus z$ implies $y = z$ for all $y, z \in G$. A magma is called left cancellative if all its elements are left cancellative. Now, a commutative semigroup can be embedded into a group if

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and only if it is cancellative [14, p. 36]. Even stronger, a finite cancellative commutative semigroup is already group [14, p. 36]. Finally, a commutative semigroup can be decomposed into equivalence classes of cancellative subsemigroups if and only if the following implication holds for all $x, y \in G$ [13, p. 315]:

$$2x = x \dotplus y = 2y \Rightarrow x = y.$$

Here, $y \sim z$ iff there exist $m, n \in \mathbb{N}$ such that $my = nz$.

**Remark 2.** *If $G$ is a dense subset of a topological space $\bar{G}$, then $[\![.]\!]$ defined on $G$ in Theorem 1 can be extended to $\bar{G}$ if $\langle ., . \rangle$ is continuous on $(G \backslash G_s) \times (G \backslash G_s)$.*

## 4. Proofs

*4.1. Properties of the wide-left-modularity*

**Lemma 1.** *Let $(G, \dotplus, [\![.]\!])$ be a wide-left-modular, hemi-unital magma. Then, for all $x, y, z, a, b \in G$ and $\varepsilon, \tilde{\varepsilon} \in G_s$,*

$$(x \dotplus y) \dotplus z \dotplus b \mathrel{|\!\!=\!\!|} y \dotplus (x \dotplus z) \dotplus (\varepsilon \dotplus b), \quad \forall \varepsilon \in G_s, \tag{21}$$
$$x \dotplus y \mathrel{|\!\!=\!\!|} y \dotplus x, \tag{22}$$
$$\varepsilon \dotplus x \mathrel{|\!\!=\!\!|} x, \tag{23}$$
$$a \dotplus (\varepsilon \dotplus (\tilde{\varepsilon} \dotplus x)) \dotplus b \mathrel{|\!\!=\!\!|} a \dotplus x \dotplus b \tag{24}$$
$$a \dotplus x \dotplus y \mathrel{|\!\!=\!\!|} a \dotplus y \dotplus x \tag{25}$$
$$a \dotplus (\varepsilon \dotplus (x \dotplus y)) \mathrel{|\!\!=\!\!|} a \dotplus y \dotplus (\tilde{\varepsilon} \dotplus x) \tag{26}$$
$$x \dotplus y \dotplus z \mathrel{|\!\!=\!\!|} x \dotplus (y \dotplus \varepsilon) \dotplus z \tag{27}$$
$$x \dotplus y \mathrel{|\!\!=\!\!|} x \dotplus \varepsilon \dotplus y \tag{28}$$
$$x \dotplus y \dotplus z \mathrel{|\!\!=\!\!|} y \dotplus x \dotplus (\varepsilon \dotplus z). \tag{29}$$

**Proof.** The first four equations are shown in [4]. Equation (25) holds, as

$$a \dotplus x \dotplus y \mathrel{|\!\!=\!\!|} x \dotplus (a \dotplus y) \mathrel{|\!\!=\!\!|} (a \dotplus y) \dotplus x$$

by Equation (21). Hence, by Equations (3), (21), (7), (25), (6), (7) and (24)

$$\begin{aligned} a \dotplus (\varepsilon \dotplus (x \dotplus y)) &\mathrel{|\!\!=\!\!|} \varepsilon \dotplus a \dotplus (x \dotplus y) \dotplus \varepsilon \dotplus 0 \\ &\mathrel{|\!\!=\!\!|} \varepsilon \dotplus a \dotplus x \dotplus (\varepsilon \dotplus y) \\ &\mathrel{|\!\!=\!\!|} \varepsilon \dotplus a \dotplus (\varepsilon \dotplus y) \dotplus x \\ &\mathrel{|\!\!=\!\!|} a \dotplus (\varepsilon \dotplus (\varepsilon \dotplus y) \dotplus x) \\ &\mathrel{|\!\!=\!\!|} a \dotplus y \dotplus (\tilde{\varepsilon} \dotplus x). \end{aligned}$$

Furthermore,

$$x \dotplus y \dotplus z \mathrel{|\!\!=\!\!|} x \dotplus y \dotplus z \dotplus (0 \dotplus \varepsilon) \mathrel{|\!\!=\!\!|} x \dotplus (y \dotplus \varepsilon) \dotplus z$$

by Equations (3) and (7). Equation (28) follows from (3) and (25). The last equation follows from Equations (28), (21) and (7),

$$\begin{aligned} x \dotplus y \dotplus z &\mathrel{|\!\!=\!\!|} (x \dotplus y) \dotplus \varepsilon \dotplus z \\ &\mathrel{|\!\!=\!\!|} y \dotplus (x \dotplus \varepsilon) \dotplus (\varepsilon \dotplus z) \\ &\mathrel{|\!\!=\!\!|} y \dotplus x \dotplus (\varepsilon \dotplus z) \end{aligned}$$

□

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**Proposition 7.** *Let $(G, \dotplus, [\![.]\!])$ be a wide-left-modular, hemi-unital magma. Then, for all $x, b \in G, \varepsilon \in G_s$ and $n, i \in \mathbb{N}_0$,*

$$n * x \dotplus i * x \dotplus b \quad \boxminus \quad \begin{cases} (n \dotplus i) * x \dotplus (\varepsilon \dotplus b) & n \text{ even} \\ (n \dotplus i) * x \dotplus b & n \text{ odd} \end{cases}. \tag{30}$$

*In particular,*

$$(2n) * x \dotplus b \quad \boxminus \quad (2n) * x \dotplus (\varepsilon \dotplus b). \tag{31}$$

**Proof.** We show the first assertion by means of induction over $n$ for $i \in \mathbb{N}$. The case $n = 0$ follows instantly from Equation (21), the case $n = 1$ is trivial. By induction, assume that Equation (30) holds for all $i \in \mathbb{N}$ for some $n \in \mathbb{N}$. By Equations (8) and (21), we have

$$n * x \dotplus i * x \dotplus b \quad \boxminus \quad (n-1) * x \dotplus (i+1) * x \dotplus (\varepsilon \dotplus b)$$

Equation (30) follows from the induction hypothesis and Equation (24). We use again Equations (21), (7), (30), and (24) to see that

$$\begin{aligned} n * x \dotplus 0 \dotplus b \quad &\boxminus \quad (x \dotplus (n-1) * x) \dotplus 0 \dotplus b \\ &\boxminus \quad (n-1) * x \dotplus (x \dotplus 0) \dotplus (\varepsilon \dotplus b) \\ &\boxminus \quad (n-1) * x \dotplus x \dotplus (\varepsilon \dotplus b) \\ &\boxminus \quad \begin{cases} n * x \dotplus (\varepsilon \dotplus b) & n \text{ even} \\ n * x \dotplus b & n \text{ odd} \end{cases}. \end{aligned}$$

With Equation (28) we deduce Equation (31). □

### 4.2. Properties of the Hemi-Associativity

**Lemma 2.** *Let $(G, \dotplus, [\![.]\!])$ be a hemi-associative, hemi-unital magma. Then, all parentheses can be removed within $[\![.]\!]$ without changing the value.*

**Proof.** See [4], Proposition 2 and Corollary 1. □

### 4.3. Joint Properties

**Corollary 1.** *Let $(G, \dotplus, [\![.]\!])$ be a hemi-unital magma. Let $x \in G, i, j, k, \ell \in \mathbb{N}_0$. Assume one of the following conditions*

1. *$\dotplus$ is hemi-associative;*
2. *$\dotplus$ is wide-left-modular and $\star$ equals $*$*

*holds. Then,*

$$j \star (i \star x) \dotplus b \quad \boxminus \quad (ji) \star x \dotplus b \tag{32}$$

$$j \star (i \star x) \dotplus \ell \star (k \star x) \quad \boxminus \quad (ij + k\ell) \star x. \tag{33}$$

**Proof.** Lemma 2 yields immediately both equations, if $\dotplus$ is hemi-associative. In case $\dotplus$ is wide-left-modular, we show Equation (32) by induction over $j$. Since there is nothing to show for $i = 0$, we

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let $i \in \mathbb{N}$. The cases $j = 0, 1$ are trivial and the case $j = 2$ follows from Equation (30). If $i$ is even we additionally use Equation (31). We have for $j > 2$, by Equations (7), (30), (31), (26) and (21),

$$\begin{aligned} j * (i * x) \dotplus b &\boxminus i * x \dotplus (i * x \dotplus (j-2) * (i * x)) \dotplus b \\ &\boxminus i * x \dotplus i * x \dotplus (b \dotplus (j-2) * (i * x)) \\ &\boxminus (2i) * x \dotplus (0 \dotplus (b \dotplus (j-2) * (i * x))) \\ &\boxminus (2i) * x \dotplus (j-2) * (i * x) \dotplus (0 \dotplus b) \\ &\boxminus (j-2) * (i * x) \dotplus ((2i) * x \dotplus (0 \dotplus b)). \end{aligned}$$

At this point we can apply the induction hypothesis and then Equations (21) and (30) to finish the induction.

$$\begin{aligned} j * (i * x) \dotplus b &\boxminus ((j-2)i) * x \dotplus ((2i) * x \dotplus (0 \dotplus b)) \\ &\boxminus (2i) * x \dotplus ((j-2)i) * x \dotplus (0 \dotplus b) \\ &\boxminus (ji) * x \dotplus b. \end{aligned}$$

Equation (33) follows from Equations (22) and (30):

$$\begin{aligned} j * (i * x) \dotplus \ell * (k * x) &\boxminus (ji) * x \dotplus \ell * (k * x) \\ &\boxminus \ell * (k * x) \dotplus (ji) * x \\ &\boxminus (\ell k) * x) \dotplus (ji) * x \\ &\boxminus (\ell k + ji) * x. \end{aligned}$$

□

**Proposition 8.** *Let $(G, \dotplus, [\![.]\!])$ be a wide-left-modular, hemi-unital magma. Further, assume that one of the following conditions holds:*

1. $\star$ *equals* $*$;
2. $\dotplus$ *is hemi-associative.*

*Then, for all $x, y \in G$, $i, j \in \mathbb{N}_0$, we have*

$$i \star x \dotplus j \star y \dotplus (x \dotplus y) \boxminus (i+1) \star x \dotplus (j+1) \star y, \tag{34}$$

$$i \star (j \star (x \dotplus y)) \boxminus (ij) \star x \dotplus (ij) \star y, \tag{35}$$

$$(ij) \star (x \dotplus y)) \boxminus i \star (j \star x) \dotplus j \star (i \star y), \tag{36}$$

**Proof.** In case of hemi-associativity, the Equations (34)-(36) are immediate, as $\dotplus$ is also hemi-commutative. We have for $i, j \in \mathbb{N}, b, c, x, y \in G$ by means of Equations (21), (7), (24) and (25),

$$\begin{aligned} (i * x \dotplus j * y) \dotplus ((x \dotplus y) \dotplus b) \dotplus c &\boxminus (x \dotplus y) \dotplus (i * x \dotplus j * y) \dotplus b \dotplus (0 \dotplus c) \\ &\boxminus (x \dotplus y) \dotplus i * x \dotplus (b \dotplus j * y) \dotplus (0 \dotplus c) \\ &\boxminus y \dotplus (i+1) * x \dotplus (0 \dotplus (b \dotplus j * y)) \dotplus c \\ &\boxminus (i+1) * x \dotplus (y \dotplus (0 \dotplus (b \dotplus j * y))) \dotplus (0 \dotplus c) \dotplus 0 \\ &\boxminus (i+1) * x \dotplus y \dotplus (0 \dotplus c) \dotplus (b \dotplus j * y) \\ &\boxminus (i+1) * x \dotplus (j+1) * y \dotplus (0 \dotplus c) \dotplus b \\ &\boxminus (i+1) * x \dotplus (j+1) * y \dotplus b \dotplus (0 \dotplus c). \end{aligned} \tag{37}$$

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We have, by Equations (27) and (28)

$$
\begin{aligned}
(i * x \dotplus j * y) \dotplus (x \dotplus y) \dotplus c &\boxminus (i * x \dotplus j * y) \dotplus ((x \dotplus y) \dotplus 0) \dotplus c \\
&\boxminus (i+1) * x \dotplus (j+1) * y \dotplus 0 \dotplus (0 \dotplus c) \\
&\boxminus (i+1) * x \dotplus (j+1) * y \dotplus (0 \dotplus c),
\end{aligned}
$$

which shows Equation (34). Equation (35) follows iteratively with Equation (37) from the fact that $i * (j * (x \dotplus y)) \boxminus (ij) * (x \dotplus y))$ by Corollary 1. Equations (35), (32) and (22) imply, that, for all $x, y \in G, i, j \in \mathbb{N}_0$, we have

$$
\begin{aligned}
(ij) * (x \dotplus y) &\boxminus (ij) * x \dotplus (ij) * y \\
&\boxminus i * (j * x) \dotplus (ij) * y \\
&\boxminus (ij) * y \dotplus i * (j * x) \\
&\boxminus i * (j * x) \dotplus j * (i * y),
\end{aligned}
$$

i.e., Equation (36) holds true. □

*4.4. Proofs for Section 3*

Subsequently we abbreviate sign $[\![\cdot]\!]$ by $e$.

**Proof of Proposition 1.** We have

$$
[\![(x \dotplus y) \dotplus z]\!] \quad = \quad e\langle x \dotplus y, z\rangle + [\![x \dotplus y]\!] + [\![z]\!] = e\langle x \dotplus y, z\rangle + e\langle x, y\rangle - [\![x]\!] - [\![y]\!] - [\![z]\!].
$$

In case $\dotplus$ is hemi-associative, the assertion follows from the following equation:

$$
[\![x \dotplus (y \dotplus z)]\!] \quad = \quad e\langle x, y \dotplus z\rangle + e\langle y, z\rangle - [\![x]\!] - [\![y]\!] - [\![z]\!].
$$

In case $\dotplus$ is wide-left-modular, the assertion follows from Equation (21) and the following equation:

$$
[\![y \dotplus (x \dotplus z)]\!] \quad = \quad e\langle y, x \dotplus z\rangle + e\langle x, z\rangle - [\![x]\!] - [\![y]\!] - [\![z]\!].
$$

□

**Proof of Proposition 2.** Immediate from the equation $x \dotplus y \boxminus 0 \dotplus (x \dotplus y) \dotplus 0 \boxminus y \dotplus x$ and Equation (22), respectively. □

**Proof of Proposition 3.** Equation (11) follows immediately from Proposition 2 in case of wide-left-modularityor hemi-commutativity. In case of hemi-associativity, we use Lemma (2). □

**Proof of Proposition 4.** Equality (13) follows immediately from

$$
[\![(i+k) \star x]\!] - [\![(i+k-1) \star x]\!] = [\![x]\!] + e\langle (i+k-1) \star x, x\rangle
$$

by summation. Equation (14) follows from Equation (13) by choosing $i = 1$. □

**Proof of Proposition 5.** Equation (9) implies that

$$
\begin{aligned}
&\langle (i-k-1)x \dotplus x, (j+k)x\rangle - \langle (i-k-1)x, (j+k+1)x\rangle \\
&= \langle x, (j+k)x\rangle - \langle (i-k-1)x, x\rangle.
\end{aligned}
$$

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Equation (10) yields

$$\langle x \dotplus (i-k-1) * x, (j+k) * x \rangle - \langle (i-k-1) * x, x \dotplus (j+k) * x \rangle$$
$$= \langle x, (j+k) * x \rangle - \langle x, (i-k-1) * x \rangle.$$

Due to Equation (11), we have in both cases that

$$\langle (i-k) \star x, (j+k) \star x \rangle - \langle (i-k-1) \star x, (j+k+1) \star x \rangle$$
$$= \langle x, (j+k) \star x \rangle - \langle x, (i-k-1) \star x \rangle$$

for $0 \leqslant k \leqslant i-2$. Summing up both sides from $k = 0$ to $k = i-2$ yields (15). It follows immediately from (15) that

$$\langle i \star x, (ni) \star x \rangle = \sum_{k=in}^{(n+1)i-1} \langle x, k \star x \rangle - \sum_{k=1}^{i-1} \langle x, k \star x \rangle, \qquad \forall x \in G; i, n \in \mathbb{N}.$$

Summing up yields Equation (16). □

**Proof of Proposition 6.** The first equality of the proposition follows immediately from Proposition 8. We show Equation (17). In both cases, hemi-associativity and $\star$ being $*$, Equation (36) and repeated application of Equation (14) give

$$\begin{aligned}
& e\sum_{k=1}^{rs-1} \langle k \star (x \dotplus y), (x \dotplus y) \rangle \\
= \quad & [\![(rs) \star (x \dotplus y)]\!] - rs[\![x]\!] - rs[\![y]\!] - rse\langle x, y \rangle \\
= \quad & [\![r \star (s \star x) + s \star (r \star y)]\!] - rs[\![x]\!] - rs[\![y]\!] - rse\langle x, y \rangle \\
= \quad & e\langle r \star (s \star x), s \star (r \star y) \rangle + [\![r \star (s \star x)]\!] + [\![s \star (r \star y)]\!] - rs[\![x]\!] - rs[\![y]\!] - rse\langle x, y \rangle \\
= \quad & e\langle r \star (s \star x), s \star (r \star y) \rangle + e\sum_{k=1}^{r-1} \langle k \star (s \star x), s \star x \rangle + r[\![s \star x]\!] \\
& + e\sum_{k=1}^{s-1} \langle k \star (r \star x), r \star x \rangle + s[\![r \star x]\!] - rs[\![x]\!] - rs[\![y]\!] - rse\langle x, y \rangle \\
= \quad & e\langle r \star (s \star x), s \star (r \star y) \rangle - rse\langle x, y \rangle + e\sum_{k=1}^{r-1} \langle k \star (s \star x), s \star x \rangle \\
& + e\sum_{k=1}^{s-1} \langle k \star (r \star y), r \star y \rangle + re\sum_{k=1}^{s-1} \langle k \star x, x \rangle + se\sum_{k=1}^{r-1} \langle k \star y, y \rangle.
\end{aligned}$$

□

**Proposition 9.** *Let $(G, \circ, \dotplus, [\![.]\!])$ be an entropy-driven magma with comparable elements. If one of the following conditions*

1. *$\dotplus$ is hemi-associative,*
2. *$\dotplus$ is wide-left-modular and $\star$ equals $*$,*

*is satisfied, then, for all $x, y \in G$ and $n, m, \tilde{n}, \tilde{m} \in \mathbb{N}$ with $n \star x = m \star y$ and $\tilde{n} \star x = \tilde{m} \star y$, we have*

$$\sum_{k=1}^{m-1} (\langle k \star (\tilde{m} \star y), \tilde{m} \star y \rangle - \tilde{m}\langle k \star y, y \rangle) = \sum_{k=1}^{\tilde{m}-1} (\langle k \star (m \star y), m \star y \rangle - m\langle k \star y, y \rangle).$$

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**Proof.** Equation (14) delivers

$$m[\![y]\!] + e\sum_{k=1}^{m-1}\langle k\star y, y\rangle = [\![m\star y]\!] = [\![n\star x]\!] = n[\![x]\!] + e\sum_{k=1}^{n-1}\langle k\star x, x\rangle.$$

Hence,

$$[\![y]\!] = \frac{n}{m}[\![x]\!] + \frac{e}{m}\sum_{k=1}^{n-1}\langle k\star x, x\rangle - \frac{e}{m}\sum_{k=1}^{m-1}\langle k\star y, y\rangle,$$

so that the assumptions of the proposition imply

$$\tilde{m}ne[\![x]\!] + \tilde{m}\textstyle\sum_{k=1}^{n-1}\langle k\star x, x\rangle - \tilde{m}\sum_{k=1}^{m-1}\langle k\star y, y\rangle \tag{38}$$
$$= m\tilde{n}e[\![x]\!] + m\textstyle\sum_{k=1}^{\tilde{n}-1}\langle k\star x, x\rangle - m\sum_{k=1}^{\tilde{m}-1}\langle k\star y, y\rangle. \tag{39}$$

On the other hand, by Equations (32) and (14),

$$\begin{aligned} e[\![(\tilde{m}m)\star y]\!] &= e[\![\tilde{m}\star(m\star y)]\!] = \tilde{m}e[\![n\star x]\!] + \sum_{k=1}^{\tilde{m}-1}\langle k\star(m\star y), m\star y\rangle \\ &= \tilde{m}ne[\![x]\!] + \tilde{m}\sum_{k=1}^{n-1}\langle k\star x, x\rangle + \sum_{k=1}^{\tilde{m}-1}\langle k\star(m\star y), m\star y\rangle. \end{aligned}$$

Similarly,

$$e[\![(m\tilde{m})\star y]\!] = m\tilde{n}e[\![x]\!] + m\sum_{k=1}^{\tilde{n}-1}\langle k\star x, x\rangle + \sum_{k=1}^{m-1}\langle k\star(\tilde{m}\star y), \tilde{m}\star y\rangle.$$

Combining the preceding equations with Equation (39) finalizes the proof. □

**Proof of Theorem 1.** Let $[\![x]\!] = E$ and $[\![\varepsilon]\!] = 0$, $\varepsilon\in G_s$. By Proposition 4, we necessarily have

$$m(y)[\![y]\!] + f_{m(y)}(y) = [\![m(y)\star y]\!] = [\![n(y)\star x]\!] = n(y)[\![x]\!] + f_{n(y)}(x), \quad y\in G\backslash G_s,$$

i.e., $[\![y]\!] = F_{m(y),n(y)}(y)$. By definition of $M_{x,e}$, we have $[\![y]\!] \geqslant 0$ for all $y\in G$. The definition is unambiguous due to condition (20). Equation (19) implies that $\langle y, z\rangle = [\![y \dotplus z]\!] - [\![y]\!] - [\![z]\!]$. □

### *4.5. Some Counter-Intuitive Results*

The subsequent Equations (40) and (41) demonstrate that in this framework, an equality can be true in many cases, but for the missing ones, the assertion can be indeed false.

**Lemma 3.** *Let $(G, \dotplus, [\![.]\!])$ be a hemi-unital magma, $x\in G$ and $i, j, k\in\mathbb{N}$. If one of the conditions,*

1. *$\dotplus$ is hemi-associative,*
2. *$\dotplus$ is wide-left-modular and $\star$ equals $\cdot$,*
3. *$\dotplus$ is wide-left-modular and $i$ is odd,*
4. *$\dotplus$ is wide-left-modular and $j$ is even*

*holds, then*

$$i\star x \dotplus j\star x \dotplus k\star x \quad \boxminus \quad (j+k-1)\star x \dotplus (i+1)\star x. \tag{40}$$

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**Proof.** Lemma 2 yields immediately the assertion, if $\dotplus$ is hemi-associative. First, we show that Equation (40) is true for $k = 1$. In case $\star$ is $*$ and $i$ is odd, then by Equations (21), and (30),

$$\begin{aligned} i * x \dotplus j * x \dotplus x &\boxminus j * x \dotplus (i * x \dotplus x) \\ &\boxminus (i * x \dotplus x) \dotplus j * x \\ &\boxminus (i + 1) * x \dotplus j * x. \end{aligned}$$

In case $j$ is even, we use Equations (29), (30) and (22).

$$\begin{aligned} i * x \dotplus j * x \dotplus x &\boxminus j * x \dotplus i * x \dotplus (0 \dotplus x) \\ &\boxminus (j + i) * x \dotplus x \\ &\boxminus (j + i + 1) * x \\ &\boxminus j * x \dotplus (i + 1) * x. \end{aligned}$$

If $\star$ is $\cdot$, then the case for $k = 1$ follows from Equation (21). For $k \geqslant 2, z \in G$, Equations (3) and (7) yield

$$z \dotplus jx \dotplus kx \boxminus z \dotplus jx \dotplus ((k-1)x \dotplus x) \dotplus 0 \boxminus z \dotplus (j + k - 1)x \dotplus x.$$

Equation (40) follows now from Equation (21). In case of $*$-summation, Equation (30) yields that, for $z \in G$, we have

$$i * x \dotplus j * x \dotplus z \quad \boxminus \quad \begin{cases} (j+i) * x \dotplus z, & i \text{ odd} \\ (j+i) * x \dotplus (\varepsilon \dotplus z), & i \text{ even} \end{cases}.$$

If $i$ is odd, Equation (40) follows after the next application of Equation (30). If $i$ is even, $j$ is even and we use Equation (31) and then (30). □

To verify that the equality may not hold outside of those cases, consider $G = \mathbb{R}$ together with $x \dotplus y = x - y$ and $[\![x]\!] = |x|$. As a result, $\dotplus$ is wide-left-modular and we choose $\star$ as $*$. We further choose $i = 2, j = 1, k = 1$ and $x = 1$. Then,

$$|(1-1) - 1 - 1| \;=\; 2 \;\neq\; 0 \;=\; |1 - (1 - (1-1))|.$$

**Corollary 2.** *Let $(G, \dotplus, [\![.]\!])$ be an entropy-driven magma with comparable elements and $i, j, k \in \mathbb{N}$. If one of the conditions,*

1. *$\dotplus$ is hemi-associative,*
2. *$\dotplus$ is wide-left-modular and $\star$ equals $\cdot$,*
3. *$\dotplus$ is wide-left-modular and $i$ is odd,*
4. *$\dotplus$ is wide-left-modular and $j$ is even*

*holds, then*

$$\begin{aligned} \langle i \star x \dotplus j \star x, k \star x\rangle &= \langle (j + k - 1) \star x, (i+1) \star x\rangle + \langle i \star x, x\rangle - \langle i \star x, j \star x\rangle \\ &\quad + \sum_{\ell=1}^{k-1} [\langle (j + \ell - 1) \star x, x\rangle - \langle \ell \star x, x\rangle]. \end{aligned} \tag{41}$$

**Proof.** Equations (40) and (13) yield

$$
\begin{aligned}
&\langle i \star x \dotplus j * x, k * x\rangle - \langle (j+k-1) * x, (i+1) * x\rangle \\
&= c(e[\![(j+k-1) * x \dotplus (i+1) * x]\!] - e[\![i * x \dotplus j * x]\!] - e[\![k * x]\!]) - \langle (j+k-1) * x, (i+1) * x\rangle \\
&= c(e[\![(j+k-1) * x]\!] + e[\![(i+1) * x]\!] - e[\![i * x]\!] - e[\![j * x]\!] - e[\![k * x]\!]) - \langle i * x, j * x\rangle \\
&= c(e[\![(j+k-1) * x]\!] + e[\![x]\!] - e[\![j * x]\!] - e[\![(1+k-1) * x]\!]) + \langle i * x, x\rangle - \langle i * x, j * x\rangle \\
&= \sum_{\ell=1}^{k-1}[\langle (j+\ell-1) * x, x\rangle - \langle \ell * x, x\rangle] + \langle i * x, x\rangle - \langle i * x, j * x\rangle.
\end{aligned}
$$

□

## 5. Discussion & Conclusions

Usually, generalizations of the inner product, particularly to the Banach space, keep at least partially the property of the linearity and abandon the the polarization equality. The loss of the polarization equality is inevitable, if the standard properties for a norm shall be kept, since the Fréchet-von-Neummann-Jordan theorem states, that a Banach space is a Hilbert space iff the polarization equality holds [15]. The approach by [2,3] keeps the polarization equality and accepts potentially unusual properties of the hemi-inner product. An advantage of the latter approach is that it is applicable to very basic algebraic structures. In this paper, we show that the hemi-inner product is symmetric and a 2-cocyle, under weak assumptions (Propositions 1 and 2). Furthermore, we present some identities, which are well known for a Hilbert space $H$, and which hold in a more general, entropy-driven framework. For instance, the equations

$$
\begin{aligned}
\|nx\|_H^2 - n\|x\|_H^2 &= 2\sum_{k=1}^{n-1}\langle kx, x\rangle_H, \\
\langle mx, nx\rangle_H &= \sum_{k=1}^{m+n-1}\langle kx, x\rangle_H - \sum_{k=1}^{n-1}\langle kx, x\rangle_H - \sum_{k=1}^{m-1}\langle kx, x\rangle_H,
\end{aligned}
$$

may carry over, cf. Propositions 3 and 5. Further, under stronger assumptions, yet weaker than those of a group, an entropy can be reconstructed from a hemi-inner product, albeit non-uniquely. The functions $x \mapsto [\![x]\!]$ and $x \mapsto \langle x, x\rangle$ are proportional in special cases, cf. Examples 1–4, but not in general, cf. Example 5.

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## References

1. Born, M. Physical reality. *Phil. Quart.* **1953**, *3*, 139–149.
2. Schlather, M. An algebraic generalization of the entropy and its application to statistics. *arxiv* **2024**, *2404.05854*.
3. Schlather, M. An algebraic generalization of the entropy and its application to statistics. *Submitted to Journal of Applied Statistics. Special Issue* **2025**. THE UPDATED MANUSCRIPT (AVAILABLE ON REQUEST) DOES NOT

CONTAIN SECTION 3.3 OF SCHLATHER (2024), WHICH IS THE BASIS OF THIS PAPER – THIS COMMENT IS DELETED BEFORE FINAL PUBLISHING.
4. Schlather, A.; Schlather, M. Depicting Falsifiability in Algebraic Modelling **2025**. MANUSCRIPT IS BEING SUBMITTED TO MDPI AND IS AVAILABLE ON REQUEST IF NOT YET AVAIABLE ON THE MDPI PREPRINT SERVER – THIS COMMENT IS DELETED BEFORE FINAL PUBLISHING.
5. Serre, J.P. *Local fields*; Springer, 2013.
6. Chilès, J.P.; Delfiner, P. *Geostatistics. Modeling Spatial Uncertainty*; John Wiley & Sons: New York, 1999.
7. Gneiting, T.; Sasvári, Z. The characterization problem for isotropic covariance functions. *Math. Geol.* **1999**, *31*, 105–111.
8. Kullback, S.; Leibler, R. On information and sufficiency. *Ann. Math. Stat.* **1951**, *22*, 79–86.
9. Schlather, M.; Ditscheid, C. An Intrinsic Characterization of Shannon's and Rényi's Entropy. *Entropy* **2024**, *26*, 1051.
10. Clifford, A.; Preston, G. *The Algebraic Theory of Semigroups, Vol. 1*; Amer. Math. Soc., 1961.
11. Stoev, S.; Taqqu, M. Extremal stochastic integrals: a parallel between max-stable processes and $\alpha$-stable processes. *Extremes* **2005**, *8*, 237–266.
12. Röttger, F.; Engelke, S.; Zwiernik, P. Total positivity in multivariate extremes. *Ann. Stat.* **2023**, *51*, 962–1004.
13. Ljapin, E. *Semigroups*; Vol. 3, Am. Math. Soc., 1968.
14. Grillet, P. *Commutative Semigroups*; Vol. 2, Springer, 2013.
15. Blanchard, P.; Brüning, E. *Mathematical Methods in Physics: Distributions, Hilbert Space Operators, Variational Methods*; Vol. 69, Birkhäuser, 2015.