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Canonical Commutation Relation Derived from Witt Algebra

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Article

Canonical Commutation Relation Derived from Witt Algebra

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Abstract: From an abstract definition of operators inspired on the oscillators of Virasoro, an algebra is derived. It encompasses the structure of Virasoro with null charge central or Witt algebra. The applied formalism has yielded commutators with a dependence on integer numbers, and it follows a kind of Witt-like algebra. Indeed, the quantum mechanics evolution operator for the case of quantum harmonic oscillator was identified. Also, Schrödinger equation was systematically derived, inside the present framework. When operators are expressed in the framework of Hilbert space states, the resulting Witt-like algebra seems to be proportional to well-known canonical commutation relation. This has demanded to develop a formalism based on abstract and physical operators as well as well-defined rules of commutation. The Witt-like was also redefined through the direct usage of uncertainty principle. It might to be suggesting that Witt algebra encloses not only quantum mechanics fundamental commutator, but also different relations among quantum mechanics observables.

Keywords: malnutrition; inflammation elderly; mortality; albumin; C Reactive Protein; Albumin; PCR/Alb ratio

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1. Introduction

1.1. Background

In the context of Veneziano model [1-2] new gauge conditions were formulated and being them associated to a set of operators that in string theory are called oscillators of Virasoro defined as a sum of products between no-commutative modes and that can be written as

$$\mathcal{L}_n = \frac{1}{2}\sum_{p\in Z}\alpha_{n-p}\alpha_p. \tag{1}$$

Them define the so-called Virasoro algebra with central charge c and this algebra can be written below as:

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n}. \tag{2}$$

With $\delta_{m,-n}$ well-known as delta of Krönecker Operators as given above have played a critic role in the formulation of dual models that expresses the covariant form of harmonic oscillator containing an infinite modes satisfying $[\alpha^{\mu}_p, \alpha^{\nu}_q]$=$\delta_{p,q}g^{\mu,\nu}$ as seen in the work of R. C. Brower in [3]. Virasoro algebra from Lie generators and its link to Lie group because the $c-$number was studied by P. V. Alstine in [4] where it was also analyzed as to its direct relation to quantum mechanics Jacobi theory, yielding interesting relationships involving the oscillators as new representations of $c-$number. In [5] R. Akhoury and Y. Okada have investigated the role of Virasoro oscillators as function of classical variables inside the framework of Hamiltonian such as $H = \sum_i \pi_i \dot{q}_i - L(X^\mu, X^\nu)$ with the Lagrangian that defines the action of closed bosonic string theory. The resulting oscillators, have yielded a copy of Virasoro algebra at the flat space. Here, it were also reformulated oscillators that have been defined

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in order to annihilate ghosts, fact that demanded to construct a series of commutation relations (not like Virasoro algebra). Y. Saito in [6] investigated the action of Virasoro oscillators inside the Becchi-Rouet-Stora (BRS) formalism in the form of subsidiary conditions specifically as seen at the action of them onto the Fock's space for $n > 0$:

$$(\mathcal{L}_n + \alpha_0 \delta_{n,0}) \left|\text{phys}\right\rangle = 0. \tag{3}$$

Furthermore, Saito observed that under conditions dictated by Virasoro algebra, them does not eliminate negative states of a squeezed string. Such conditions have derived the redefinition of Virasoro oscillators in a new version being this a kind of extension from standard structure of algebra. Indeed, complex forms were founded in according to the ones got by Scherk as seen in [7]. Variations on the Virasoro oscillators have been obtained in [8] through the modifications of Kac-Moody algebra arriving to modified Virasoro oscillators containing also Kac-Moody generators, yielding other classes of Virasoro algebra (Equation 2.7 Ref.[7]). Here it was seen the role of central charge to interpret the true roles of oscillators to some extent (see [9] for example)

1.2. Contribution of Paper

In [10], S. Katagiri and collaborators have postulated the idea that Virasoro generators despite of the fact that them are dimensionless entities in string theories, can be expressed as a linear combination of momentum and position operators (see in [10] Section 4.1, Equation 1, for instance) such as $\mathcal{L}_i = -\frac{i}{2}(\hat{\mathbf{x}}^{i+1}\hat{\mathbf{p}} - \hat{\mathbf{p}}\hat{\mathbf{x}}^{i+1})$ with i integer number and $\mathcal{L}_i$ satisfies algebra Equation 2. It is interesting the link between dimensionless operators (Virasoro oscillators) and Quantum Mechanics (QM) observables, since it might to enclose important implications that are crossing an unproven string theory and tested QM theory. As it is well-known, QM is based in a sophisticated mathematical formulation of physics operators acting on wave functions, in fully agreement to QM postulates [11]. The central equation of QM theory known as Schrödinger equation written as:

$$\mathbf{H}(\hat{\mathbf{x}}, \hat{\mathbf{p}}) \left|\Psi(\mathbf{r}, t)\right\rangle = \mathcal{E} \left|\Psi(\mathbf{r}, t)\right\rangle \tag{4}$$

being $\mathbf{H}$ is the Hamiltonian operator [12-15] often it is depending on $\hat{\mathbf{x}}$ and $\hat{\mathbf{p}}$ the position and momentum operators, respectively, satisfying the well-known canonical commutation relation (CCR) that can be written as:

$$[\hat{\mathbf{x}}, \hat{\mathbf{p}}] = i\hbar, \tag{5}$$

and $\hbar$ the Planck's constant [16-19]. This relation has the level of being main piece in the formulation of quantum theory. Moreover, other relations can be derived from that. An important aspect linked to Equation 2 is the unknown mathematical procedure that yields that. Some attempts have been done in past. In this manner, the purpose of paper is establishing a kind of artificial origin to Equation 1 through the so-called Witt's algebra [20-25] that is defined as:

$$[\Xi_m, \Xi_n] = (m - n)\Xi_{m+n}, \tag{6}$$

with Ξ_q known as transverse Virasoro operators that have played a noteworthy role in the development of strings theory. While, m and n integer numbers Equation 3 is also called Virasoro algebra without central extension or the Witt algebra [26-31]. Witt's algebra can be again rewritten as:

$$[\mathcal{L}(u, m), \mathcal{L}(u, n)] = g(m, n)\mathcal{A}(u, \Xi_q), \tag{7}$$

with $g(m, n)$ a function of integer numbers, and $\mathcal{A}(u, \Xi_q)$ function of commutators such as the ones of Equation 2, and u being an physical observable.

This paper has as central objective to demonstrate that exists alternative ways that would allow to arrive to the Canonical Commutation Relation (CCR) in a closed-form. The scheme to be employed is inspired in the Virasoro algebra. In this way, new types of Virasoro-like oscillators $\mathcal{L}_q$ are proposed. It is actually a kind of generalization of convolution integration having a polynomial profile because the integer q that can be written as:

$$\mathcal{L}_q(s) = \frac{d}{ds}\int \delta(w-s)f(w)g_q(s,w)dw, \tag{8}$$

with q integer number and "$\delta(w-s)$" Dirac delta function. In praxis, these oscillators might be fulfilling the algebra:

$$[\mathcal{L}_p, \mathcal{L}_q] = (p-q)\mathcal{L}_{p+q}, \tag{9}$$

known as Witt algebra and indeed being it a particular case of Virasoro algebra. The proposal of this paper is to establish that:

$$[\mathcal{L}_p, \mathcal{L}_q] \equiv \mathcal{A}([\hat{\mathbf{x}}, \hat{\mathbf{p}}]), \tag{10}$$

by which one can see that Witt algebra might be a functions of CCR through function $\mathcal{A}$ that depends on QM commutators $[\hat{\mathbf{x}}, \hat{\mathbf{p}}]$. In addition it shall be demonstrated that $\mathcal{L}_q(s)$ is a pure QM expression in the sense that it has a clear ground based in wave function and physical observables. Because this, one would expect a set of QM relations including CCR, for instance. Also, extra commutators can be derived, but without physics meaning. Thus oscillator Equation 8 can be see as a kind of abstract expression, instead of being perceived as one belonging to QM or having a well-defined profile. However, the inclusion of derivative in Equation 5 it's in fact inspired in the QM momentum operator:

$$\frac{d}{ds} \Rightarrow \frac{\hbar}{i}\frac{d}{ds}. \tag{11}$$

with s having units of distance. Clearly, although it is not explicitly written above, that derivative can be understood to some extent as the action onto convolution in the sense that:

$$\mathcal{L}_q(s) = \frac{\hbar}{i}\frac{d}{ds}.\int \delta(w-s)f(w)g_q(s,w)dw, \tag{12}$$

so that one can see that unlike Equation 8 $\mathcal{L}_q(s)$ from Equation 9 is proportional to momentum operator $\mathcal{L}_q(s) \approx \mathbf{P}$, since $\hbar$ has units of momentum times distance. Also, one can see that $\int \delta(w-s)f(w)g_q(s,w)dw$ is dimensionless. The rest of paper is as follows: Second section presents the extended formulation of Equation 5 and its connection to Witt's algebra. In second section, DeWitt operators are presented, third section gives the interpretation of these operators. Finally, conclusion of paper, is drawn.

2. Witt Operators

We begin the present debate with the consideration of unphysical operators: $\mathcal{A}_m$ and $\mathcal{A}_n$ with m and n integer numbers, then it is said that both satisfy Witt algebra [32-36] if:

$$[\mathcal{A}_m, \mathcal{A}_n] = (m-n)\mathcal{A}_{m+n}. \tag{13}$$

It is evident that commutator is understood that follows the rule:

$$[\mathcal{A}_m, \mathcal{A}_n] = \mathcal{A}_m\mathcal{A}_n - \mathcal{A}_n\mathcal{A}_m, \tag{14}$$

so that it is obvious that:

$$\mathcal{A}_m\mathcal{A}_n - \mathcal{A}_n\mathcal{A}_m = m\mathcal{A}_{m+n} - n\mathcal{A}_{m+n}. \tag{15}$$

So far, no any information about operators $\mathcal{A}_m$ and $\mathcal{A}_n$ has been explicitly manifested. In other words, these operators are not necessarily to be subjected to be a real or complex quantity. Nevertheless, in virtue to Equation 12 one can see its complex character for instance (although it can also be defined as pure real quantities). It is important to observe that such operators are abstract entities without any concrete role. Inspired on the structure shown in Equation 12 with $\hbar = 1$, one can define a kind of Witt operators (being them complex quantities) as:

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m, \tag{16}$$

and its corresponding "partner" (essentially follows the change $K \to L$) in the sense that:

$$\mathcal{L}_L = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{17}$$

It should be noted that subindex K and L are seen inside integration through polynomials x_m^K and x_m^L. Also, it is seen in both the presence of delta of Krönecker as well as Dirac delta function [37-39]. Indeed, it should be noted that $\frac{1}{\sqrt{2}}$ plays the role as a kind of normalization constant although it is not exactly its genuine function as shall be seen later. Because complex number one can claim at a first instance that both operators are pure complex entities. The reader can be aware that when Equation 16 and Equation 17 are solved independently, $[\mathcal{L}_K\mathcal{L}_L - \mathcal{L}_L\mathcal{L}_K] = 0$. Nevertheless, the case when one operates onto other and vice verse the commutator yields $[\mathcal{L}_K\mathcal{L}_L - \mathcal{L}_L\mathcal{L}_K] \neq 0$. In this manner and by following the order seen in Equation 14, the the next step is to calculate $[\mathcal{L}_K, \mathcal{L}_L]$ = $\mathcal{L}_K \otimes \mathcal{L}_L$ - $\mathcal{L}_L \otimes \mathcal{L}_K$, demanding to calculate the products:

$$\mathcal{L}_K \otimes \mathcal{L}_L, \tag{18}$$
$$\mathcal{L}_L \otimes \mathcal{L}_K, \tag{19}$$

with $\otimes$ denoting the product of operators. In this manner one can carry out the products firstly Equation 18 as follows:

$$\mathcal{L}_K \otimes \mathcal{L}_L = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m \otimes x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{20}$$

This can be also written as:

$$\left[x_\ell \delta_\ell^m \left(\frac{i}{\sqrt{2}}\right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K dx_m x_\ell \delta_\ell^m\right] \otimes \left(\frac{i}{\sqrt{2}}\right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{21}$$

To note that product $x_\ell \delta_\ell^m$ has passed to be inside bracket left-side. Subsequent operations have as starting point the direct usage of delta of Krönecker by affecting the quantity x_ℓ in both extremes, yielding for them $\delta_\ell^m x_\ell = x^m$, that subsequently ℓ opts by m in the whole Equation 21. Thus one gets:

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left[\left(\frac{i}{\sqrt{2}}\right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+2} dx_m\right] \otimes \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^L dx_m. \tag{22}$$

Same operation as done to left-side bracket is applied to right-side. On the other side it is obvious that $\left(\frac{i}{\sqrt{2}}\right)\left(\frac{i}{\sqrt{2}}\right) = -\frac{1}{2}$ by which the product $\mathcal{L}_K \otimes \mathcal{L}_L$ is flipping to real space. In this manner, one arrives to:

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right)\left[\frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^{K+2}dx_m\right] \otimes \delta_\ell^m \frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^L dx_m. \tag{23}$$

From above it is clear that polynomial x^m were absorbed by integrations in both sides, so that one has below that:

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right)\frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^{K+2}dx_m \frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^L dx_m. \tag{24}$$

Integrations containing the Dirac delta functions and their subsequent derivatives are solved in a straightforward manner. For right-side one gets:

$$\frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^L dx_m = \frac{d}{dx}x^L = Lx^{L-1}. \tag{25}$$

Same procedure is applied to left-side as follows:

$$\frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^{K+2}dx_m = \frac{d}{dx}x^{K+2} = (K+2)x^{K+1}, \tag{26}$$

so that one arrives to:

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right)(K+2)x^{K+1}Lx^{L-1} = \left(-\frac{1}{2}\right)L(K+2)x^{K+L}. \tag{27}$$

Now to solve product $\mathcal{L}_L \otimes \mathcal{L}_K$ reader should be aware that it follows as done to $\mathcal{L}_K \otimes \mathcal{L}_L$. With this view one can write down:

$$\mathcal{L}_L \otimes \mathcal{L}_K = \left(-\frac{1}{2}\right)x^m\frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^L dx_m x^m \frac{d}{dx}\int_{-\infty}^{\infty}\delta(x-x_m)x_m^K dx_m, \tag{28}$$

So that, as above one arrives to:

$$\mathcal{L}_L \otimes \mathcal{L}_K = \left(-\frac{1}{2}\right)(L+2)x^{L+1}Kx^{K-1} = \left(-\frac{1}{2}\right)K(L+2)x^{K+L}. \tag{29}$$

Thus, Equation 27 and Equation 29 allows to calculate commutator $[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_K \otimes \mathcal{L}_L$ - $\mathcal{L}_L \otimes \mathcal{L}_K$. Therefore, by putting altogether one can demonstrate that $[\mathcal{L}_K, \mathcal{L}_L]$ is in fully agreement to Witt algebra $\mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K =$

$$\begin{gathered}\left(-\frac{1}{2}\right)L(K+2)x^{K+L} - \left(-\frac{1}{2}\right)K(L+2)x^{K+L} = \left(-\frac{1}{2}\right)(LK+2L-KL-2K)x^{K+L} \\ = \left(-\frac{1}{2}\right)(2L-2K)x^{K+L}\mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K = (K-L)x^{K+L} \Rightarrow [\mathcal{L}_K, \mathcal{L}_L] = (K-L)x^{K+L},\end{gathered} \tag{30}$$

that satisfies the Witt-like algebra:

$$[\mathcal{L}_K, \mathcal{L}_L] = (K-L)x^{K+L}. \tag{31}$$

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Equation 31 fits exactly to Witt algebra if holds he following proposal:

$$x^{K+L} \Rightarrow \mathcal{L}_{K+L}, \tag{32}$$

that can be validated through the usage of Equation 16, for example, with the substitution $K \to K + L$

$$\mathcal{L}_{K+L} = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+L} dx_m \tag{33}$$

$$= \left(\frac{i}{\sqrt{2}}\right) \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^{K+L+1} dx_m = \left(\frac{i}{\sqrt{2}}\right) (K + L + 1) x^{K+L+1} \tag{34}$$

$$\Rightarrow \mathcal{L}_{K+L} = \left(\frac{i}{\sqrt{2}}\right) (K + L) x^{K+L+1} + \left(\frac{i}{\sqrt{2}}\right) x^{K+L+1}. \tag{35}$$

One can see that while Equation 35 is divided by $\frac{ix}{\sqrt{2}}$ one gets:

$$\Rightarrow \frac{\mathcal{L}_{K+L}}{\frac{ix}{\sqrt{2}}} = (K + L) x^{K+L} + x^{K+L} \Rightarrow \frac{\mathcal{L}_{K+L}}{\frac{ix}{\sqrt{2}}} - x^{K+L} = (K + L) x^{K+L}. \tag{36}$$

Aside one can see from above right-side Equation 36 one gets directly the following:

$$\frac{\mathcal{L}_{K+L}}{\frac{ix}{\sqrt{2}}} = (K + L + 1) x^{K+L}. \tag{37}$$

Finally, Equation 32 is satisfied under the condition derived of above written as:

$$x = \frac{\sqrt{2}}{i(K + L + 1)}, \tag{38}$$

establishing the fact that under proposal given by Equation 32 $K + L \neq 1$. One can see that if this inequality is violated therefore one can see in a straightforward manner inside Witt algebra the following:

$$[\mathcal{L}_K, \mathcal{L}_L]\Big|_{K+L=1} = (K - L) x^{K+L}\Big|_{K+L=1} \Rightarrow [\mathcal{L}_{1-L}, \mathcal{L}_L] = (1 - 2L) x, \tag{39}$$

yielding another type of algebra differing notably from Witt algebra. It is easy to check that the commutation is null only when $L = \frac{1}{2}$, that cannot be sustained in a scenario of Witt or Virasoro algebra.

2.1. Identification of Symmetry

Interestingly, Witt commutator in last term of Equation 29 can also be derived in a straightforward manner through the assumption that exists symmetry between operators, in the sense that when Equation 27 opts by $K \to L$ and $L \to K$, on gets Equation 29. Thus, has that:

$$\mathcal{L}_K \otimes \mathcal{L}_L = \left(-\frac{1}{2}\right) L(K + 2) x^{K+L} \tag{40}$$

$$K \to L$$

$$L \to K$$

$$\mathcal{L}_L \otimes \mathcal{L}_K = \left(-\frac{1}{2}\right) K(L + 2) x^{L+K}, \tag{41}$$

yielding directly $[\mathcal{L}_K, \mathcal{L}_L] = (K - L) x^{K+L}$ providing the fulfilling of Witt algebra. One can argue that this operation of symmetry might be considered as a sufficient condition to arrive to Witt algebra.

3. Quantum Mechanics Interpretation

Since DeWitt operators Equation 16 and Equation 17 exhibit same structure, then one can carry out kind of "mathematical radiography" in order to interpret them inside QM territory. Thus, for example one can rewrite again Equation 16 as:

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \delta(x - x_m) x_m^K, dx_m, \tag{42}$$

Moreover, by taking into account these definitions through the usage of bra and kets as states of Hilbert space, one gets below:

$$\delta(x - x_m) = \langle x|x_m\rangle, \tag{43}$$

$$x_m^K = \langle x_m|K\rangle. \tag{44}$$

The braket $\langle x_m|K\rangle$ is now defined as a polynomial of Kth-order, so that one can rewrite Equation 42 as:

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \langle x|x_m\rangle \langle x_m|K\rangle, dx_m. \tag{45}$$

Here, it is noteworthy to note that one can appeal to well-known completeness relationship as commonly used in QM:

$$\int_{-\infty}^{\infty} |x_m\rangle \langle x_m| \, dx_m = \mathbb{I}, \tag{46}$$

that applies to integration Equation 45 as follows:

$$\mathcal{L}_K = x_\ell \left(\frac{i}{\sqrt{2}}\right) \delta_\ell^m \frac{d}{dx} \int_{-\infty}^{\infty} \langle x| \, |x_m\rangle \langle x_m| \, dx_m \, |K\rangle, = x_\ell \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) \frac{d}{dx} \langle x| \left[\int_{-\infty}^{\infty} |x_m\rangle \langle x_m| \, dx_m\right] |K\rangle. \tag{47}$$

In this way, operator can again be rewritten as:

$$\mathcal{L}_K = x_\ell \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) \frac{d}{dx} \langle x| \, \mathbb{I} \, |K\rangle = \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) x_\ell \frac{d}{dx} \langle x|K\rangle. \tag{48}$$

With the definition of function $\langle x|K\rangle = \Phi_K(x)$, it allows to arrive to a form of operator as:

$$\mathcal{L}_K = \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) x_\ell \frac{d}{dx} \Phi_K(x), \tag{49}$$

in conjunction with its partner given by:

$$\mathcal{L}_L = \left(\frac{i\delta_\ell^m}{\sqrt{2}}\right) x_\ell \frac{d}{dx} \Phi_L(x). \tag{50}$$

In this way, when are incorporated the Planck's constants in both Equation 49 and Equation 50 the one arrives to:

$$\mathcal{L}_K = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right) x_\ell \frac{\hbar}{i} \frac{d}{dx} \Phi_K(x), \tag{51}$$

$$\mathcal{L}_L = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right) x_\ell \frac{\hbar}{i} \frac{d}{dx} \Phi_L(x). \tag{52}$$

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It's not difficult to observe above that of momentum operator $\mathbf{p}=\frac{\hbar}{i}\frac{d}{dx}$ has been identified. With this one can rewrite Equation 51 and Equation 52 as follows:

$$\mathcal{L}_K = \left(-\frac{\delta^m_\ell}{\sqrt{2}\hbar}\right) x_\ell \mathbf{p}\Phi_K(x), \tag{53}$$

$$\mathcal{L}_L = \left(-\frac{\delta^m_\ell}{\sqrt{2}\hbar}\right) x_\ell \mathbf{p}\Phi_L(x). \tag{54}$$

Consider for example Equation 53, so that both sides are multiplied by $\frac{\mathbf{p}}{M\sqrt{2}}$, in this manner one gets:

$$\frac{\mathbf{p}}{M\sqrt{2}}\mathcal{L}_K = \left(-\frac{\delta^m_\ell}{\hbar}\right) x_\ell \frac{\mathbf{p}^2}{2M}\Phi_K(x) = \left(-\frac{x_m}{\hbar}\right)\frac{\mathbf{p}^2}{2M}\Phi_K(x). \tag{55}$$

One can write again Equation 55 under the argumentation of eigenvalues equations such as:

$$\frac{\mathbf{p}^2}{2M}\Phi_K(x) = -\frac{\hbar\mathbf{p}}{M\sqrt{2}x_m}\mathcal{L}_K. \tag{56}$$

Furthermore, one can claim that Equation 56 is a kind of Schrödinger equation established for free particle whose Hamiltonian operating onto $\Phi_K(x)$: is given by the equation:

$$\mathbf{H}\Phi_K(x) = \frac{\mathbf{p}^2}{2M}\Phi_K(x). \tag{57}$$

From Equation 56 and Equation 57 one arrives to:

$$\mathbf{H}\Phi_K(x) = -\frac{\hbar\mathbf{p}}{M\sqrt{2}x_m}\mathcal{L}_K. \tag{58}$$

It should be noted that emerges a kind of frequency in the sense that:

$$\frac{\mathbf{p}}{M\sqrt{2}x_m} \equiv \omega, \tag{59}$$

by encompassing the balance of physical units in both sides of Equation 58. Thus one gets below that:

$$\mathbf{H}\Phi_K(x) = -\hbar\omega\mathcal{L}_K. \tag{60}$$

Equation 60 offers an interesting scenario to speculate about the physical meaning of that equation. Thus, while is assumed for instance[1] $\Phi_K(x)$=$x^{-(\frac{K}{2}+1)}$ and guided by main definitions Equation 16 and Equation 17 (having a derivative), then one can also assume the relationship:

$$\mathcal{L}_K = \frac{d}{dx}\Phi_K(x), \tag{61}$$

fact that allows to arrive to the Schrödinger equation with the quantized energies of harmonic oscillator as:

$$\mathbf{H}\Phi_K(x) = \hbar\omega(\frac{K}{2}+1)\Phi_K(x). \tag{62}$$

[1] It is allowed in virtue to polynomial character of Equation 16 and Equation 17.

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3.1. Derivation of Schrödinger Equation

Despite of the fact that to some extent procedures above have been done inside a purely abstract (algebraic) scheme, some noteworthy aspects associated to tangible physics such as the harmonic oscillator (e.g. Equation 62) presented above, might be consistently derived. For example, consider the case that states are canceling each other:

$$\mathcal{L}_K + \Phi_K(x) = 0, \tag{63}$$

implying the triviality $\mathcal{L}_K \Rightarrow -\Phi_K(x)$, then from Equation 58 one arrives to:

$$\mathbf{H}\Phi_K(x) = \frac{\hbar\mathbf{p}}{M\sqrt{2}x_m}\Phi_K(x). \tag{64}$$

In this manner, one should be guarantee that $\frac{\hbar\mathbf{p}}{M\sqrt{2}x_m}$ must be consistent with units of energy. It can be so only if:

$$\frac{\mathbf{p}}{Mx_m\sqrt{2}} = \frac{1}{t} \equiv \omega, \tag{65}$$

having units of frequency. In this way, one can multiply by left-side in both terms of Equation 59 by $Mx_m^2\omega$ yielding:

$$\sqrt{\frac{1}{8}}\omega x_m\mathbf{p} = \frac{1}{2}Mx_m^2\omega^2 = \mathcal{E}_{\text{HAR}}, \tag{66}$$

the classical energy of harmonic oscillator again, and units are checked both sides. Besides, from Equation 53 for example, one has again that:

$$\mathcal{L}_K = \left(-\frac{\delta_\ell^m}{\sqrt{2}\hbar}\right)x_\ell\mathbf{p}\Phi_K(x) = \left(-\frac{1}{\sqrt{2}\hbar}\right)x_m\mathbf{p}\Phi_K(x). \tag{67}$$

With the usage of Equation 65 and Equation 66 one arrives to:

$$\mathcal{L}_K = \left(-\frac{2}{\hbar\omega}\right)\sqrt{\frac{1}{8}}\omega x_m\mathbf{p}\Phi_K(x). \tag{68}$$

It is easy to see that now operator is proportional to energy of harmonic oscillator, so that one can write down that:

$$\mathcal{L}_K = \left(-\frac{2}{\hbar\omega}\right)\mathcal{E}_{\text{HAR}}\Phi_K(x). \tag{69}$$

Furthermore, based in the approximation Equation 65:

$$\omega = \frac{1}{\Delta t}. \tag{70}$$

Besides, one can extend the meaning of Equation 63 so that one has two roots:

$$(\Phi_K(x) - \mathcal{L}_K)\left(\Phi_K(x) - \frac{i}{2}\Psi_K(x)\right) = 0, \tag{71}$$

by allowing to arrive in the reformulation of field $\Phi_K(x)$ as:

$$\Phi_K(x) \Rightarrow \frac{i}{2}\Psi_K(x), \tag{72}$$

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then operator Equation 69 can be written as:

$$\mathcal{L}_K = \left(-\frac{i\mathcal{E}_{\text{HAR}}\Delta t}{\hbar}\right)\Psi_K(x). \tag{73}$$

Note that if Δt passes to left-side then one has below:

$$\frac{\mathcal{L}_K}{\Delta t} = \left(-\frac{i\mathcal{E}_{\text{HAR}}}{\hbar}\right)\Psi_K(x), \tag{74}$$

whose right-side can also be perceived as the time-derivative of a kind of evolution operator in the sense as:

$$\frac{\mathcal{L}_K}{\Delta t} \equiv \frac{\partial}{\partial t}\mathrm{U}(t-t_0) = \frac{\partial}{\partial t}\mathrm{Exp}\left[-i\frac{i\mathcal{E}_{\text{HAR}}(t-t_0)}{\hbar}\right] = \left(-\frac{i\mathcal{E}_{\text{HAR}}}{\hbar}\right)\Psi_K(x), \tag{75}$$

by which it was assumed $\Delta = t - t_0$. By taking into account first and last term of Equation 75 (from left to right) it can also be written in a more familiar manner as:

$$i\hbar\frac{1}{\Delta t}\mathcal{L}_K = \mathcal{E}_{\text{HAR}}\Psi_K(x). \tag{76}$$

The fact by which the kinetic part is missing, might be associated to the approximation under the assumption[2] of $\frac{\mathcal{E}_{\text{KI}}}{\mathcal{E}_{\text{HAR}}} \approx 0$

$$\mathcal{E}_{\text{HAR}} \approx \mathcal{E}_{\text{HAR}}\left(1 + \frac{\mathcal{E}_{\text{KI}}}{\mathcal{E}_{\text{HAR}}}\right) \Rightarrow \mathcal{E}_{\text{HAR}} + \mathcal{E}_{\text{KI}}. \tag{77}$$

In infinitesimal times implying $\frac{1}{\Delta} \to \frac{\partial}{\partial t}$ so that one can employ time derivative in left-side of Equation 57, therefore one arrives:

$$i\hbar\frac{\partial}{\partial t}\mathcal{L}_K = (\mathcal{E}_{\text{HAR}} + \mathcal{E}_{\text{KI}})\,\Psi_K(x) = \mathcal{E}\Psi_K(x). \tag{78}$$

One can see that if operator $\mathcal{L}_K$ coincide to field $\Psi_K(x)$ (or $\mathcal{L}_K = \Psi_K(x)$), then because this then one can write down the Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\Psi_K(x) = \mathcal{E}\Psi_K(x). \tag{79}$$

It should be noted that under the equality $\mathcal{L}_K = \Psi_K(x)$ and Equation 72 one also arrives to:

$$(\mathcal{L}_K - \Psi_k)(\mathcal{L}_K - \frac{i}{2}\Psi_k) = 0, \tag{80}$$

leading to quadratic equation:

$$\mathcal{L}_K^2 - \left(\frac{i}{2}+1\right)\Psi_K\mathcal{L}_K - \frac{i}{2}\Psi_K^2 = 0. \tag{81}$$

by which holds the commutation $[\mathcal{L}_K, \Psi_K] = 0$, clearly not necessarily to be fulfilled.

---

2 That can also be understood as the inequality: $x_m \gg \frac{\mathrm{P}}{M\omega}$ in the classical limit applying systems with a large amplitude of oscillation

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4. Results

As shown above both evolution operator and Schrödinger equation might to be revealing some aspects that are not evident from Witt algebra as well as from the operators initially defined by Equation 13 and Equation 14. In this manner, one can argue that these central operators and involved algebra would be intrinsically associated to aspects of quantum mechanics, more than to be simple abstract operators belonging to unphysical Witt algebra. Under this view, Equation 49 and Equation 50 can be harnessed to be used directly, with $\mathbf{p} = \frac{\hbar}{i}\frac{d}{dx}$, in commutator such as $[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_K \otimes \mathcal{L}_L - \mathcal{L}_L \otimes \mathcal{L}_K =$

$$
\begin{aligned}
[\mathcal{L}_K, \mathcal{L}_L] &= \left[\frac{-\delta^m_\ell x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_K(x)\right] \otimes \left[\frac{-\delta^m_\ell x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_L(x)\right] - \left[\frac{-\delta^m_\ell x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_L(x)\right]\left[\frac{-\delta^m_\ell x_\ell}{i\sqrt{2}}\frac{d}{dx}\Phi_K(x)\right] && (82)\\
&= \left[\frac{\delta^m_\ell x_\ell}{i\sqrt{2}}\frac{\delta^m_\ell x_\ell}{i\sqrt{2}}\right] \otimes \left[\frac{d}{dx}\Phi_K(x)\frac{d}{dx}\Phi_L(x)\right] - \left[\frac{\delta^m_\ell x_\ell}{i\sqrt{2}}\frac{\delta^m_\ell x_\ell}{i\sqrt{2}}\right] \otimes \left[\frac{d}{dx}\Phi_L(x)\frac{d}{dx}\Phi_K(x)\right] && (83)\\
&= \left[\frac{x_m^2}{2}\right] \otimes \left[\frac{d}{dx}\Phi_L(x)\frac{d}{dx}\Phi_K(x)\right] - \left[\frac{x_m^2}{2}\right] \otimes \left[\frac{d}{dx}\Phi_K(x)\frac{d}{dx}\Phi_L(x)\right] && (84)\\
&= \left[\frac{x_m^2}{2}\right]\left(\frac{d}{dx}\Phi_L(x)\frac{d}{dx}\Phi_K(x) - \frac{d}{dx}\Phi_K(x)\frac{d}{dx}\Phi_L(x)\right). && (85)
\end{aligned}
$$

Whereas, not any rule of commutation between $\Phi_K(x)$ and $\Phi_L(x)$ has been explicitly established, then it is assumed that $[\Phi_K(x), \Phi_L(x)] \neq 0$.

On the other side, one can employ the momentum-based definitions as seen in Equation 67 for both cases: $\mathcal{L}_K = \frac{-\delta^m_\ell x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x)$ and $\mathcal{L}_L = \frac{-\delta^q_\ell x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x)$) inside Equation 82 so that one gets:

$$
\begin{aligned}
[\mathcal{L}_K, \mathcal{L}_L] &= \frac{-\delta^m_\ell x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x)\frac{-\delta^q_\ell x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x) - \frac{-\delta^q_\ell x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x)\frac{-\delta^m_\ell x_\ell}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x) && (86)\\
&= \frac{x_m}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x)\frac{x_q}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x) - \frac{x_q}{\sqrt{2}\hbar}\mathbf{p}\Phi_L(x)\frac{x_m}{\sqrt{2}\hbar}\mathbf{p}\Phi_K(x) && (87)\\
&= \frac{1}{2\hbar^2}\left[x_m\mathbf{p}x_q\mathbf{p}\Phi_K(x)\Phi_L(x) - x_q\mathbf{p}x_m\mathbf{p}\Phi_L(x)\Phi_K(x)\right]. && (88)
\end{aligned}
$$

The pass from Equation 86 to Equation 87 has demanded to accept the validity of these commutators:

$$
\begin{aligned}
[\Phi_K, x_q] &= 0, && (89)\\
[\Phi_L, x_m] &= 0. && (90)
\end{aligned}
$$

From above Equation 88 might be also written as[3]:

$$
\begin{aligned}
[\mathcal{L}_K, \mathcal{L}_L] &= \frac{1}{2\hbar^2}\left[x_m\mathbf{p}x_q\mathbf{p}\Phi_K(x)\Phi_L(x) - x_q\mathbf{p}x_m\mathbf{p}\Phi_K(x)\Phi_L(x) + x_q\mathbf{p}x_m\mathbf{p}\Phi_K(x)\Phi_L(x) - x_q\mathbf{p}x_m\mathbf{p}\Phi_L(x)\Phi_K(x)\right] && (91)\\
&= \frac{1}{2\hbar^2}\left[[x_m\mathbf{p}, x_q\mathbf{p}]\Phi_K(x)\Phi_L(x) + x_q\mathbf{p}x_m\mathbf{p}[\Phi_K(x), \Phi_L(x)]\right]. && (92)
\end{aligned}
$$

Clearly, from $[x_m\mathbf{p}, x_q\mathbf{p}]$ in Equation 92 it is needed the usage of identity $[AB, CD]$=$A[B, C]D$+$[A, C]BD$ + $CA[B, D]$ + $C[A.D]B$. By applying this in a straightforward manner one gets the following:

$$
[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2\hbar^2}\left[(x_q[x_m, \mathbf{p}]\mathbf{p} - x_m[x_q, \mathbf{p}]\mathbf{p})\Phi_K(x)\Phi_L(x) + x_q\mathbf{p}x_m\mathbf{p}\{\Phi_K(x), \Phi_L(x)\}\right], \tag{93}
$$

[3] It will be added and subtracted $x_q\mathbf{p}x_m\mathbf{p}\Phi_K(x)\Phi_L(x)$.

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demonstrating the validity of hypothesis formulated in Equation 7. **Equation 93 becomes the main result of this paper**. The general formulation of CCR given by $[x_I, p_J]$=$i\hbar\delta_{I,J}$ stop us to go through commutators in Equation 93, essentially because neither $x_{m,q}$ nor $\mathbf{p}$ have been explicitly specified. In this way, various scenarios might to emerge for the choice of a concrete component of $\mathbf{p}$, as well as for x_m and x_q. The particular case when $m = q$ reduces Equation 93 to:

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2}\left(\frac{x_m \mathbf{p}}{\hbar}\right)^2 [\Phi_K(x), \Phi_L(x)]\,. \tag{94}$$

Because commutator $[\mathcal{L}_K, \mathcal{L}_L]$ has acquired now a certain physical meaning because the position and momentum operators, then Equation 94 can also be rewritten as function of observables measurements such as:

$$[\mathcal{L}_K, \mathcal{L}_L] = \frac{1}{2}\left(\frac{\Delta x_m \Delta\mathbf{p}}{\hbar}\right)^2 [\Phi_K(x), \Phi_L(x)]\,. \tag{95}$$

Inspired at the uncertainty principle, one can impose the following restriction:

$$\Delta x_m \Delta\mathbf{p} \geq \sqrt{2}\hbar, \tag{96}$$

yielding an alternative redefinition of Witt algebra through new operators $\Phi_K(x)$ and $\Phi_L(x)$ as:

$$[\mathcal{L}_K, \mathcal{L}_L] = [\Phi_K(x), \Phi_L(x)]\,, \tag{97}$$

entering in a total contradiction with Equation 63. This suggest to keep Equation 96 as $\Delta x_m \Delta\mathbf{p} \geq \hbar$ that allows to write Equation 95 down as:

$$[\mathcal{L}_K, \mathcal{L}_L] = \left[-i\frac{\Phi_K(x)}{\sqrt{2}}, i\frac{\Phi_L(x)}{\sqrt{2}}\right], \tag{98}$$

by which emerge the complex version of Equation 63 in the sense of:

$$\mathcal{L}_K + i\frac{\Phi_K(x)}{\sqrt{2}} = 0, \tag{99}$$

$$\mathcal{L}_L - i\frac{\Phi_L(x)}{\sqrt{2}} = 0. \tag{100}$$

5. Conclusion

Along this paper, it was presented a mathematical methodology that allows to demonstrate that schemes based at Witt algebra are consistently linked to quantum mechanics canonical commutation relation (CCR), despite of the fact that not any assumption associated to quantum variables nor quantum mechanics formalism was deeply considered. As seen in Equation 16 and Equation 17, operators have been defined built on the basis of Krönecker delta and Dirac delta function, as well as the operations of derivative and integration. After closed-form operations as seen in previous sections these operators have turned out to fulfill the Witt algebra. Although some abstract procedures have been applied, it was seen that all that have acquired sense when quantum mechanics definitions were shortly applied. Particularly, the momentum operator emerged in a spontaneous manner. It has played a noteworthy importance as noted in last section of paper. In this manner, based rigorously on the presented formalism (by which it might be extended and improved) Witt algebra has turned out to be proportional to well-known CCR. Certainly, more formalism and operations would have to be added in order to claim a robust proportionality between DeWitt (Virasoro central charge [40]) algebra and quantum mechanics canonical commutation relation.

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