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*Article*

# The General Semimartingale Market Model

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**Abstract:** This paper develops a unified framework for mathematical finance under general semimartingale models that allow for dividend payments, negative asset prices, and unbounded jumps. We present a rigorous approach to the mathematical modeling of financial markets with dividend-paying assets by defining appropriate concepts of numéraires, discounted processes, and self-financing trading strategies. While most of the mathematical results are not new, this unified framework has been missing in the literature. We carefully examine the transition between nominal and discounted price processes and define appropriate notions of admissible strategies that work naturally in both settings. By establishing the equivalence between these models and providing clear conditions for their applicability, we create a mathematical foundation that encompasses a wide range of realistic market scenarios and can serve as a basis for future work on mathematical finance and derivative pricing.

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## 1. Introduction

The theory of modern financial mathematics in its present form has its origin in the dissertation "Théorie de la Spéculation" by L.F. Bachelier from 1900 [1]. In this publication, one finds the first mathematical description of Brownian motion as a stochastic process (although not under that name). Bachelier's goal was to derive theoretical values for various types of options on certain goods by modelling prices of goods using a Brownian motion and comparing these prices with actual market prices. He proposed as option prices the expected value of the payment arising from the option. The crucial shortcoming in Bachelier's modelling was that the prices of goods could become negative.

Bachelier's work was forgotten for a long time. It was only after the development of the stochastic integral and the introduction of geometric Brownian motion as a pricing model in the 1960s that financial mathematics revived [2–7].

In 1973, Fischer Black and Myron Scholes made the decisive breakthrough [8] by developing the famous Black-Scholes equation and formula. Since then, financial mathematics has become a huge field of research, and numerous models have been proposed and analysed. The progress and advancement of stochastic analysis and stochastic integral, mainly by [9–21] also opened up numerous new possibilities for financial mathematics. In particular, the modern approach of option pricing according to the duplication principle has established itself as a standard. This approach is a natural application of the martingale theory and representation theorem. Here [22] can be considered as a cornerstone.

However, at that time, while many different models had been examined and studied, there was no overarching theory that combined all these models to lay the groundwork for modelling financial markets. A significant breakthrough in the general theory of financial mathematics was achieved in the 1990s by [23] and [24] by presenting a very general financial market model that included almost all of the known models and proving the connection between arbitrage and mathematical conditions on the existence of specific probability measures. Since then, most publications have referred to this model. The financial market, as discussed in [24], can be seen as the general market, which comprises almost all models of frictionless markets that are used in practice. Therefore, the results are universal, and the general set-up of the market is, without a doubt, the most important market model in Mathematical Finance. However, there are some gaps and shortcomings in the literature.

- The model in [24] assumes a discounted set up (sometimes also referred as normalized set up). However, most models used in practice are described in non-discounted terms in order to be able to verify its assumption with real-world observations. The question how and under which assumptions non-discounted set ups can be transformed to discounted set-ups has not been described in general terms, but only for specific models.
- The model as it is presented in [24] does not consider dividends or additional cash-flows, and therefore excludes some essential models, such as models for pricing and setting of futures. In many cases, it is possible to transform dividend-paying models to non-dividend models (see for example [25], Section 2.3). Therefore, an extension of the initial model to include dividends is desirable.
- Some basic properties of market models are often assumed to be true without validating them for this very general model. This applies to the notion of admissible strategies, discounted processes, numéraires and so on. In particular, since the general market model allows for negative prices some of the available properties get more complicated or even completely devalidated.

The goal of this paper is to close the gaps mentioned above and to define a general market model that considers dividend payments in a real-world set-up. Then introduce the notion of a numéraire and transform the model to a discounted set-up. The special conditions and technical requirements are investigated and motivated. Furthermore, the literature on these topics and some examples are reviewed.

## 2. The General Semimartingale Model with Dividends

In this section, we are going to define a very general market model with dividends. Our discussion primarily centers on time-continuous models. Although discrete models can be viewed as a subset of time-continuous models, they are typically easier to navigate mathematically. Nevertheless, time-continuous models are arguably more popular in financial mathematics, especially within portfolio theory. One reason why continuous-time models are popular is that they often give clean, unique solutions to optimization problems. In contrast, discrete-time models frequently lack unique solutions and only approximate strategies are available, which then, in turn, again complicates calculations.

For a given $\mathbb{R}^d$-valued semimartingale $S$, the space $L(S)$ is defined as the set of possible integrands for $S$ for the general vector-valued stochastic integrals for a semimartingale integrator. Furthermore for a semimartingale $S, \varphi \in L(S)$ and $t \in \mathbb{R}_+$

$$\varphi \bullet S_t = \varphi_0^\top S_0 + \int_{(0,t]} \varphi \mathrm{d}S_u$$

denotes the stochastic integral at time $t$. For the definition of the general vector valued integral, see [26], or [27].

Consider a financial market that comprises $d+1$ tradable securities. The price processes of these securities are modelled by the $d+1$-dimensional process

$$S_t = (S_t^0, S_t^1, \dots, S_t^d)_{t\in\mathbb{R}_+}.$$

These processes also have associated cumulative dividend processes, termed

$$D = (D_t^0, D_t^1, \dots, D_t^d)_{t\in\mathbb{R}_+},$$

which are adapted to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

We assume that the market is both frictionless and competitive, where frictionless means there are neither transaction costs nor taxes or trading constraints such as short sale restrictions, borrowing limits, and margin requirements. Furthermore, assets are assumed to be infinitely divisible. We call a market competitive when traders function as price takers. Hence, trades can engage in trading activities involving any desired quantity of shares without impacting the market price. This ensures that there is no presence of liquidity risk.

Furthermore, we assume

- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ is a filtered probability space with probability measure $\mathbf{P}$.
- The processes $S_t^i$ and $D_t^i$ are semimartingales for all $i = 0, \ldots, d$.
- The filtration $\mathbb{F}$ satisfies the usual conditions and the $\sigma$ -algebra $\mathcal{F}_0$ is trivial, that is, $A \in \mathcal{F}_0$ implies $\mathbf{P}(A) = 0$ or $\mathbf{P}(A) = 1$.

**Remark 1.** *Markets with dividends can often be transformed into dividend-free ones. However, these transformations usually need additional assumptions to hold that are not universally applicable. For instance, [25] presumes all cash flows to be positive and that the numéraire follows a deterministic continuous FV process.*

The process $D_t^i$ denotes the cumulative dividend payments of the $i$-th share up until the time $t$. We don't assume monotonicity; hence, dividend payments might even be negative. Dividend processes that potentially have negative increments are crucial for the study of certain products such as futures as they can simply be seens cash flows (for example margin payments).

Generally, dividend processes are categorized into three types:

The first type comprises dividend processes with continuous paths, which, while mathematically convenient, isn't particularly realistic.

The second assumes discrete dividend payments, with the dividend process $D$ being a pure jump process $(D = \sum \Delta D)$. This is more aligned with reality, though not straightforward mathematically.

The third type allows for any mixture of the two - combining both continuous dividend streams and discrete payments. This is of course the most general case and the one we consider in this paper.

Mathematically speaking, it's reasonable to presume price processes as semimartingales because the most widely accepted definition of the general stochastic integral only incorporates semimartingales as integrators. This assumption also finds economic justification; for instance, if the price process is locally bounded, adapted, and the market adheres to the 'No Free Lunch With Vanishing Risk' principle (a variant of 'No Arbitrage'), then the price process is a semimartingale, as demonstrated in [28] (Theorem 8) and or [23] (p. 504-507). Another economic rationale is provided in [29], where Constantinos Kardaras and Eckhard Platen show that in markets where only simple predictable trading strategies are permitted, where short-selling is disallowed and no-arbitrage principles hold, price processes are always semimartingales [29].

Requiring all price processes to be semimartingales excludes fractional Brownian motions with an Hurst parameter $H \neq \frac{1}{2}$. As of now, no consistent no-arbitrage theory for these processes in a frictionless, continuously trading market exists. Yet, in markets with transaction costs, such arbitrage possibilities typically vanish. In these settings fractional Brownian motions are considered realistic and reasonable, for example, in [30].

### *2.1. Self-Financing Trading Strategies*

**Definition 1.** *(a) A $d+1$-dimensional process $\varphi = (\varphi^0, \ldots, \varphi^{d+1}) \in L(S)$ is called a trading strategy.*
*(b) The wealth process of the investor is defined as*

$$V_t := \sum_{i=0}^{d} \varphi_t^i \left( S_t^i + \Delta D_t^i \right), \quad t \in \mathbb{R}_+, \tag{1}$$

where $\varphi_t^i$ represents the number of the $i$-th security that an investor holds in his portfolio at $t$.

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**Definition 2.** *A trading strategy* $\varphi = (\varphi^0, \varphi^1, \ldots, \varphi^d)$ *is called self-financing if the wealth process* $V_t(\varphi)$ *satisfies*

$$V_t = \varphi_s \bullet (S + D)_t, \quad \text{for all } t \in \mathbb{R}_+. \tag{2}$$

Note (2) deals with higher dimensional processes and hence multi-dimensional integration.

**Remark 2.** *A trading strategy* $\varphi$ *is self-financing if the changes in the associated wealth process* $V(\varphi) = \sum_{i=0}^{d} \varphi^i (S^i + \Delta D^i)$ *are the result of changes in the price of the assets included. All shifts in the portfolio are cost-neutral and there is no external capital injection or withdrawal. This definition and explanation can be better illustrated by considering time-discrete processes (as a special case of the time-continuous processes).*

*To keep it simple, we consider only a one-dimensional process in this illustration. In the discrete model, let* $\varphi_n$ *be the number of securities purchased at price* $S_{n-1}$ *(i.e. after the jump* $S_{n-1} - S_{n-2}$*. Since the dividends accrued in the previous period* $\varphi_{n-1}\Delta D_{n-1}$ *still have to be reinvested, the self-financing condition is*

$$\varphi_n S_{n-1} = \varphi_{n-1}(S_{n-1} + \Delta D_{n-1}),\ n = 1, 2, \tag{3}$$

*It can be shown that* (3) *is equivalent to*

$$\varphi_n(S_n + \Delta D_n) = \varphi_{n-1}(S_{n-1} + \Delta D_{n-1}) + \varphi_n(\Delta S_n + \Delta D_n), \quad n = 1, 2, \ldots \tag{4}$$

(4) *is the discrete-time special case of condition* (2).

**Remark 3.** *The variable* $\varphi_t^i$ *represents the quantity of securities i held by the agent within the portfolio, specifically within the time frame of* $t-1$ *to* $t$ *(or, in continuous time* $t-$ *to* $t$*). The jumps referred to as* $\Delta S_t^i$ *and* $\Delta D_t^i$ *should be considered as synchronous. The term* $S^i$ *can be conceptualized as the ex-dividend price, which refers to the price of a security after the dividend payment has been made.*

*The dividend payout for the agent at time t is*

$$\sum_{i=0}^{d} \varphi_t^i \Delta D_t^i.$$

*This is now invested in the* $d+1$ *securities immediately after* $t$*. So, as before, there is no permanent cash holding. Therefore, only* $\Delta D$ *and not* $D$ *appears in* (1).

### 2.2. Numéraire

In practice, comparing two assets at different times based on their nominal size is unusual and makes no sense. Therefore, a benchmark should be introduced that enables us to produce comparative values. We call this benchmark numéraire.

**Definition 3.** *A predictable semimartingale* $(N_t)_{t \in \mathbb{R}_+}$ *which satisfies*

$$\inf_{t \in [0,T]} N_t > 0, \quad \text{for all } T \in \mathbb{R}_+ \ \mathbf{P} - a.s. \tag{5}$$

*is called a numéraire.*

**Remark 4.** *The predictability is necessary to ensure that* $\frac{1}{N}$ *can be used as integrand. This is important to define the discounted dividend process and to map the self-financing condition of price processes to the self-financing condition of their discounted counterparts.*

If one writes asset values as multiples of the numéraire, they can also be compared independently of the time. The discounted processes are denoted by $\hat{S}^i$ or $\hat{V}$, which means:

$$\hat{S}^i := \frac{S^i}{N} \quad \text{and} \quad \hat{V} := \frac{V}{N}. \tag{6}$$

For further analysis, one needs the following result.

**Lemma 1.** *Let $N$ be a numéraire, then $\frac{1}{N}$ is bounded on each compact interval and in particular locally bounded.*

**Proof.** By definition of a numéraire, $\inf_{t\in[0,T]} N_t > 0$ a.s. for all $T \in \mathbb{R}_+$. Therefore, on any compact interval $[a, b]$, there exists $\delta > 0$ such that $N_t > \delta$ a.s. for all $t \in [a, b]$, which immediately implies $\frac{1}{N_t} < \frac{1}{\delta}$ a.s. on $[a, b]$. □

The definition of the numéraire is not unique in the literature. For instance, in [31], [32], or [33], it is only required that $N$ should be a positive semimartingale. This notion is also used in [34], which is regarded as the standard reference for a numéraire in an abstract context. However, to ensure that discounted price processes remain semimartingales, the numéraire's left limit must also be greater than 0. Even stricter conditions are demanded in specific settings such as [35]. Additionally, distinctions between strong and weak numéraire concepts are discussed in [36]. Our interpretation is equivalent to the one presented in [37]. Recent publications have started to describe financial market models without the concept of a numéraire. Nevertheless, more intricate market assumptions, as described in [38], are necessary to maintain the feasibility of discounting price processes.

Drawing parallels to (6), it is essential to introduce both a discounted dividend process, $\hat{D}^i$, and a discounted wealth process, $\hat{V}$. The process $\hat{D}_t$ denotes cumulative dividends up to a given time. However, each dividend gets discounted by the numéraire value, $N$, at its payment time and not by its value at $t$. This approach ensures that any change in the discounted dividend process, $\hat{D}^i$, occurs only when the dividend process, $D^i$, changes. In scenarios like $\frac{D^i}{N}$, these properties would not hold since a mere change in the numéraire $N$ could lead to a change in $\frac{D^i}{N}$.

Therefore, the following definition naturally arises.

**Definition 4.** *By*

$$\hat{D}^i_t := \frac{1}{N} \bullet D^i_t, \; i = 0, \ldots, d. \tag{7}$$

*the discounted dividend processes is denoted. Furthermore,*

$$\hat{V}(\varphi) := \frac{V}{N}(\varphi) = \sum_{i=0}^{d} \varphi^i (\hat{S}^i + \Delta \hat{D}^i)$$

*denotes the discounted wealth process.*

**Remark 5.** *By Theorem A1, one has $\Delta \hat{D}^i = \frac{\Delta D^i}{N}$.*

While it is commonly assumed that $D$ is a pure jump process, we want to avoid making this assumption here and rather stay as general as possible.

However, if we operate under the assumption that the dividend process is a pure jump process (meaning $D = \sum \Delta D$ is valid), then, by Theorem A1, we arrive at

$$\hat{D}^i = \frac{1}{N} \bullet D^i_t = \frac{1}{N} \bullet \left( \sum_{0\le s\le t} \Delta D^i_s \right) = \sum_{0\le s\le t} \Delta \left( \frac{1}{N} \bullet D^i{}_s \right) = \sum_{0\le s\le t} \Delta \hat{D}^i_s.$$

.

In this scenario, $\hat{D}$ also becomes a pure jump process, only changing when $D$ does.

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Economically speaking, (7) is logical as it considers the varying times at which increases in $D$ are paid out as dividends. For instance, let's say $N = \exp(rt)$ and there was just one dividend payment of amount $K$. Intuitively, a dividend disbursed earlier (at $t = 0$) holds more value than one distributed later (at $T = 1$). If we were to assign the value $\hat{D}^i = \frac{D^i}{N}$, the timing of this singular dividend payment wouldn't matter for $\hat{D}$. The result would be $\hat{D}_1 = \frac{K}{\exp(r)}$. However, with our definition of the discounted dividend process, and by Theorem A1, we get:

$$\hat{D}_0 = K \quad \text{and} \quad \hat{D}_1 = \frac{K}{\exp(r)} < K.$$

The following theorem is extremely beneficial, as it offers the technical requirements to explore properties in discounted settings using the conventional tools of stochastic analysis.

**Theorem 1.** *The processes $N, S^0, \ldots, S^d, D^0, \ldots, D^d$ are semimartingales and $N$ is a numéraire if and only if the processes $\frac{1}{N}, \hat{S}^0, \ldots \hat{S}^d, \hat{D}^0, \ldots, \hat{D}^d$ are semimartingales and $\frac{1}{N}$ is a numéraire. If, furthermore, $\varphi$ is self-financing, then the discounted wealth process $\hat{V}(\varphi)$ is a semimartingale.*

**Proof.** "$\Rightarrow$" Let $N$ be a numéraire and $S^0, \ldots, S^d$ semimartingales. We define

$$T^n := \inf\left\{t \geq 0\,;\, N_t \leq \frac{1}{n}\right\}.$$

Now $(T^n)$ is by (5) a localizing sequence and we examine the processes

$$N_t^n := (N^n)_t^{T^n-} := \begin{cases} N_t & \text{for } t < T^n, \\ N_{T_n-} & \text{for } t \geq T^n. \end{cases}$$

Let $f_n$ be a convex function, which satisfies $f_n(x) = \frac{1}{x}$ for $x \geq \frac{1}{n}$. By Remark A1, we obtain that $f_n(N^n)$ is a semimartingale and since $N^n \geq \frac{1}{n}$ holds, $\frac{1}{N^n}$ is also a semimartingale. Thus $\frac{1}{N^n}$ is prelocally a semimartingale and hence, by Theorem A5, a semimartingale. Since $N$ is a semimartingale and therefore in particular càdlàg, we also have

$$\mathbf{P}\left(\sup_{t\in[0,T]} N_t < \infty\right) = 1 \quad \text{for all } T \in \mathbb{R}_+$$

and hence

$$\inf_{t\in[0,T]} \frac{1}{N_t} > 0 \quad \text{for all } T \in \mathbb{R}_+ \quad \mathbf{P} - \text{almost surely.}$$

Thus $\frac{1}{N}$ is a numéraire and $\hat{S}^i = \frac{S^i}{N}$ is by Theorem A2 also a semimartingale.

"$\Leftarrow$" Let $\frac{1}{N}, \hat{S}^0, \ldots, \hat{S}^d$ be semimartingales and $\frac{1}{N}$ a numéraire.

If we proceed as above, we obtain that $N = \frac{1}{\frac{1}{N}}$ is a semimartingale and by arguing as above, it follows that $N$ is càdlàg. Hence $N$ is a numéraire. Since

$$\hat{\hat{S}}^i = \frac{\hat{S}^i}{\frac{1}{N}} = S^i \quad i = 0, \ldots d$$

holds, we obtain that $S^i$ are also semimartingales.

Now we assume $\varphi$ to be self-financing. Then we have $\hat{V}_t = \frac{\int_0^t \varphi \mathrm{d}S_u}{N_t}$, which is a semimartingale by Theorem A2. □

Most papers tacitly treat nominal models, their discounted counterparts and settings with or without dividends as interchangeable: one simply divides by a numéraire or subtracts cumulative dividends and all desired results still hold. In fact, this equivalence is valid for virtually every statement

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we need in this section. The next theorem, however, shows that a mild additional requirement, namely a specific orthogonality between the numéraire scaled dividend streams and the numéraire itself, is necessary to transfer the self financing property from the nominal to the discounted world and back.

To state the condition we recall the decomposition of a semimartingale $X = X^c + X^d$ into its continuous part $X^c$ and its purely discontinuous part $X^d$.

**Definition 5.** *For two semimartingales $X$ and $Y$ we set*

$$[X,Y]^c_t := [X^c, Y^c]_t, \qquad t \in [0,T].$$

*In particular, $[X]^c := [X,X]^c$.*

With the above definition, we get

$$[X,Y]_t = [X,Y]^c_t + \sum_{0 \le s \le t} \Delta X_s \Delta Y_s. \tag{8}$$

The following theorem states that the self-financing property stays valid regardless of whether the associated wealth process is viewed in nominal or discounted terms.

**Theorem 2.** *Let $\varphi = (\varphi^0, \varphi^1, \ldots, \varphi^d)$ be a trading strategy, $V$ be the associated wealth process and $N$ a numéraire with $[N, D^i]^c = 0$ for all $i \in \{0, \ldots, d\}$. Then $\varphi$ is self-financing if and only if*

$$\hat{V}_t(\varphi) = \sum_{i=0}^{d} \varphi^i \bullet (\hat{S}^i + \hat{D}^i)_t, \quad t \in [0,T], \tag{9}$$

*where $\hat{v}_0 := \frac{v_0}{N_0}$*

**Proof.** We show the one-dimensional case. The multi-dimensional case follows from the linearity of the integral and the fact that any strategy can be approximated by component-wise integrable strategy.

We have $\Delta V = \varphi \Delta S$ and therefore, it is easy to see that

$$V_- = \varphi S_- \quad \text{and} \quad \hat{V}_- = \varphi \hat{S}_-.$$

We first assume

$$\hat{V}_t(\varphi) = \varphi \bullet (\hat{S} + \hat{D})_t.$$

By Theorem 1, $\hat{V} = \hat{V}(\varphi)$ is a semimartingale and with Theorem A1, we obtain

$$\Delta \hat{V} = \varphi \Delta (\hat{S} + \hat{D}).$$

By definition, we also have

$$\hat{V} = \varphi(\hat{S} + \Delta \hat{D}),$$

and therefore,

$$\hat{V}_- = \hat{V} - \Delta \hat{V} = \varphi(\hat{S} - \Delta \hat{S}) = \varphi \hat{S}_- \tag{10}$$

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With Theorem A4, we obtain

$$\begin{aligned} V_t = \hat{V}_t N_t &= N_- \bullet \hat{V}_t + \hat{V}_- \bullet N_t + [N, \hat{V}]_t \\ &= N_- \bullet (\varphi \bullet (\hat{S} + \hat{D}))_t + (\varphi \hat{S}_-) \bullet N_t + [N, \varphi \bullet (\hat{S} + \hat{D})]_t \\ &= \varphi \bullet (N_- \bullet (\hat{S} + \hat{D}) + \hat{S}_- \bullet N + [N, \hat{S} + \hat{D}])_t \\ &= \varphi \bullet (N_- \bullet \hat{S} + (N - \Delta N) \bullet \hat{D} + \hat{S}_- \bullet N + [N, \hat{S}] + [N, \hat{D}])_t \\ &= \varphi \bullet \Big( \underbrace{N_- \bullet \hat{S} + \hat{S}_- \bullet N + [N, \hat{S}]}_{= N\hat{S}} + N \bullet \hat{D} - \sum \Delta N \Delta \hat{D} + \underbrace{[N, \hat{D}]^c}_{=0} + \sum \Delta N \Delta \hat{D} \Big)_t \\ &= \varphi \bullet (N\hat{S} + N \bullet \hat{D})_t \\ &= \varphi \bullet \left( N\hat{S} + N \bullet \left( \frac{1}{N} \bullet D \right) \right)_t \\ &= \varphi \bullet (S + D)_t \end{aligned}$$

Hence $\varphi$ is self-financing. The converse follows analogously with $N' = \frac{1}{N}$ and $\hat{V} = VN'$.

Since each trading strategy can be approximated with component-wise integrable trading strategy, the general result follows by taking limits. □

**Definition 6.** *Let $S = (S^0, \dots, S^d)$ be a $d+1$-dimensional semimartingale, $D = (D^1, \dots, D^d)$ a $d$-dimensional semimartingale, $S^0$ a numéraire and $\mathbf{Q}$ a probability measure that is equivalent to $\mathbf{P}$.*

*(a)* $\mathbf{Q}$ *is called an equivalent martingale measure if $\hat{S}^i + \hat{D}^i := \frac{S^i}{S^0} + \frac{1}{S^0} \bullet D^i$ is a $\mathbf{Q}$-martingale for all $i \in \{1, \dots, d\}$.*

*(b)* $\mathbf{Q}$ *is called an equivalent local martingale measure if $\hat{S}^i + \hat{D}^i$ is a local $\mathbf{Q}$-martingale for all $i \in \{1, \dots, d\}$.*

*(c)* $\mathbf{Q}$ *is called an equivalent sigma martingale measure if $\hat{S}^i + \hat{D}^i$ is a sigma martingale under $\mathbf{Q}$ for all $i \in \{1, \dots, d\}$.*

For the sake of completeness, we give an example in which a continuous dividend payment is made.

**Example 1.**
- *Suppose a corporation pays a continuous-time dividend at rate $S_t\delta$ per share, where $\delta \in \mathbb{R}_+ \setminus \{0\}$. So $D_t = \delta \int_0^t S_u \mathrm{d}u$. Under a martingale measure $\mathbf{Q}$, then, the discounted price process $e^{-rt}S_t$ is no longer a martingale but the process $e^{-rt}S_t + \delta \int_0^t e^{-ru} S_u \mathrm{d}u$.*
- *With a tracker certificate on a share, the dividend distributions of the share are automatically invested in new shares of the same company. If the certificate starts with one share (or the value $S_0$), then in the above example, the replicating portfolio at time $t$ consists of $\exp(\delta t)$ shares whose discounted value is*

$$\tilde{S}_t := e^{(\delta - r)t} S_t.$$

*This can be seen from the fact that $\varphi_t := \exp(\delta t)$ solves the differential equation $\varphi' = \delta\varphi$ with $\varphi_0 = 1$ and the dividend payment $\delta \varphi_t S_t \mathrm{d}t$ can finance the purchase of $\delta \varphi_t \mathrm{d}t = \varphi'_t \mathrm{d}t$ shares. Alternatively, one shows that the strategy $(0, \varphi)$ satisfies the self-financing condition from (2). Indeed, with the bank account as the numéraire, the following holds for this strategy with Integration by Parts*

$$\begin{aligned} \tilde{S}_t = e^{\delta t} \hat{S}_t &= \tilde{S}_0 + \int_{(0,t]} e^{\delta u} \mathrm{d}\hat{S}_u + \delta \int_{(0,t]} \hat{S}_u e^{\delta u} \mathrm{d}u \\ &= \tilde{S}_0 + \int_0^t e^{\delta u} \mathrm{d}(\hat{S} + \hat{D})_u. \end{aligned} \tag{11}$$

*Thus (9) is satisfied and it follows by Theorem 2 that $(0, \varphi)$ is self-financing. Since integrals of locally bounded integrands are again local martingales according to Theorem A3, $\tilde{S}$ is a local martingale if $\hat{S} + \hat{D}$ is one, and since conversely from (11)*

$$\hat{S}_t + \hat{D}_t = \hat{S}_0 + \hat{D}_0 + \int_0^t e^{-\delta u} \mathrm{d}\tilde{S}_u,$$

*$\hat{S} + \hat{D}$ is also a local martingale if $\tilde{S}$ is one. Thus, it holds that for a measure $\mathbf{Q}$, the process $\hat{S} + \hat{D}$ is a $\mathbf{Q}$-local martingale if and only if the process $\tilde{S}$ is a $\mathbf{Q}$-local martingale.*

- *In the Black-Scholes model, if there is a continuous dividend payoff of the above kind, then*

$$\tilde{S}_t = S_0 \exp\left((\mu + \delta - r)t + \sigma B_t - \frac{1}{2}\sigma^2 t\right)$$

*The change of measure in the Black-Scholes model is thus given by*

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\mathbf{P}} = \exp\left(\frac{r - \mu - \delta}{\sigma} B_T - \frac{1}{2}\frac{(r - \mu - \delta)^2}{\sigma^2} T\right)$$

*and the process $B_t^{\mathbf{Q}} := B_t + \frac{\mu+\delta-r}{\sigma} t$ is a standard Brownian motion under $\mathbf{Q}$. Putting this into the price process $S_t$ yields*

$$\begin{aligned} S_t &= S_0 \exp\left(\mu t + \sigma B_t - \frac{1}{2}\sigma^2 t\right) \\ &= S_0 \exp\left(\mu t + \sigma B_t^{\mathbf{Q}} + (r - \mu - \delta)t - \frac{1}{2}\sigma^2 t\right) \\ &= S_0 \exp\left((r - \delta)t + \sigma B_t^{\mathbf{Q}} - \frac{1}{2}\sigma^2 t\right) \end{aligned}$$

*So, after the change in measure, the dividend payment leads to a reduction in the drift of the share.*

### 2.3. Admissible Strategies

The class of possible trading strategies involves strategies that would be difficult to realize. Therefore, these should be further restricted the class and exclude the strategies that would require the investor to have an infinite line of credit with the bank. This also prohibits the so-called duplication strategies, which otherwise offer arbitrage opportunities in most markets, which was already noted by [22].

**Definition 7.** *A self-financing strategy $\varphi = (\varphi^0, \varphi^1, \ldots, \varphi^d) \in L(S)$ is called admissible, if there exists an $\alpha \in \mathbb{R}_+$, such that one has*

$$V_t \geq -\alpha \quad \text{for all } t \in [0, \infty)$$

*for the corresponding wealth process $V$.*

*By $\Phi(S)$, we denote the set of admissible strategies belonging to the price process $(S_t)_{t \in \mathbb{R}_+}$.*

A note on the definition of admissibility

The definition of admissibility is not unique in the literature. There are several common definitions. It shall be noted that the models used in the literature given here usually do not consider dividends. However, the definition of admissibility is independent of the dividend process.

A self-financing strategy $\varphi \in L(S)$ is called admissible, if

(a) there exist $a, b > 0$ such that $a \leq \hat{V}_t \leq b$ almost surely [39,40];
(b) $\hat{V}_t$ is a martingale under an equivalent martingale measure $\mathbf{Q}$ for the corresponding wealth process $V$ [41–43];

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(c) $\hat{V}(\varphi)$ is square integrable [32];
(d) One has $V_t \geq 0$ for the corresponding wealth process $V$ [33,44];
(e) there exists an $\alpha \in \mathbb{R}_+$, such that one has $V_t \geq -\alpha$ for all $t \in [0, \infty)$ for the corresponding wealth process $V$ [45,46];
(f) there exists an $\alpha \in \mathbb{R}_+$, such that one has $\hat{V}_t \geq -\alpha$ for all $t \in [0, \infty)$ for the corresponding wealth process $V$ [23,24];
(g) $\hat{V}(\varphi)_t \geq -a\mathbf{E}_\mathbf{Q}[w \mid \mathcal{F}_t]$ for each $t \geq 0$, each equivalent sigma martingale measure $\mathbf{Q}$ a positive number $a$ and a random variable $w \geq 1$ such that $\mathbf{E}_{\mathbf{Q}_0}[w] < \infty$ for an equivalent martingale measure $\mathbf{Q}_0$, as in [24], Section 5.

In some textbooks, further distinctions are made, for example in [31] tame is defined according to (d) and admissible according to (b) and in [47] tame is defined according to (e), admissible according to (d) and a strategy satisfying condition (b) is called regular. In the original proof of the Second Fundamental Theorem of Asset Pricing [22,48] an admissible strategy needs to satisfy both (d) and (b).

(a) is the most restrictive condition and implies (b), (c), (e) and (f). Furthermore, it excludes many common strategies such as any hedging strategies for a European call option in the Black Scholes model.

(b) is hard to justify from an economic standpoint. However, it is necessary for the Second Fundamental Theorem of Asset Pricing to hold.

(c) and (g) are rather technical assumptions and hard to justify economically.

(f) is a slightly less restrictive definition and is, for example, used in the original literature about the First Fundamental Theorem of Asset Pricing. However, it is still hard to justify, why your credit line is limited in discounted terms instead of nominal terms. It is, therefore, somehow unrealistic.

(d) and (e) are reasonable from an economic standpoint. For almost they can be treated interchangeably. However, it is more challenging to prove basic properties for the mathematical model. It should be mentioned that these two definitions are the only ones that are numéraire independent and, therefore, the only ones that can potentially be applied to a model without a fixed numéraire as defined in [38]. As these are the most realistic definitions and (e) is slightly more general than (d), we choose (e) for our definitions.

It shall be noted that in all of the above-mentioned cases the set of admissible strategies might still be small or even empty. For example, consider a compound Poisson process $X$ with unbounded jumps. Clearly, any strategy $\varphi$, making sure that $\int \varphi \mathrm{d}X$ is bounded from below, must vanish almost surely. The same holds for a process $S_t := X\mathbb{1}_{[1,\infty)}$ with $X$ being a normally distributed real random variable.

**Remark 6.** *Under all of the above-mentioned conditions, the discounted value process $\hat{V}(\varphi)$ is a local martingale and a supermartingale. For (a), one can conclude with Ansel-Stricker that it is a martingale. The same holds for (b) by definition. For (d) and (f), it follows from Ansel-Stricker (and the fact that numéraires are locally bounded). For (c), we get the result with Theorem A3. For (g), one obtains the result, if one applies Ansel-Stricker on $\hat{V}(\varphi)_t + a\mathbf{E}_\mathbf{Q}[w \mid \mathcal{F}_t]$. For (e), see Proposition 4.*

Despite its simplicty, the following Theorem 4 is an extremely crucial result as, without it, statements such as the Fundamental Theorems of Asset Pricing could not be transformed from the discounted to the non-discounted setup. We need the following recent result for it to hold.

**Theorem 3** ([49], Theorem 6)**.** *Let $X$ be a one-dimensional sigma martingale. Then the following are equivalent:*

*(i) $X$ is a local martingale.*
*(ii) There exist a local martingale $M$ and a càdlàg finite variation process $A$ such that $\Delta X \geq \Delta(M + A)$.*
*(iii) There exist a local martingale $M$ and a càdlàg process $A$ (with $\sup_{s \leq t} A_s$ locally integrable) for which $\Delta X \geq \Delta(M + A)$.*
*(iv) There exist a local martingale $M$ and a càdlàg finite variation process $A$ such that $X \geq M + A$.*

(v) *There exist a local martingale $M$ and a càdlàg process $A$ (with $\sup_{s\le t} A_s$ locally integrable) satisfying $X \ \ge \ M + A$.*

**Theorem 4.** *Let $\varphi$ be an admissible strategy. If $\mathbf{Q}$ is an equivalent sigma martingale measure for $\hat{S}+\hat{D}$, then $\varphi \bullet (\hat{S}+\hat{D})$ is a $\mathbf{Q}$-supermartingale.*

**Proof.** This is a direct consequence of Theorem 2 and Theorem 3. □

## 3. Fair Prices

In this section, we will deal with the pricing of derivatives. As usual, our goal is to determine the prices in such a way that it cannot lead to arbitrage, in order to obtain a model that is as realistic as possible.

**Definition 8.** *(a) A claim with expiration date $T$ is a non-negative random variable $X \in L^1(\Omega, \mathcal{F}_T, \mathbf{P})$.*

*(b) An admissible trading strategy $\phi \in \Phi$ is called a hedge for a claim $X$ if*

$$V_T(\phi) \ge X \quad a.s.$$

*(c) A claim is called attainable if there exists an admissible trading strategy $\phi \in \Phi$ such that*

$$V_T(\phi) = X \quad a.s.$$

*This corresponding trading strategy is called a perfect hedge.*

*(d) A financial market model is called complete if for every claim there exists a perfect hedge.*

$X$ is usually a function of a price process $S : X = f(S)$. For instance, $X := (S_T - K)^+$ for a European call option with strike price $K$ and expiration date $T$.

**Theorem 5.** *Let $X$ be a claim maturing at time $T$ and let $S^0$ be a numéraire such that $[S^0, D]^c = 0$. Then $X$ is attainable if and only if there exists an admissible strategy $\varphi = (\varphi^0, \dots, \varphi^d) \in \Phi(S)$ satisfying*

$$\frac{X}{S_T^0} \ = \ \big(\varphi \bullet (\hat{S}+\hat{D})\big)_T,$$

*where $\hat{S}^i := S^i/S^0$ and $\hat{D}^i := (1/S^0) \bullet D^i$ are the discounted price and dividend processes, respectively, and $\hat{S}+\hat{D} := (\hat{S}^1+\hat{D}^1, \dots, \hat{S}^d+\hat{D}^d)$.*

**Proof.** Assume that the claim $X$ is attainable. Hence there is an admissible self-financing strategy $\varphi$ with terminal wealth $V_T(\varphi) = X$. Set $\hat{V} := V/S^0$; by admissibility $\hat{V}$ is well–defined and càdlàg. Since $\varphi$ is self-financing, Theorem 2 yields

$$\hat{V}_t \ = \ \big(\varphi \bullet (\hat{S}+\hat{D})\big)_t, \qquad t \ge 0.$$

Evaluating this identity at $t = T$ and using $\hat{V}_T = X/S_T^0$ proves the representation in the statement.

Conversely, suppose an admissible strategy $\varphi$ satisfies

$$\frac{X}{S_T^0} = \big(\varphi \bullet (\hat{S}+\hat{D})\big)_T.$$

Define $V := \varphi \bullet (S+D)$; by Theorem 2 the strategy $\varphi$ is self-financing and $\hat{V}_t = (\varphi \bullet (\hat{S}+\hat{D}))_t$ for every $t$. Hence $V_T = S_T^0 \hat{V}_T = S_T^0 \big(\varphi \bullet (\hat{S}+\hat{D})\big)_T = X$, so $\varphi$ replicates the claim $X$. Therefore $X$ is attainable. □

Since a financial market model is formally given by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ and the price processes $S^0, S^1, \ldots, S^d$ of tradable securities, completeness (just like absence from arbitrage) also depends on the filtration $\mathbb{F}$, i.e., the investor's flow of information.

By Theorem 3 we have

$$\mathbf{E}_{\mathbf{Q}}(\hat{V}_T(\varphi)) \leq \hat{V}_0 \tag{12}$$

for any admissible strategy $\varphi$. That means with an admissible strategy in a fair game (risk-neutral probability measure), a trader cannot win on average. In discrete time or with another definition of an admissible strategy, $\hat{V}$ would even be a $\mathbf{Q}$-martingale and therefore, on average, one would never lose anything. In our terminology, however, $\hat{V}$ is only a supermartingale. This raises the question of whether there is a strategy in which we would lose on average and whether we could replace the $\leq$ in (12) with a $<$.

In discrete time (which can also be assumed as a special case of continuous time), such a strategy would be given by the reverse doubling strategy. We consider a game in which we can win with a certain probability and can lose with a certain probability. Under the risk-neutral probability measure, both would occur with a probability of $\frac{1}{2}$. We start with a one euro bet and double the bet every time we win, stopping the game as soon as we lose once. With an infinite time horizon, we would therefore lose one euro on average. Thus, we also see that with an infinite time horizon, there are certainly strategies by which we can win on average. However, these are not admissible. The time when we would lose the first time is distributed geometrically, and therefore, the expected value of the game is negative.

While, as already demonstrated, there are no arbitrage opportunities in a market with suicide strategies when an equivalent sigma martingale measure exists (note that a strategy derived from a suicide strategy by switching signs offers an arbitrage opportunity but is itself not an admissible strategy since possible losses are not limited, just as profits are not limited in a suicide strategy), such strategies are problematic in derivative pricing. It's not sufficient to simply "remove" these strategies from the set of admissible strategies.

To illustrate, consider a claim $X$ and an admissible strategy $\phi$ with

$$V_T(\phi) = \phi_T^0 S_T^0 + \phi_T^1 S_T = X.$$

It is intuitive to set the fair price of the claim at time 0 as the value of the replicating portfolio $V_0(\phi)$. However, if $\tilde{\phi} = (\tilde{\phi}^1, \tilde{\phi}^2)$ is an admissible suicide strategy with

$$\tilde{\phi}_0^0 S_0^0 + \tilde{\phi}_0^1 S_0^1 = 1 \quad \text{and} \quad \tilde{\phi}_T^0 S_T^0 + \tilde{\phi}_T^1 S_T^1 = 0,$$

then $\hat{\phi} = (\phi^0 + \tilde{\phi}^0, \phi^1 + \tilde{\phi}^1)$ is also an admissible strategy with

$$V_T(\hat{\phi}) = (\phi_T^0 + \tilde{\phi}_T^0) S_T^0 + (\phi_T^1 + \tilde{\phi}_T^1) S_T^1 = V_T(\phi) + V_T(\tilde{\phi}) = X.$$

Thus, the strategy $\hat{\phi}$ is also a replicating strategy for the claim $X$. However, it holds that

$$V_0(\hat{\phi}) = V_0(\phi) + V_0(\tilde{\phi}) = V_0(\phi) + 1 \neq V_0(\phi).$$

Therefore, the price of a replicating portfolio is generally not unique, and neither is the fair price. Since $\hat{\phi}$ is not itself a suicide strategy, it is also not sufficient to just remove these from the set of admissible strategies to achieve a unique "hedge price".

Our goal now is to compute an interval in which an "economically sensible" price should lie. We will subsequently show that the so-called risk-neutral prices always lie within this interval and therefore represent "sensible" prices. Since these are relatively easy to calculate and, depending on the nature of the underlying probability measure, further economic justifications can be found for this type of price, most pricing formulas are based on risk-neutral prices.

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**Definition 9.** *Let there be a market in which an equivalent sigma martingale measure* $\mathbf{Q}$ *exists.*

(a) *A perfect hedge* $\phi$ *is called a martingale hedge if* $\hat{V}(\phi)$ *is a* $\mathbf{Q}$*-martingale.*

(b) *Let* $\Phi$ *denote the set of all admissible strategies.*

  (i) *The superhedging price or seller's arbitrage price* $\pi_V$ *of a claim* $X$ *is given by*

$$\pi_V(X) := \inf\{V_0(\phi) \,:\, \phi \in \Phi \text{ and } V_T(\phi) \geq X \text{ a.s.}\}.$$

  (ii) *The buyer's arbitrage price is defined as*

$$\pi_B(X) := \sup\{V_0(\phi) \,:\, -\phi \in \Phi \text{ and } V_T(\phi) \leq X \text{ a.s.}\}.$$

(c) *For a specific equivalent sigma martingale measure* $\mathbf{Q}$*, we call*

$$\tilde{\pi}^{\mathbf{Q}}(X) := \mathbf{E}_{\mathbf{Q}}\left(\frac{S_0^0}{S_T^0}X\right)$$

*the risk-neutral price with respect to measure* $\mathbf{Q}$ *of* $X$*.*

The naming of the seller's arbitrage price is intuitive because, if you sell a claim at the seller's arbitrage price, you can (at least approximately due to the infimum) construct a portfolio at precisely these costs such that at time $T$ you have made a risk-free profit of $V_T(\phi) - X \geq 0$.

On the other hand, the buyer's arbitrage price corresponds to the highest price a claim can assume so that the buyer can hedge against any possible loss (and possibly make a risk-free profit). Assuming a claim $X$ has a market price $X_0$ with $X_0 < \pi_B(X)$, then there exists (at least approximately) a strategy $\psi$ with $-\psi \in \Phi$ for which

$$X_0 + V_0(\psi) < \pi_B(X) + V_0(\psi) \leq 0,$$

but at the same time

$$X_T + V_T(\psi) = X_T - V_T(-\psi) \geq 0.$$

This creates an arbitrage opportunity for the buyer of the claim.

It is important to note that unlike the superhedging and buyer's arbitrage prices, which are independent of the choice of equivalent martingale measure, the risk-neutral price $\tilde{\pi}^{\mathbf{Q}}(X)$ depends explicitly on the selected measure $\mathbf{Q}$. In incomplete markets, where multiple equivalent sigma martingale measures exist, different choices of $\mathbf{Q}$ will generally yield different risk-neutral prices.

Calculating a superhedging price using the above definition proves to be challenging, whereas determining the risk-neutral price with respect to a specific martingale measure is relatively straightforward. Therefore, financial products are often priced using risk-neutral pricing methods. The following theorem provides an economic justification for this approach by showing that risk-neutral prices always lie within the arbitrage-free price bounds, and coincide with the superhedging price when a martingale hedge exists.

**Theorem 6.** *Let* $\mathbf{Q}$ *be any equivalent sigma martingale measure.*

(a) *It always holds that*

$$\pi_B(X) \leq \tilde{\pi}^{\mathbf{Q}}(X) \leq \pi_S(X).$$

(b) *If a martingale hedge* $\phi$ *exists, then*

$$\pi_S(X) = \tilde{\pi}^{\mathbf{Q}}(X). \tag{13}$$

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**Proof.** (a) Since $\hat{V}(\phi)$ according to Theorem 3 is a $\mathbf{Q}$-supermartingale for all admissible $\phi$ and $\mathcal{F}_0$ is the trivial $\sigma$-algebra, it holds for $\phi \in \Phi$ with $V_T(\phi) \geq X$ that

$$\begin{aligned} V_0(\phi) &= S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{V_0(\phi)}{S_0^0}\right) = S_0^0 \mathbf{E}_{\mathbf{Q}}(\hat{V}_0(\phi)) \\ &\geq S_0^0 \mathbf{E}_{\mathbf{Q}}(\hat{V}_T(\phi)) = S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{V_T(\phi)}{S_T^0}\right) \\ &\geq S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{X}{S_T^0}\right) = \mathbf{E}_{\mathbf{Q}}\left(\frac{S_0^0}{S_T^0} X\right) = \tilde{\pi}^{\mathbf{Q}}(X). \end{aligned} \tag{14}$$

Thus, $\pi_S(X) \geq \tilde{\pi}^{\mathbf{Q}}(X)$.
For the second inequality, we proceed analogously and obtain for $-\phi \in \Phi$ with $V_T(\phi) \leq X$

$$\begin{aligned} V_0(-\phi) &\geq S_0^0 \mathbf{E}_{\mathbf{Q}}(\hat{V}_T(-\phi)) \\ &= -S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{V_T(\phi)}{S_T^0}\right) \\ &\geq -S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{X}{S_T^0}\right) \\ &= -\tilde{\pi}^{\mathbf{Q}}(X). \end{aligned}$$

Taking the supremum, we get $\pi_B(X) \geq -(-\tilde{\pi}^{\mathbf{Q}}(X)) = \tilde{\pi}^{\mathbf{Q}}(X)$.

(b) Now let $\phi$ be a martingale hedge and therefore $\hat{V}(\phi)$ is a $\mathbf{Q}$-martingale. It follows

$$\begin{aligned} \pi_S(X) \leq V_0(\phi) &= S_0^0 \mathbf{E}_{\mathbf{Q}}(\hat{V}_0(\phi)) \\ &= S_0^0 \mathbf{E}_{\mathbf{Q}}(\hat{V}_T(\phi)) \quad \text{(by martingale property)} \\ &= S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{V_T(\phi)}{S_T^0}\right) \\ &= S_0^0 \mathbf{E}_{\mathbf{Q}}\left(\frac{X}{S_T^0}\right) \quad (\text{since } V_T(\phi) = X) \\ &= \tilde{\pi}^{\mathbf{Q}}(X). \end{aligned}$$

Together with ((a)), this now leads to (13).
□

So far, we have only defined the fair price for the time $t = 0$. However, this concept can be extended to any time $t \in [0, T]$. It should be noted, however, that the fair price $\pi_t(X)$ is then a random variable and not a fixed value. Theorem 6 then applies analogously when we replace the expected values with the conditional expectations given $\mathcal{F}_t$.

**Remark 7.** *Finding a mathematically precise as well as economically meaningful definition for the fair price was and remains one of our main objectives. So far, we have found such a definition only for the case of a market in which in which an equivalent sigma martingale measure exists exists and for the case where a martingale hedge exists. The latter is indeed a restriction, as in markets with transaction costs or dividends, a hedge often does not exist, and even if one exists, it does not necessarily have to be a martingale hedge. For an explanation, see for example [50] or [51,52].*

*While $\mathbf{E}_{\mathbf{Q}}(X \mid \mathcal{F}_t)$ for a contingent claim $X$ defines a martingale, there may not be an admissible strategy that replicates this process as a wealth process. This is particularly important to note since the discounted price processes $\hat{S}^i$ with respect to the equivalent martingale measure $\mathbf{Q}$ are only local martingales and not necessarily $L^2$-martingales, which would be required for a direct application of the martingale representation theorem.*

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We now show an application of the above theorem.

**Example 2.** *Let $S_t^0 = e^{rt}$ (a deterministic money market account with constant interest rate $r$). We consider both a European Call option $C$ and a European Put option $P$, both with the same strike price $K$ and expiration date $T$, and for both a martingale hedge exists. Let $\mathbf{Q}$ be the equivalent martingale measure under which both options admit martingale hedges.*

*Now, we have:*

$$\begin{aligned}\tilde{\pi}^{\mathbf{Q}}(C) - \tilde{\pi}^{\mathbf{Q}}(P) &= \mathbf{E}_{\mathbf{Q}}\Big(e^{-rT}(S_T - K)^+\Big) - \mathbf{E}_{\mathbf{Q}}\Big(e^{-rT}(K - S_T)^+\Big) \\ &= e^{-rT}\mathbf{E}_{\mathbf{Q}}\big((S_T - K)^+ - (K - S_T)^+\big) \\ &= e^{-rT}\mathbf{E}_{\mathbf{Q}}(S_T - K) \quad (\text{since } (S_T - K)^+ - (K - S_T)^+ = S_T - K) \\ &= e^{-rT}\mathbf{E}_{\mathbf{Q}}(S_T) - e^{-rT}K \\ &= e^{-rT} \cdot e^{rT}S_0 - e^{-rT}K \quad (\text{since } \mathbf{E}_{\mathbf{Q}}(S_T) = e^{rT}S_0 \text{ under } \mathbf{Q}) \\ &= S_0 - e^{-rT}K.\end{aligned}$$

*This corresponds exactly to the well-known Put-Call Parity for $t = 0$. Furthermore, since both options admit martingale hedges, by Theorem 6 (b), we have $\pi_S(C) = \tilde{\pi}^{\mathbf{Q}}(C)$ and $\pi_S(P) = \tilde{\pi}^{\mathbf{Q}}(P)$, which confirms that the risk-neutral price equals the seller's arbitrage price in this case.*

## Appendix A. Referenced Results

This appendix contains well-known results that were referenced in the paper. Their proofs can be found in [27] and [49].

**Theorem A1.** *Let $X \in \mathcal{S}^d$ be a d-dimensional topological semimartingale and $H \in L(X)$. Then the following hold:*

*(a)* *For $\alpha, \beta \in \mathbb{R}$ and $J \in L(X)$,*

$$(\alpha H + \beta J) \in L(X) \quad \text{and} \quad (\alpha H + \beta J) \bullet X \;=\; \alpha H \bullet X + \beta J \bullet X.$$

*Hence $L(X)$ is a vector space.*

*(b)* *If $Y \in \mathcal{S}^d$ and $H \in L(X) \cap L(Y)$, then $H \in L(X + Y)$ and*

$$H \bullet (X + Y) \;=\; H \bullet X \;+\; H \bullet Y.$$

*(c)* *$(\Delta(H \bullet X)_s) = H_s^\top \Delta X_s$ almost surely, for all $s \geq 0$.*

*(d)* *$(H \bullet X)^T = (H\mathbb{1}_{[\![0,T]\!]}) \bullet X = (H \bullet X^T)$ for any stopping time $T$.*

*(e)* *Let $X = (X^1, \ldots, X^d)$, $H = (H^1, \ldots, H^d)$ with $H^i \in L(X^i)$, and $K = (K^1, \ldots, K^d)$ be predictable. Then*

$$(K \bullet (H \bullet X)) \;=\; (KH) \bullet X$$

*whenever the relevant integrals exist. Concretely, $K^i \in L(H^i \bullet X^i)$ if and only if $K^i H^i \in L(X^i)$.*

*(f)* *If $X$ is an FV semimartingale, then $H \bullet X$ coincides with the usual pathwise Riemann–Stieltjes integral.*

*(g)* *If $\mathbf{Q} \ll \mathbf{P}$, the notion of integrability $H \in L(X)$ does not change, and $H \bullet X$ agrees $\mathbf{Q}$-a.s. with the same process defined under $\mathbf{P}$.*

**Theorem A2.** *Let $X, Y$ be two semimartingales. Then the product process $XY$ is again a semimartingale, and the following decomposition holds:*

$$XY = X_- \bullet Y + Y_- \bullet X + [X, Y]. \tag{A1}$$

**Theorem A3.** *A sigma martingale $X$ is a local martingale if and only if $X$ is locally integrable.*

**Theorem A4.** *Let $X, Y \in \mathcal{S}^d$. Then $[X, Y]$ is, in each component, a finite variation process and a semimartingale, and it satisfies the following properties:*

*(a)* $[X, Y]_0 = X_0 Y_0$ *and* $\Delta[X, Y]_{i,j} = \Delta X^i \Delta Y^j$.

*(b)* *For any stopping time $T$, we have*

$$[X^T, Y] = [X, Y^T] = [X^T, Y^T] = [X, Y]^T.$$

*(c)* *The quadratic variation $[X, X]$ is a positive, increasing process.*

*(d)* *If $X$ is an FV process, then*

$$[X, Y]_t = X_0 Y_0^\top + \sum_{0<s\le t} \Delta X_s \Delta Y_s^\top.$$

**Theorem A5.** *(a) A $d$-dimensional process $X$ is a topological semimartingale if and only if each of its $d$ components is a topological semimartingale in the one-dimensional sense.*

*(b) A process that is locally a topological semimartingale or prelocally a topological semimartingale is automatically a topological semimartingale (i.e. the property is preserved under localization).*

*(c) If $\mathbf{Q} \ll \mathbf{P}$, then any topological semimartingale under $\mathbf{P}$ remains a topological semimartingale under $\mathbf{Q}$.*

**Remark A1.** *The space of semimartingales exhibits notable stability properties. For instance, if $f : \mathbb{R} \to \mathbb{R}$ is a convex function and $X$ is a semimartingale, then the process $f(X)$ is again a semimartingale.*

**Lemma A1** (Ansel–Stricker Lemma)**.** *Let $X$ be a one-dimensional sigma martingale that is bounded from below (for instance $X \ge 0$). Then $X$ is a local martingale and also a supermartingale. Similarly, if $X$ is bounded from above, then $X$ is a submartingale.*

**Theorem A6.** *Let $B(t)$ be Brownian motion. Then the following processes are martingales:*

*(a)* $B(t)$

*(b)* $B(t)^2 - t$

*(c)* $\exp\left(uB(t) - \frac{u^2}{2} t\right)$*, for any* $u \in \mathbb{R}$

**Theorem A7.** *Let $B(t)$ be Brownian motion. Then*

$$\lim_{t\to\infty} \frac{B(t)}{t} = 0 \quad a.s.$$

## References


1. Bachelier, L. Théorie de la spéculation. In Proceedings of the Annales scientifiques de l'École normale supérieure, 1900, Vol. 17, pp. 21–86.
2. Itô, K. Stochastic integral. *Proc. Imp. Acad.* **1944**, *20*, 519–524. https://doi.org/10.3792/pia/1195572786.
3. Itô, K. On a stochastic integral equation. *Proc. Japan Acad.* **1946**, 22, 32–35. https://doi.org/10.3792/pja/1195572371.
4. Itô, K. Stochastic differential equations in a differentiable manifold. *Nagoya Math. J.* **1950**, *1*, 35–47.
5. Itô, K. Multiple Wiener integral. *Journal of the Mathematical Society of Japan* **1951**, *3*, 157–169.
6. Itô, K. On a formula concerning stochastic differentials. *Nagoya Math. J.* **1951**, *3*, 55–65.
7. Itô, K. *On Stochastic Differential Equations*; Vol. 4, *Memoirs of the American Mathematical Society*, American Mathematical Society: Providence, RI, 1951.
8. Black, F.; Scholes, M. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy* **1973**, *81*, 637–654.
9. Doob, J.L. *Stochastic Processes*; Vol. 7, *Wiley Classics Library*, John Wiley & Sons: New York, 1953.
10. Meyer, P.A. A decomposition theorem for supermartingales. *Illinois Journal of Mathematics* **1962**, *6*, 193–205. https://doi.org/10.1215/ijm/1255632318.
11. Meyer, P.A. Decomposition of supermartingales: The uniqueness theorem. *Illinois Journal of Mathematics* **1963**, *7*, 1–17. https://doi.org/10.1215/ijm/1255637477.

12. Kunita, H.; Watanabe, S. On square integrable martingales. *Nagoya Mathematical Journal* **1967**, *30*, 209–245. https://doi.org/10.1017/s0027763000012484.
13. Meyer, P.A. Intégrales stochastiques I. *Séminaire de Probabilités de Strasbourg* **1967**, *39*, 72–94.
14. Meyer, P.A. Intégrales stochastiques II. *Séminaire de probabilités de Strasbourg* **1967**, *1*, 95–117.
15. Meyer, P.A. Intégrales stochastiques III. *Séminaire de probabilités de Strasbourg* **1967**, *1*, 118–141.
16. Meyer, P.A. Intégrales stochastiques IV. *Séminaire de probabilités de Strasbourg* **1967**, *1*, 142–162.
17. Doléans-Dade, C.; Meyer, P.A. Intégrales stochastiques par rapport aux martingales locales. In *Séminaire de Probabilités IV, Université de Strasbourg*; Springer: Berlin, 1970; Vol. 124, *Lecture Notes in Mathematics*, pp. 77–107. https://doi.org/10.1007/BFb0066162.
18. Meyer, P.A. Un Cours sur les Intégrales Stochastiques. In *Séminaire de Probabilités 1967–1980*; Émery, M.; Yor, M., Eds.; Springer Berlin Heidelberg: Berlin, 2002; Vol. 1771, *Lecture Notes in Mathematics*, pp. 174–329. https://doi.org/10.1007/978-3-540-45530-1_11.
19. Jacod, J. *Calcul Stochastique et Problèmes de Martingales*; Vol. 714, *Lecture Notes in Mathematics*, Springer-Verlag: Berlin–Heidelberg, 1979; pp. x+280.
20. Chou, C.S.; Meyer, P.A.; Stricker, C. Sur les intégrales stochastiques de processus prévisibles non bornés. In *Séminaire de Probabilités XIV, 1978/79*; Springer: Berlin, 1980; Vol. 784, *Lecture Notes in Mathematics*, pp. 128–139. https://doi.org/10.1007/BFb0089478.
21. Jacod, J. Intégrales stochastiques par rapport à une semimartingale vectorielle et changements de filtration. In *Séminaire de Probabilités XIV, 1978/79*; Springer: Berlin, 1980; Vol. 784, *Lecture Notes in Mathematics*, pp. 161–172.
22. Harrison, J.M.; Pliska, S.R. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic processes and their applications* **1981**, *11*, 215–260. https://doi.org/10.1016/0304-4149(81)90026-0.
23. Delbaen, F.; Schachermayer, W. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* **1994**, *300*, 463–520. https://doi.org/10.1007/BF01450498.
24. Delbaen, F.; Schachermayer, W. The Fundamental Theorem of Asset Pricing for Unbounded Stochastic Processes. *Mathematische Annalen* **1998**, *312*, 215–250. https://doi.org/10.1007/s002080050220.
25. Jarrow, R. *Continuous-Time Asset Pricing Theory: A Martingale-Based Approach*, 2. ed.; Springer Finance, Springer International Publishing, 2021.
26. Cohen, S.N.; Elliott, R.J. *Stochastic Calculus and Applications*; Probability and Its Applications, Springer: New York, 2015. https://doi.org/10.1007/978-1-4939-2867-5.
27. Sohns, M. General Stochastic Vector Integration - Three Approaches. *Preprints* **2025**. Submitted to *Probability Surveys*, https://doi.org/10.20944/preprints202501.0645.v1.
28. Ansel, J.P.; Stricker, C. Lois de martingale, densités et décomposition de Föllmer Schweizer. *Annales de l'institut Henri Poincaré (B) Probabilités et Statistiques* **1992**, *28*, 375–392.
29. Kardaras, C.; Platen, E. On the semimartingale property of discounted asset-price processes. *Stochastic processes and their Applications* **2011**, *121*, 2678–2691. https://doi.org/10.1016/j.spa.2011.06.010.
30. Guasoni, P. No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Mathematical Finance* **2006**, *16*, 569–582. https://doi.org/10.1111/j.1467-9965.2006.00283.x.
31. Bingham, N.H.; Kiesel, R. *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, 2 ed.; Springer Finance, Springer: London, 2013. https://doi.org/10.1007/978-1-4471-3856-3.
32. Elliott, R.J.; Kopp, P.E. *Mathematics of Financial Markets*, 2 ed.; Springer Finance, Springer-Verlag: New York, 2005; pp. xii+352.
33. Pascucci, A. *PDE and Martingale Methods in Option Pricing (Bocconi & Springer Series)*; Springer, 2011. https://doi.org/10.1007/978-88-470-1781-8.
34. Geman, H.; Karoui, N.E.; Rochet, J.C. Changes of Numéraire, Changes of Probability Measure and Option Pricing. *Journal of Applied Probability* **1995**, *32*, 443–458. https://doi.org/10.2307/3215299.
35. Ebenfeld, S. *Grundlagen der Finanzmathematik: Mathematische Methoden, Modellierung von Finanzmärkten und Finanzprodukten*; Schäffer-Poeschel: Stuttgart, 2007.
36. Klein, I.; Schmidt, T.; Teichmann, J. No Arbitrage Theory for Bond Markets. In *Advanced Modelling in Mathematical Finance*; Springer, 2016; pp. 381–421. https://doi.org/10.1007/978-3-319-45875-5_17.
37. Qin, L.; Linetsky, V. Long-Term Risk: A Martingale Approach. *Econometrica* **2017**, *85*, 299–312. https://doi.org/10.3982/ecta13438.
38. Herdegen, M.; Schweizer, M. Strong bubbles and strict local martingales. *International Journal of Theoretical and Applied Finance* **2016**, *19*, 1650022. https://doi.org/10.1142/s0219024916500229.

doi:10.20944/preprints202505.0844.v1




39. Shiryaev, A.N. *Essentials of Stochastic Finance*; Vol. 3, *Advanced Series on Statistical Science & Applied Probability*, World Scientific Publishing Co., Inc.: River Edge, NJ, 1999; pp. xvi+834. https://doi.org/10.1142/9789812385192.
40. Shiryaev, A.N.; Cherny, A. Vector stochastic integrals and the fundamental theorems of asset pricing. *Proceedings of the Steklov Institute of Mathematics-Interperiodica Translation* **2002**, *237*, 6–49.
41. Filipovic, D. *Term-Structure Models. A Graduate Course.*; Springer, 2009. https://doi.org/10.1007/978-3-540-68015-4.
42. Biagini, F. Second Fundamental Theorem of Asset Pricing. *Encyclopedia of Quantitative Finance* **2010**, *4*, 1623–1628. https://doi.org/10.1002/9780470061602.eqf04008.
43. Musiela, M.; Rutkowski, M. *Martingale Methods in Financial Modelling*, 2 ed.; Vol. 36, *Stochastic Modelling and Applied Probability*, Springer: Berlin, 2005; pp. xx+638.
44. Klebaner, F.C. *Introduction to Stochastic Calculus with Applications*, 2 ed.; Imperial College Press: London, 2005; pp. xvi+412.
45. Øksendal, B. *Stochastic Differential Equations: An Introduction with Applications*, 6 ed.; Universitext, Springer: Berlin, 2010; p. 404. https://doi.org/10.1007/978-3-642-14394-6.
46. Kuo, H.H. *Introduction to Stochastic Integration*, 1 ed.; Universitext, Springer: New York, 2006; pp. x+278. https://doi.org/10.1007/b137005.
47. Yan, J.A. *Introduction to Stochastic Finance*, 1 ed.; Universitext, Springer: Singapore, 2018. https://doi.org/10.1007/978-981-13-1657-9.
48. Harrison, J.M.; Pliska, S.R. A stochastic calculus model of continuous trading: Complete markets. *Stochastic Processes and their Applications* **1983**, *15*, 313–316. https://doi.org/10.1016/0304-4149(83)90038-8.
49. Sohns, M. $\sigma$-Martingales: Foundations, Properties, and a New Proof of the Ansel–Stricker Lemma. *Mathematics* **2025**, *13*, 682. https://doi.org/10.3390/math13040682.
50. Cox, A.; Hobson, D. Local martingales, bubbles and option prices. *Finance and Stochastics* **2005**, *9*, 477–492. https://doi.org/10.1007/s00780-005-0162-y.
51. Jarrow, R.A.; Protter, P.; Shimbo, K. Asset Price Bubbles in Complete Markets. In *Advances in Mathematical Finance*; Fu, M.C.; Jarrow, R.A.; Yen, J.Y.J.; Elliott, R.J., Eds.; Applied and Numerical Harmonic Analysis, Birkhäuser Boston, 2007; pp. 97–121. https://doi.org/10.1007/978-0-8176-4545-8_7.
52. Jarrow, R.A.; Protter, P.; Shimbo, K., Asset Price Bubbles in Complete Markets. In *Advances in Mathematical Finance*; Fu, M.C.; Jarrow, R.A.; Yen, J.Y.J.; Elliott, R.J., Eds.; Birkhäuser Boston: Boston, MA, 2007; pp. 97–121. https://doi.org/10.1007/978-0-8176-4545-8_7.